

EE 2015

(Partial) Differential Equations and Complex Variables

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Second-order ODEs: Superposition

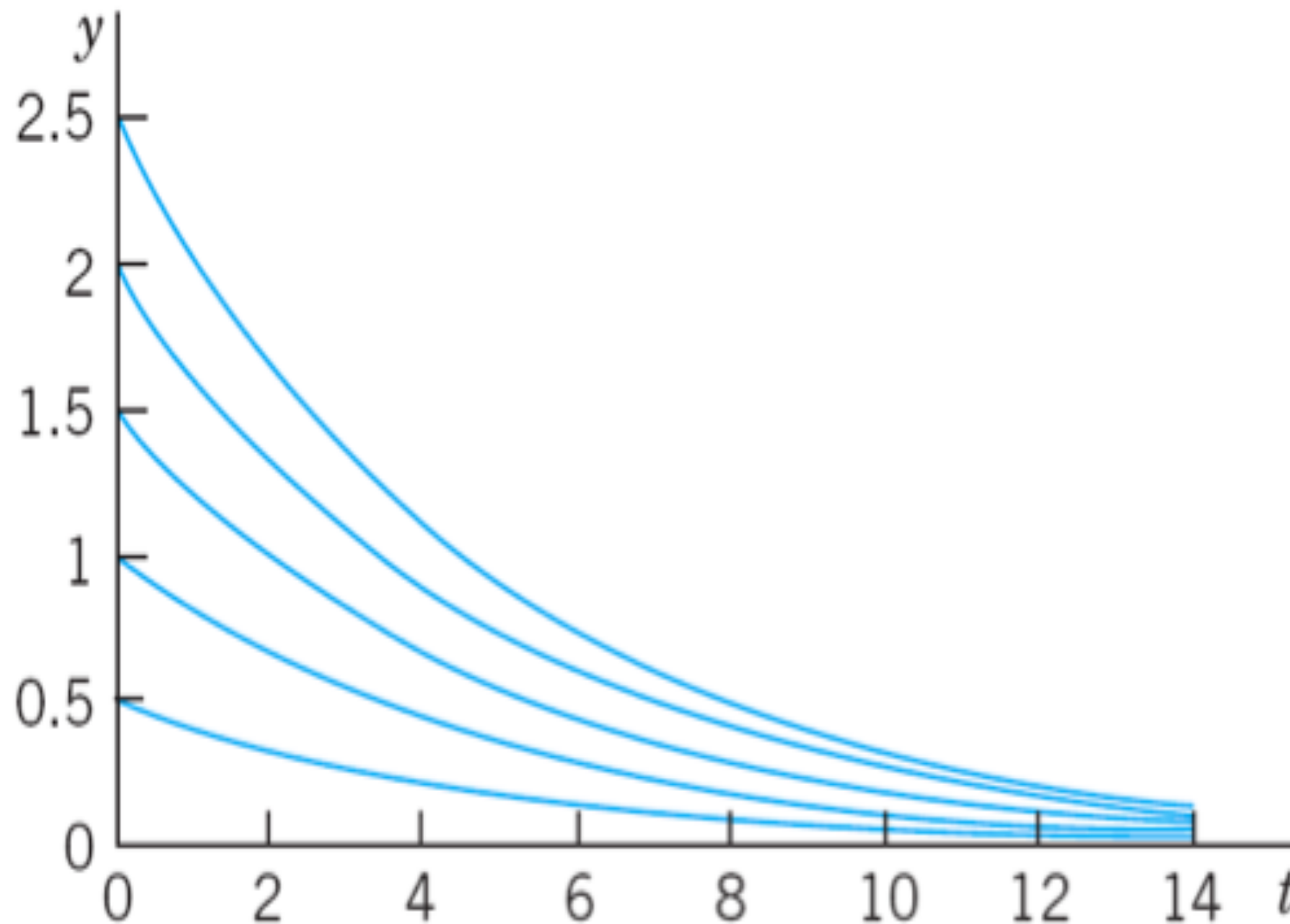
$$y'' + p(x)y' + q(x)y = r(x),$$

- **Homogeneous** ODEs of Second-order: $r(x) = 0$,
- **Non-homogeneous** ODEs of Second-order: $r(x) \neq 0$.
- Superposition principle applies to
 1. homogeneous linear ODEs ✓
 2. non-homogeneous linear ODEs ✗



Family of solutions: Exponential Decay

$$y' = -0.2y$$



Second-order ODEs: Linearly Independent

- Two functions $y_1(x)$ and $y_2(x)$ are called **linearly independent** on an interval I where they are defined if

$$c_1 y_1(x) + c_2 y_2(x) = 0, \quad \text{everywhere on } I,$$

which implies that the only solution is $c_1 = c_2 = 0$.

- If $c_1 \neq 0$ or $c_2 \neq 0$, we can divide and see that y_1 and y_2 are proportional, *linearly dependent*,

$$y_1 = -\frac{c_2}{c_1} y_2, \quad \text{or} \quad y_2 = -\frac{c_1}{c_2} y_1.$$

- In other words, the ratio between two linearly dependent functions is a constant,

$$\frac{y_2}{y_1} = \text{constant}.$$



Second-order ODEs: Initial Value Problem

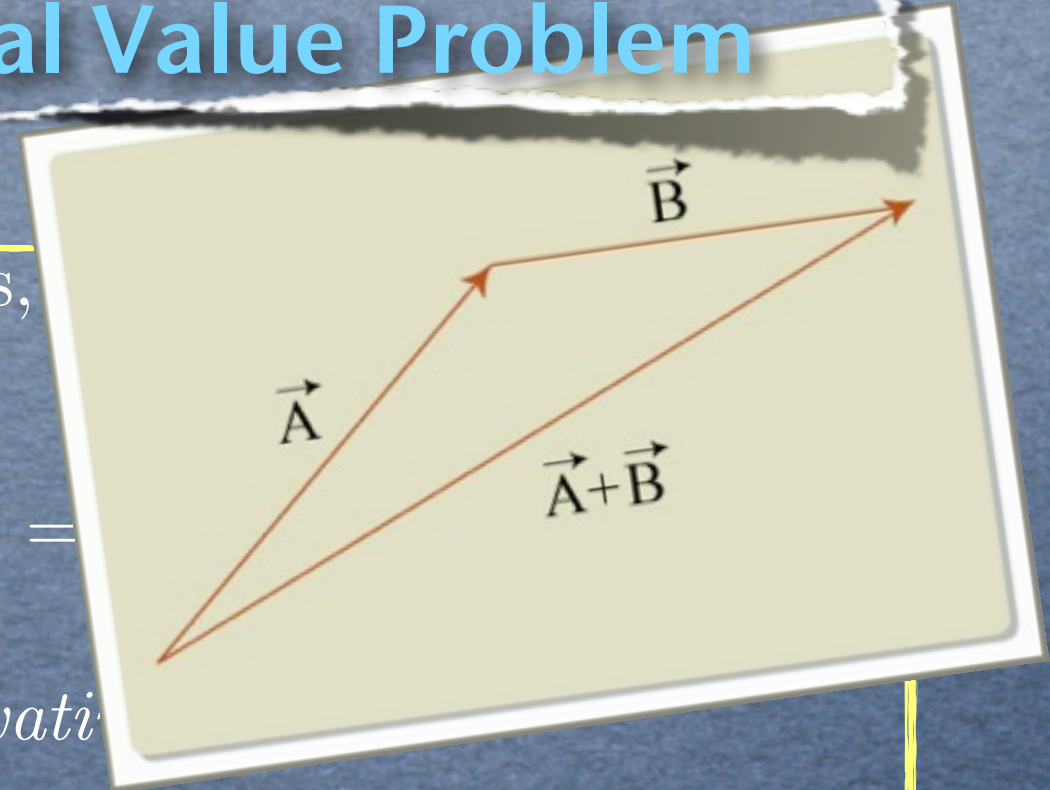
- For a Second-order **homogeneous linear** ODEs, **IVP** consists two initial conditions:

$$y(x_0) = K_0, \quad y'(x_0) =$$

- These conditions also prescribe its *first derivative*
- For these two initial conditions, there exists a **general solution** of the ODE,

$$y(x) = c_1 y_1(x) + c_2 y_2(x),$$

here c_1 and c_2 are two arbitrary constants; while y_1 and y_2 are called the **bases** of the ODE.



Second-order ODEs: Linearly Indep., orthogonal

- Two functions $y_1(x)$ and $y_2(x)$ are called **linearly independent** on an interval I where they are defined if

$$c_1 y_1(x) + c_2 y_2(x) = 0, \quad \text{everywhere on } I,$$

which implies that the only solution is $c_1 = c_2 = 0$.

- Two vectors $\vec{y}_1$ and $\vec{y}_2$ are called **linearly independent** in a vector space S where they are defined if

$$c_1 \vec{y}_1 + c_2 \vec{y}_2 = 0, \quad \text{everywhere on } S,$$

which implies that the only solution is $c_1 = c_2 = 0$.

- The **inner product** of two **orthogonal** vectors is zero,

$$\vec{y}_1 \cdot \vec{y}_2 \equiv \langle \vec{y}_1 | \vec{y}_2 \rangle = 0,$$

- Apply the same idea to two functions, then the inner product can be defined as,

$$\langle y_1(x) | y_2(x) \rangle = \int y_1(x) \cdot y_2(x) dx = 0,$$



Second-order ODEs: IVP, Example

Example:

$$y'' + y = 0, \quad \text{with} \quad y(0) = 3, \quad y'(0) = -0.5$$

Hints:

1. Find the **Basis** for the ODE:

$$\cos x, \quad \text{and} \quad \sin x.$$

2. Construct the general solutions:

$$y(x) = c_1 \cos x + c_2 \sin x.$$

Solution:

The particular solution:

$$y(x) = 3 \cos x - 0.5 \sin x.$$



Homework #3:

1. (25%) Solve the Bernoulli's Eq.:

$$2y y' + y^2 \sin x = \sin x, \quad y(0) = \sqrt{2}, \quad (1)$$

Problem 23 in the [Textbook], at p.p. 33.

2. (25%) Solve a 1st-order ODE by using Richard's method of iteration:

$$y' = y - 1, \quad y(0) = 2, \quad (2)$$

3. (25%) Verify by substitution that the given functions form a basis:

$$x^2 y'' + x y' - 4y = 0, \quad x^2 \quad \text{and} \quad x^{-2}, \quad (3)$$

and find the special solution to satisfy the initial values $y(1) = 11, y'(1) = -6$.

Problem 5 in the [Textbook], p.p. 52.



Find another Basis from the **known one** : General case

Example:

$$y'' + p(x)y' + q(x)y = 0,$$

with a known solution $y(x) = y_1$.

Hints:

1. Find another basis $y_2(x)$, which is *linearly independent* to $y_1(x)$, by substituting,

$$y_2(x) = u(x) y_1(x), \quad \text{where } u(x) \text{ is not a constant.}$$

2. The original 2nd-order ODE is reduced into a 1st-order ODE,

$$u'' y_1 + u'(2y_1' + p y_1) + u(y_1'' + p y_1' + q y_1) = 0.$$

Solution:

$$y_2 = y_1(x) \int \frac{e^{-\int p(x) dx}}{y_1^2} dx.$$



Find another Basis from the **known one** :

Example:

$$(x^2 - x)y'' - xy' + y = 0,$$

with a known solution $y_1 = x$

Hints:

1. Find another basis $y_2(x)$, which is *linearly independent* to $y_1(x)$, by substituting,

$$y_2(x) = u(x) y_1(x), \quad \text{where } u(x) \text{ is not a constant.}$$

2. The original 2nd-order ODE is reduced into a 1st-order ODE,

$$(x^2 - x)u'' + (x - 2)u' = 0.$$

Solution:

$$y_2 = x \ln |x| + 1.$$



2nd-order ODEs: Constant Coefficients

A 2nd-order **homogeneous linear** ODE with **constant coefficients**:

Hints:

$$y'' + ay' + by = 0,$$

1. Try the solution for the 1st-order linear ODE with a constant coefficient,

$$y' = \lambda y.$$

2. Substitute $y = e^{\lambda x}$, we have a **characteristic equation** (or auxiliary equation),

$$\lambda^2 + a\lambda + b = 0.$$

Solution:

$$\begin{aligned} y(x) &= c_1 y_1 + c_2 y_2, \\ &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}. \end{aligned}$$



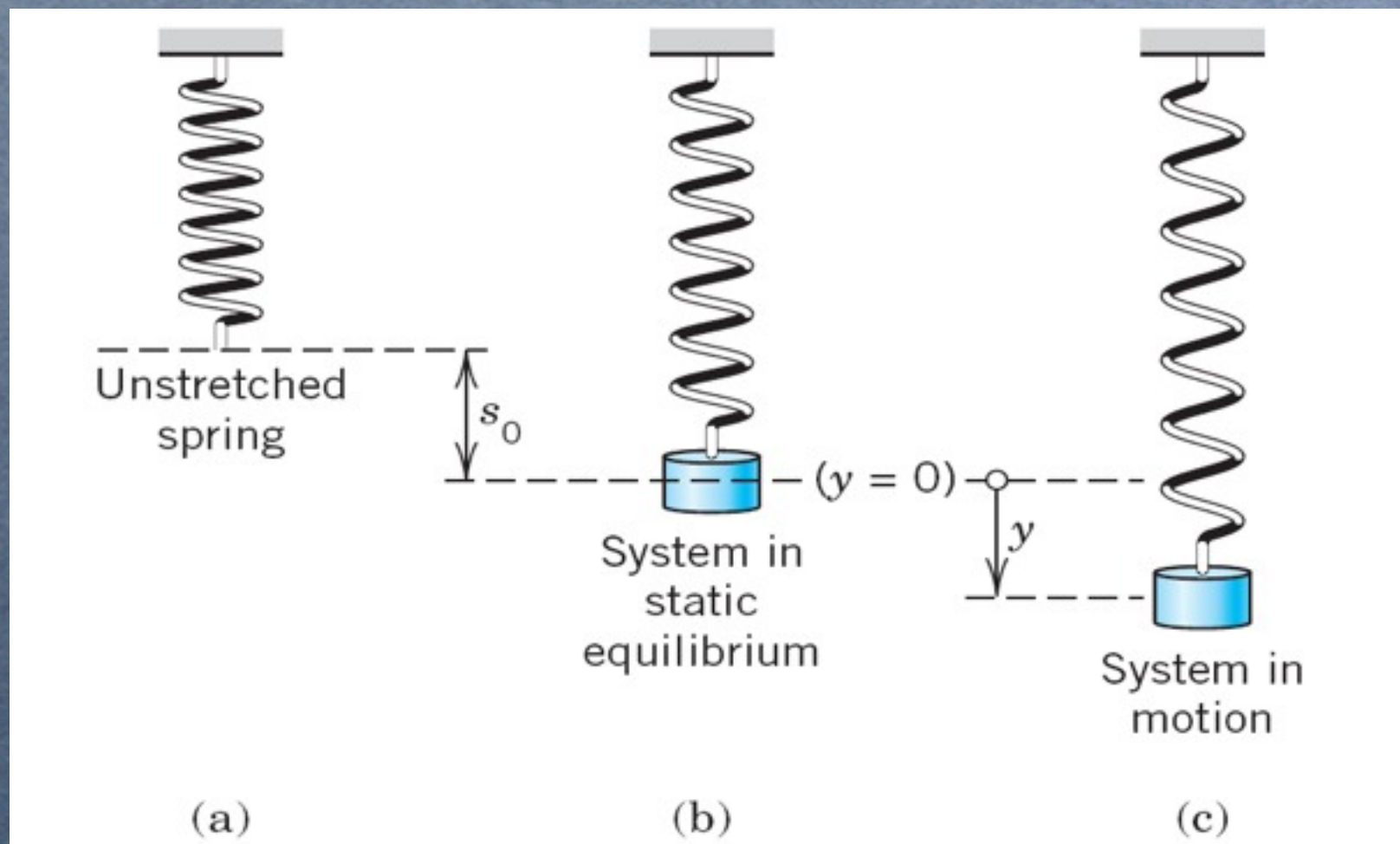
2nd-order ODEs: Constant Coefficients, cont.

Case	Roots of (2)	Basis of (1)	General Solution of (1)
I	Distinct real λ_1, λ_2	$e^{\lambda_1 x}, e^{\lambda_2 x}$	$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
II	Real double root $\lambda = -\frac{1}{2}a$	$e^{-ax/2}, xe^{-ax/2}$	$y = (c_1 + c_2 x)e^{-ax/2}$
III	Complex conjugate $\lambda_1 = -\frac{1}{2}a + i\omega,$ $\lambda_2 = -\frac{1}{2}a - i\omega$	$e^{-ax/2} \cos \omega x$ $e^{-ax/2} \sin \omega x$	$y = e^{-ax/2}(A \cos \omega x + B \sin \omega x)$

QUIZ: How to find the second basis for the case of double roots?



Constant Coef.: Undamped Mass-Spring system



$$F = m \frac{d^2 y}{dt^2} = -k y,$$

- We choose the downward direction as the positive direction.
- k is the *spring constant* by Hooke's law.



Constant Coef.: Undamped System, cont.

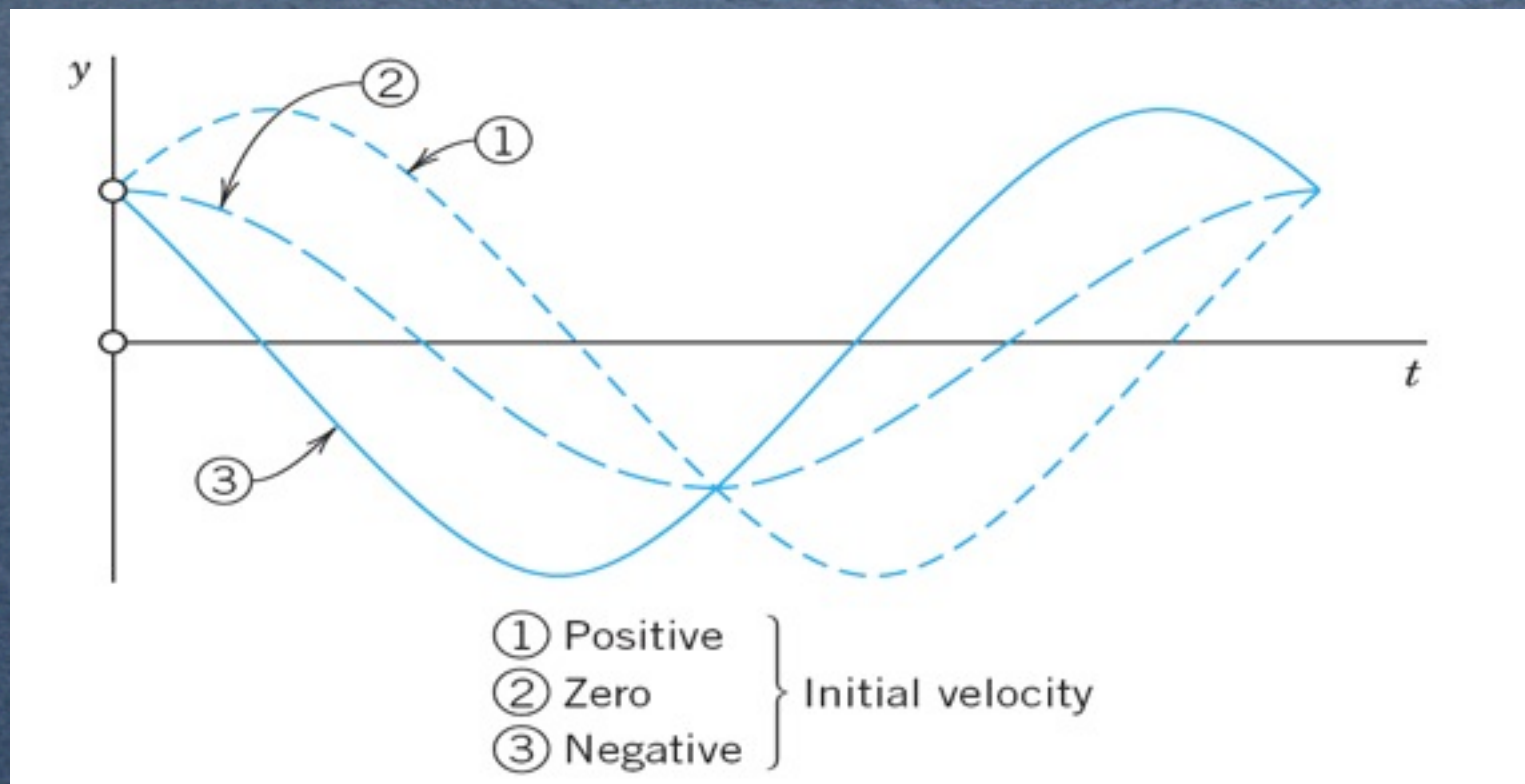
$$y'' + \omega_0^2 y = 0,$$

with the *natural frequency* $\omega_0 \equiv \sqrt{\frac{k}{m}}$.

Solution:

$$y(t) = A \cos(\omega_0 t) + B \sin(\omega_0 t),$$

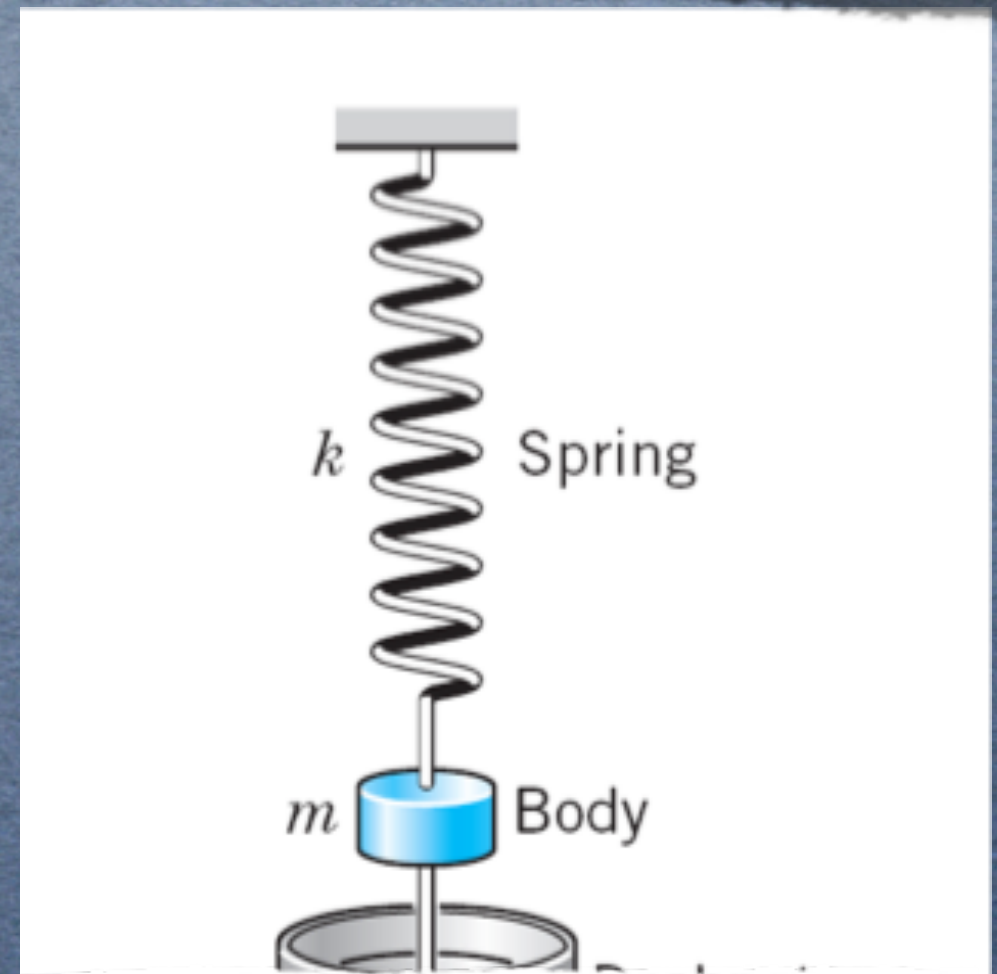
is called a **harmonic oscillation**.



Constant Coef.: Damped Mass-Spring system

$$y'' + \omega_0^2 y = -\frac{c}{m} y',$$

a **damping force** $F_d = -cy'$ is added.



QUIZ: Why the damping force is proportional to the velocity (first derivative)?



Constant Coef.: Damped system, cont.

Example:

$$y'' + \frac{c}{m}y' + \omega_0^2 y = 0,$$

Solution:

Case I. $c^2 > 4mk$. *Distinct real roots λ_1, λ_2 .* **(Overdamping)**

Case II. $c^2 = 4mk$. *A real double root.* **(Critical damping)**

Case III. $c^2 < 4mk$. *Complex conjugate roots.* **(Underdamping)**



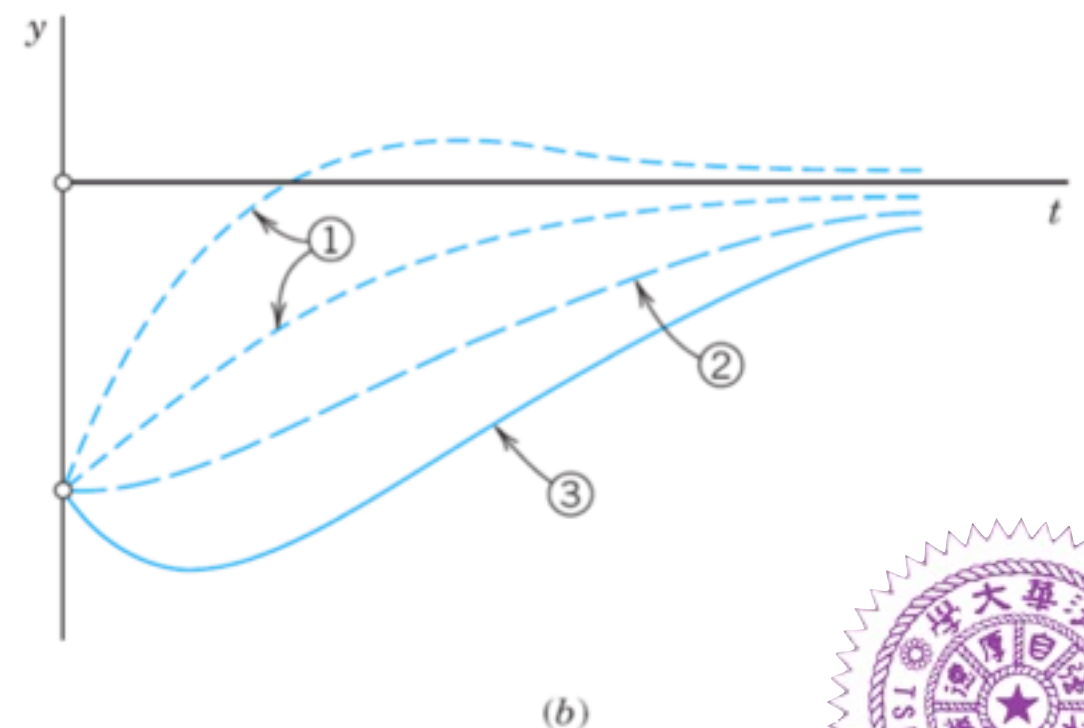
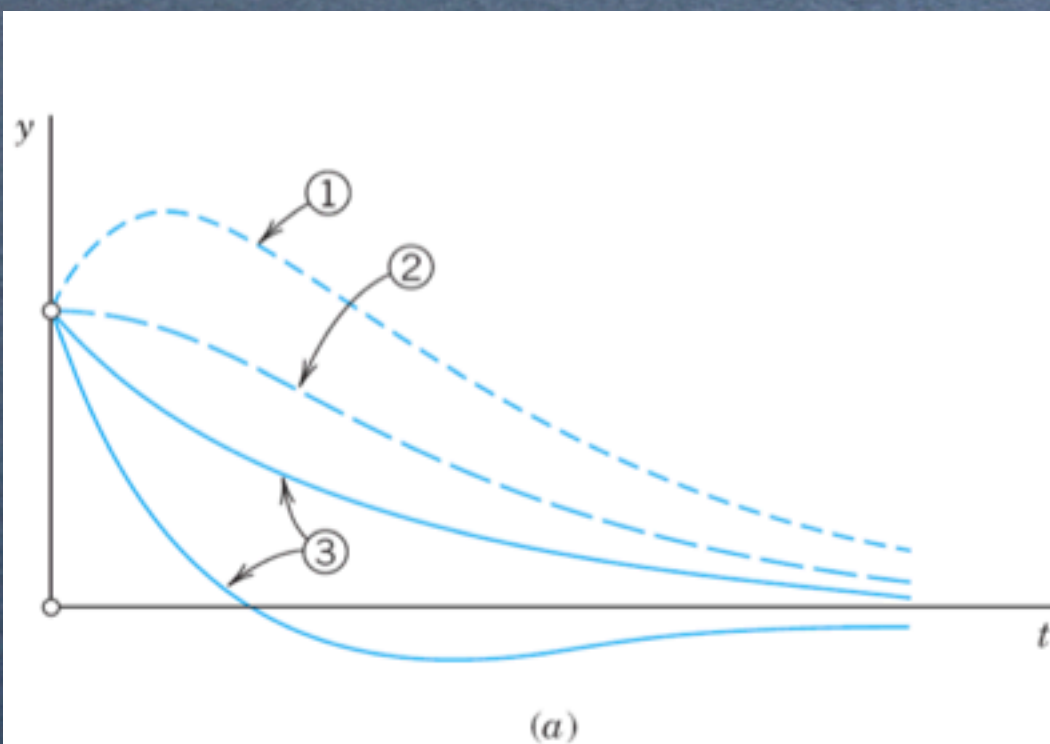
Constant Coef.: Damped system, Overdamping

Overdamping:

Distinct real roots:

$$y(t) = c_1 e^{-(\alpha-\beta)t} + c_2 e^{-(\alpha+\beta)t},$$

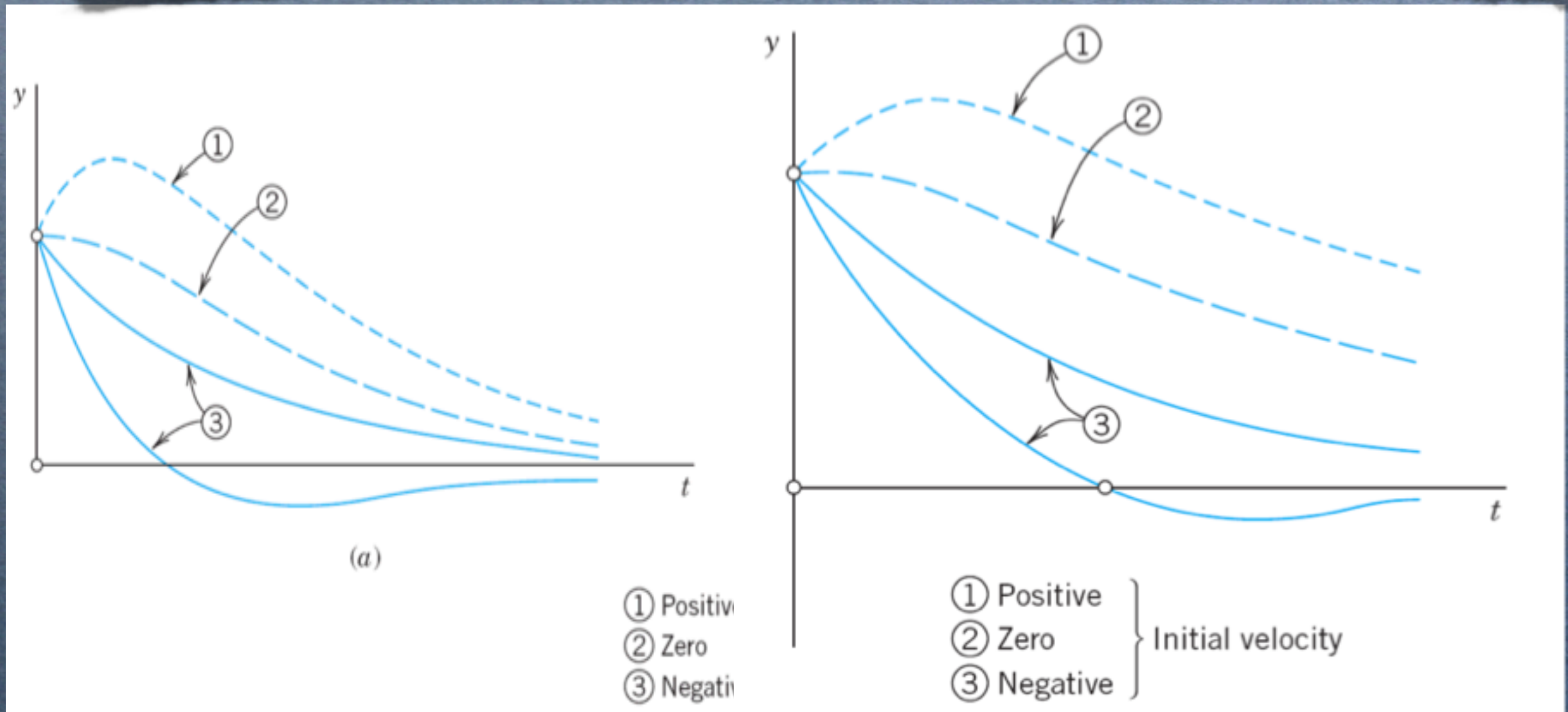
where $\alpha = \frac{c}{2m}$ and $\beta = \frac{1}{2m} \sqrt{c^2 - 4mk}$.



- ① Positive
 - ② Zero
 - ③ Negative
- } Initial velocity



Constant Coef.: Damped system, Critical damping



$$y(t) = (c_1 + c_2 t)e^{-\alpha t},$$

where $\alpha = \frac{c}{2m}$ and $\beta = \frac{1}{2m}\sqrt{c^2 - 4mk} = 0$.

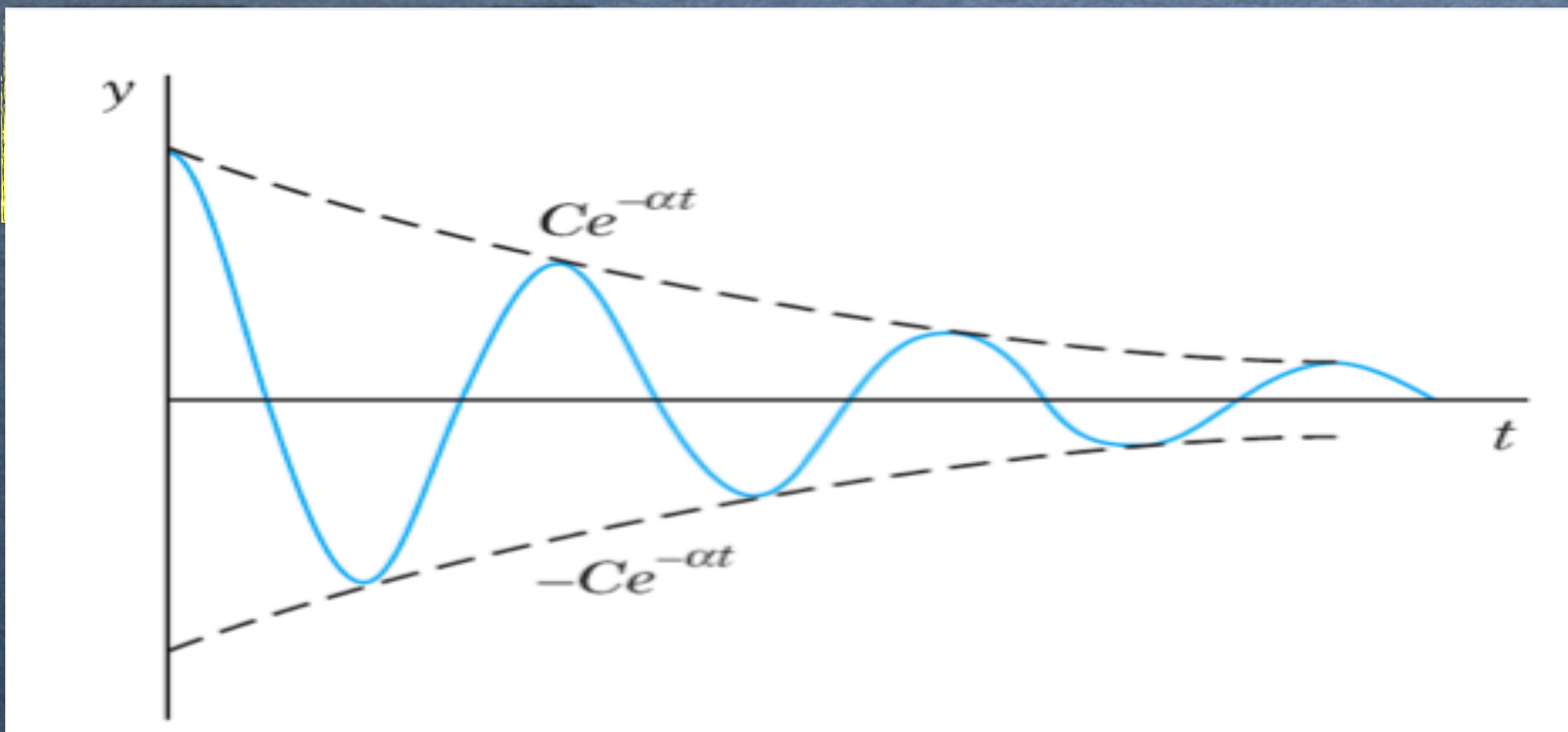


Constant Coef.: Damped system, Underdamping

Underdamping:

$$y(t) = [A \cos(\omega t) + B \sin(\omega t)]e^{-\alpha t} = C \cos(\omega t - \delta)e^{-\alpha t},$$

where $\alpha = \frac{c}{2m}$, $\omega = \frac{1}{2m} \sqrt{4mk - c^2}$, $C^2 = A^2 + B^2$, and $\delta = \tan^{-1} B/A$.



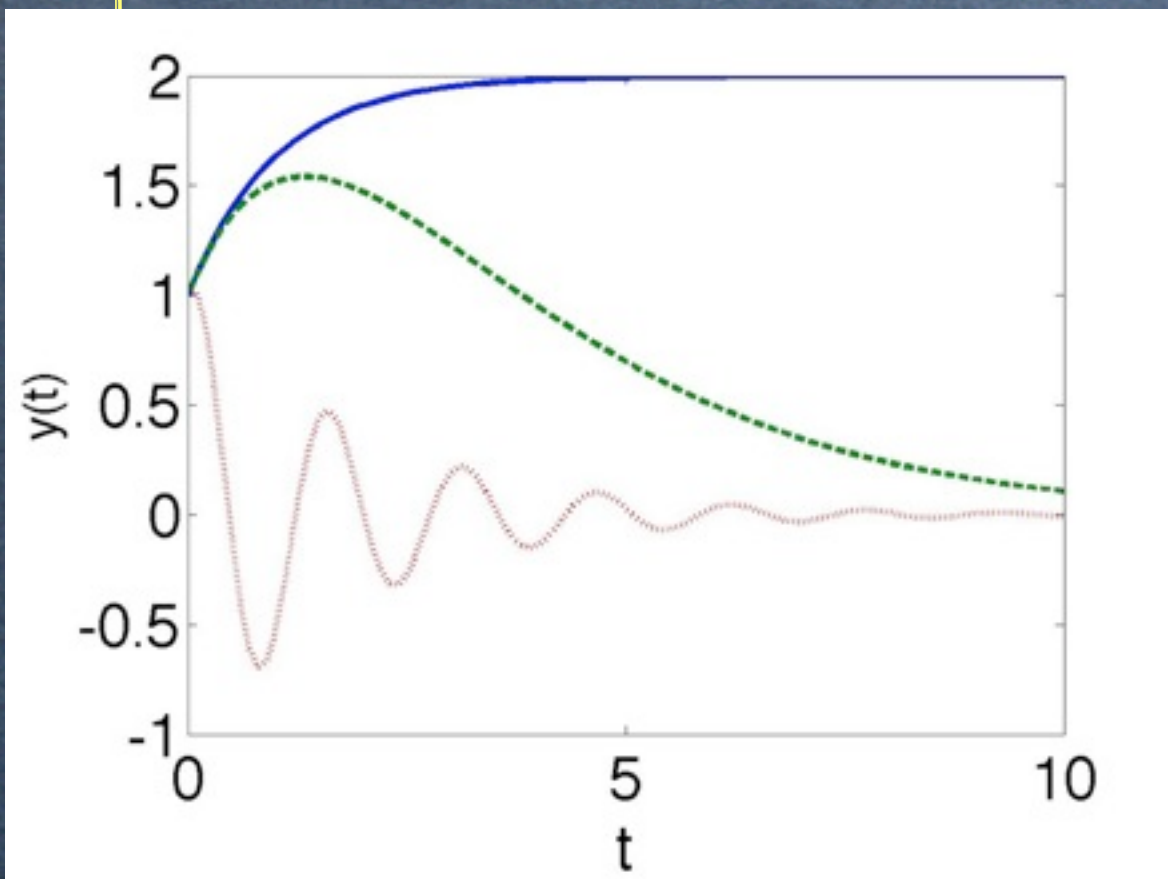
Homework #4:

1. (25%) Solve the 2nd-order ODE with a known solution:

$$x^2 y'' - 3xy' + 4y = 0, \quad y_1(x) = x^2. \quad (1)$$

2. (50%) Solve the the initial-value problem for a damped motion:

$$\frac{d^2 y}{dt^2} + 1 \frac{dy}{dt} + \omega_0^2 y = 0, \quad y(0) = 1, \quad \text{and} \quad \left. \frac{dy}{dt} \right|_{t=0} = 1. \quad (2)$$



constant value of ω_0 .

case of $\omega_0^2 = 0$.

case of $\omega_0^2 = 1$.

case of $\omega_0^2 = 17$.

the "Over-damping", "Critical-damping".



2nd-order ODEs: Differential Operators

- Operational calculus: the technique and application of operators.
- Differential Operator $\hat{D}$:

$$\hat{D}y \equiv y' = \frac{dy}{dx}, \quad \hat{D}^2y \equiv y'' = \frac{d^2y}{dx^2}, \dots$$

- A homogeneous linear ODE with constant coefficients:

$$y'' + ay' + by = 0 = (\hat{D}^2 + a\hat{D} + b\hat{I})y \equiv \hat{L}y.$$

where $\hat{I}$ is the *identity operator*, i.e., $\hat{I}y = y$, and $\hat{L}$ is a *linear operator*.

- Nonlinear operator:

$$\hat{N}(y)y.$$



2nd-order ODEs: Differential Operators, cont.

Example:

$$y'' - 3y' - 40y = (\hat{D}^2 - 3\hat{D} - 40\hat{I})y = 0.$$

Hints:

- Factorization:

$$(\hat{D}^2 - 3\hat{D} - 40\hat{I})y = (\hat{D} - 8\hat{I})(\hat{D} + 5\hat{I})y = 0.$$

- Two solutions are:

$$(\hat{D} - 8\hat{I})y = 0,$$

$$(\hat{D} + 5\hat{I})y = 0.$$

Solution:

$$y_h(x) = c_1 e^{8x} + c_2 e^{-5x}.$$



2nd-order ODEs: Euler-Cauchy Equation

A 2nd-order ODE is called Euler -Cauchy equation in the form of:

$$x^2 y'' + axy' + by = 0,$$

Hints:

Substitute $y = x^m$, we have a characteristic equation,

$$m(m-1) + am + b = 0.$$

Solution:

$$\begin{aligned} y(x) &= c_1 y_1 + c_2 y_2, \\ &= c_1 x^{m_1} + c_2 x^{m_2}. \end{aligned}$$



2nd-order ODEs: Euler-Cauchy Equation, cont.

- Case I: **Distinct real roots**: $m_{1,2} = \frac{-(a-1)}{2} \pm \frac{\sqrt{(a-1)^2 - 4b}}{2},$

$$y(x) = c_1 x^{m_1} + c_2 x^{m_2}.$$

- Case II: **Double roots**: $m = \frac{-(a-1)}{2},$

$$y(x) = (c_1 + c_2 \ln x) x^m,$$

compared with the case of constant coefficients, i.e., $y(x) = (c_1 + c_2 x) e^{\lambda x}.$

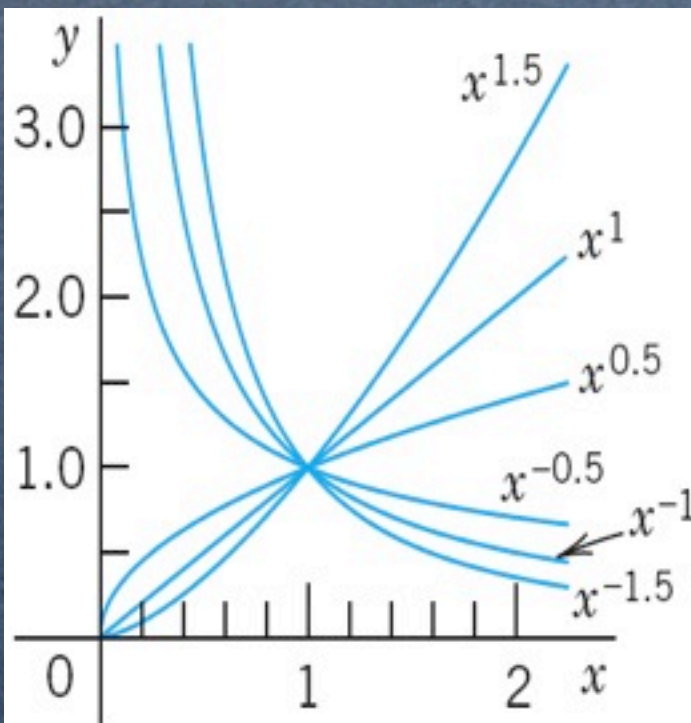
- Case III, **Complex conjugates**: $m_{1,2} = \frac{-(a-1)}{2} \pm i \frac{\sqrt{4b - (a-1)^2}}{2} = \alpha \pm i\beta,$

$$\begin{aligned} y(x) &= c_1 x^{m_1} + c_2 x^{m_2}, \\ &= x^\alpha [c_3 \cos(\beta \ln x) + c_4 \sin(\beta \ln x)]. \end{aligned}$$

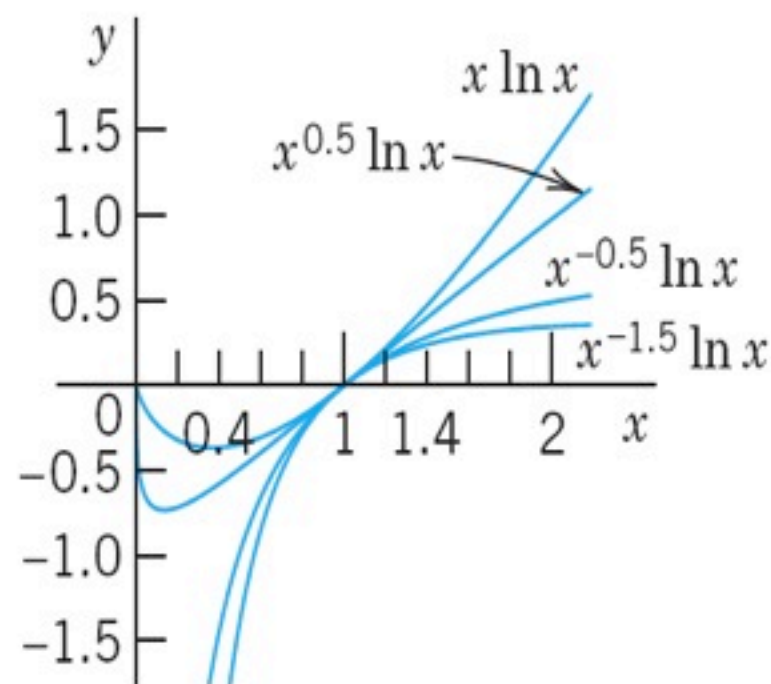


2nd-order ODEs: Euler-Cauchy Equation, cont.

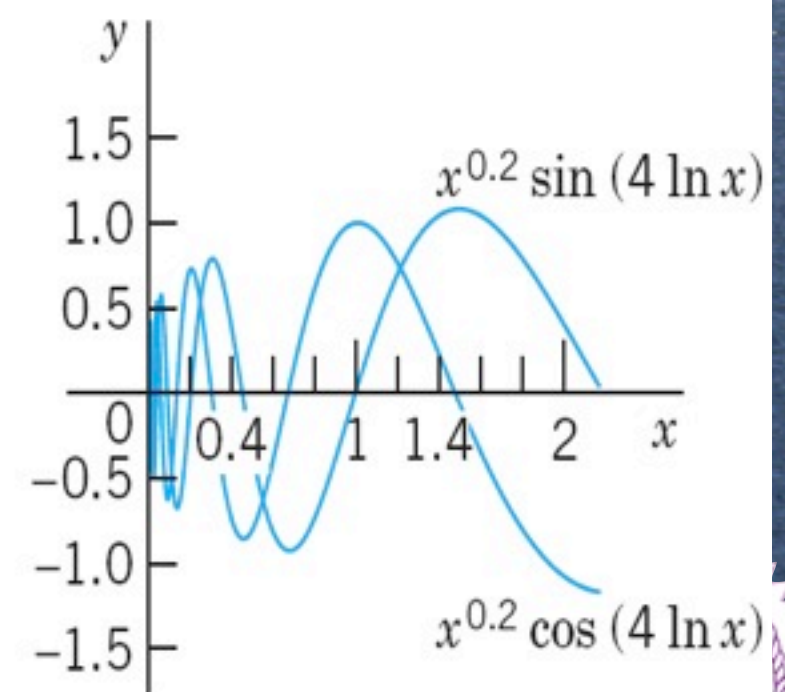
- Case I: $y(x) = c_1 x^{m_1} + c_2 x^{m_2}$,
- Case II: $y(x) = (c_1 + c_2 \ln x) x^m$,
- Case III, $y(x) = x^\alpha [c_3 \cos(\beta \ln x) + c_4 \sin(\beta \ln x)]$.



Case I: Real roots



Case II: Double root



Case III: Complex roots

2nd-order ODEs: More special functions

Bessel's Eq.:

$$x^2 y'' + xy' + (x^2 - \alpha^2)y = 0,$$

Laguerre's Eq.:

$$xy'' + (1 - x)y' + ny = 0,$$

Legendre's Eq.:

$$\frac{d}{dx} \left[(1 - x^2) \frac{d}{dx} P_n(x) \right] + n(n + 1)P_n(x) = 0,$$



2nd-order ODEs: Existence and Uniqueness

$$y'' + p(x)y' + q(x)y = 0,$$

The general Existence and Uniqueness theory of **homogeneous linear** ODEs for *Initial Value Problems, IVP*:

- If $p(x)$ and $q(x)$ are **continuous function** on some open interval I and x_0 is in I ,
- then the IVP has a unique solution $y(x)$ on the interval I .
- If $p(x)$ and $q(x)$ are **continuous function** on some open interval I , then there exists a *general solution* on I .



2nd-order ODEs: Wronskian

$$y'' + p(x)y' + q(x)y = 0,$$

Two solutions $y_1(x)$ and $y_2(x)$ on the interval I are **linearly dependent** if and only if their **Wronskian** $W = 0$ at some x_0 in I ,

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1',$$

- If there is an x_1 in I at which $W \neq 0$, then y_1 and y_2 are *linearly independent* on I .
- Every solution $y(x)$ on I is of the form:

$$y(x) = c_1 y_1(x) + c_2 y_2(x),$$

where y_1 and y_2 are the **basis** of the ODE on I , while c_1 and c_2 are suitable constants.



2nd-order ODEs: Wronskian, proof

1. Let y_1 and y_2 be linearly dependent, i.e., $y_1 = cy_2$ (c is a constant), then

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1' = cy_2 y_2' - y_2 cy_2' = 0.$$

2. Conversely, we let $W(y_1, y_2) = 0$ for some $x = x_0$ and show that this implies linearly dependence of y_1 and y_2 on I .

- Consider the linear system of equations in the unknowns c_1 and c_2 ,

$$c_1 y_1(x_0) + c_2 y_2(x_0) = 0,$$

$$c_1 y_1'(x_0) + c_2 y_2'(x_0) = 0,$$

- By eliminating c_2 , we have

$$c_1 [y_1(x_0) y_2'(x_0) - y_1'(x_0) y_2(x_0)] \equiv c_1 W(y_1, y_2) = 0.$$

- By eliminating c_1 , we have

$$c_2 W(y_1, y_2) = 0.$$

3. If $W(y_1, y_2) \neq 0$, we could divide by W and conclude that $k_1 = k_2 = 0$.
4. If $W(y_1, y_2) = 0$, division is not possible, then the system has a solution for which k_1 and k_2 are not both 0.



2nd-order ODEs: Non-homogeneous Eq.

$$y'' + p(x)y' + q(x)y = r(x),$$

Hints:

1. Find the **general solution**, $y_h(x)$, for the homogeneous ODE,

$$y_h'' + p(x)y_h' + q(x)y_h = 0,$$

2. and a **particular solution**, $y_p(x)$,

$$y_p'' + p(x)y_p' + q(x)y_p = r(x),$$

Solution:

$$\begin{aligned} y(x) &= y_h(x) + y_p(x), \\ &= c_1 y_1(x) + c_2 y_2(x) + y_p(x). \end{aligned}$$



Higher-order ODEs: Non-homogeneous

Non-homogeneous Linear ODE:

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = r(x),$$

Solution:

$$y(x) = y_h(x) + y_p(x),$$

How to find a particular solution?

- Method of Undetermined Coefficients,
- Method of Variation of Parameters.



2nd-order ODEs: Non-homogeneous, Example 1

Example:

$$y'' + y = 0.001x^2, \quad \text{with } y(0) = 0, y'(0) = 1.5,$$

Hints:

1. Find the **general solution**, $y_h(x)$, for the homogeneous ODE,

$$y_h = A \cos x + B \sin x,$$

2. and a **particular solution**, $y_p(x)$,

Solution:

$$\begin{aligned} y_p &= c_0 + c_1 x + c_2 x^2, \\ &= -0.002 + 0.001x^2. \end{aligned}$$

$$y(x) = 0.002 \cos x + 1.5 \sin x - 0.002 + 0.001x^2.$$



2nd-order ODEs: Non-homogeneous, Example 2

Example:

$$y'' + 3y' + 2.25y = -10e^{-1.5x}, \quad \text{with } y(0) = 1, y'(0) = 0,$$

Hints:

1. Find the **general solution**, $y_h(x)$, for the homogeneous ODE,

$$y_h = (A + Bx)e^{-1.5x},$$

2. and a **particular solution**, $y_p(x)$,

Solution:

$$\begin{aligned} y_p &= Cx^2e^{-1.5x} \\ &= -5x^2e^{-1.5x}. \end{aligned}$$

$$y(x) = (1 + 1.5x)e^{-1.5x} - 5x^2e^{-1.5x}.$$



2nd-order ODEs: Method of Undetermined Coefficients

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
kx^n ($n = 0, 1, \dots$)	$K_n x^n + K_{n-1} x^{n-1} + \dots + K_1 x + K_0$
$k \cos \omega x$	$\left\{ K \cos \omega x + M \sin \omega x \right.$
$k \sin \omega x$	
$ke^{\alpha x} \cos \omega x$	$\left\{ e^{\alpha x} (K \cos \omega x + M \sin \omega x) \right.$
$ke^{\alpha x} \sin \omega x$	



Homework #5:

1. (25%) Show *linearly independence* by calculating the Wronskian,

$$\cosh 2.5x, \quad \sinh 2.5x, \quad (1)$$

Problem 11 in the [Textbook], at p.p. 77.

2. (25%) Solve the 2nd-order non-homogeneous ODE :

$$y'' - 5y' + 4y = 8e^x. \quad (2)$$

3. (25%) Solve the 3rd-order non-homogeneous ODE :

$$y''' + y'' = e^x \cos x. \quad (3)$$

4. (25%) Solve the 2nd-order non-homogeneous ODE :

$$(x^2 \hat{D} - 2x \hat{D} + 2\hat{I})y = x^3 \sin x.$$

Problem 13 in the [Textbook], at p.p. 101.



2nd-order ODEs: Forced oscillation

A damped oscillator with the input, *Driving force* function $r(t)$:

$$my'' + cy' + ky = r(t).$$

- Periodic external force:

$$r(t) = F_0 \cos(\omega t), \quad F_0 > 0, \quad \text{and} \quad \omega > 0.$$

- Three cases for the general solution.
- The particular solution,

Lorentzian lineshape

$$y_p(t) = A \cos(\omega t) + B \sin(\omega t),$$
$$= F_0 \frac{m(\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2} \cos(\omega t) + F_0 \frac{\omega c}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2} \sin(\omega t).$$



2nd-order ODEs: Forced oscillation, cont.

- If the damping effect is so small, we can set $c = 0$.
- Then the particular solution becomes,

$$y_p(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t).$$

Solution:

The output is a superposition of two harmonic oscillations

$$y(t) = C \cos(\omega_0 t - \delta) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$$



2nd-order ODEs: Forced oscillation, Envelope

$$\begin{aligned} E_1(x, t) &= E_{01} \cos[\omega_1 t + \theta_1], \\ E_2(x, t) &= E_{02} \cos[\omega_2 t + \theta_2], \end{aligned}$$

- Addition of Waves of the **Different** frequency: for $E_{01} = E_{02} = E_0$ and $\Delta\theta = \theta_1 - \theta_2 = 0$,

$$E = 2E_0 \cos\left[\frac{(\omega_1 + \omega_2)t}{2}\right] \cos\left[\frac{(\omega_1 - \omega_2)t}{2}\right],$$

- **slowly-varying envelope**, with the **beat frequency**, $\Delta\omega = (\omega_1 - \omega_2)/2$,
- **carrier**, with the frequency, $\bar{\omega} = (\omega_1 + \omega_2)/2 \approx \omega_0$.



2nd-order ODEs: Forced oscillation, Resonance

- For the amplitude of the particular solution, $y_p(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$,

$$a_0 \equiv \frac{F_0}{m(\omega_0^2 - \omega^2)} = \frac{F_0}{k} \rho,$$

$$\rho \equiv \frac{1}{1 - (\omega/\omega_0)^2}, \quad \text{resonance factor.}$$

- The amplitude a_0 depends on ω and ω_0 ,
- If $\omega \rightarrow \omega_0$, a_0 goes to infinity.
- The excitation of input oscillation matches the natural frequency, $\omega = \omega_0$, is called resonance.



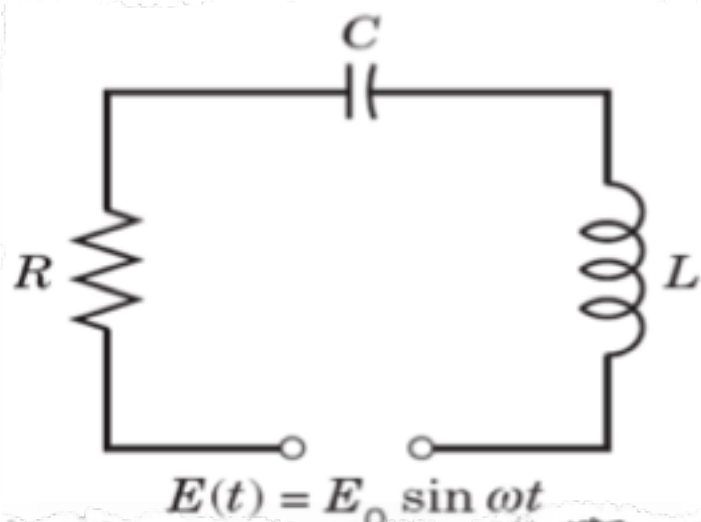
2nd-order ODEs: Electric circuit

- According to Kirchhoff's Voltage Law, an RLC -circuit with electromotive force $E(t)$ is modeled by the "integro-differential equation",

$$LI' + RI + \frac{1}{C} \int I dt = E(t) = E_0 \sin(\omega t),$$

- To get rid of the integral, we have a 2nd-order ODE for the current,

$$LI'' + RI' + \frac{1}{C}I = E'(t) = E_0 \omega \cos(\omega t),$$



Name	Symbol	Notation	Unit	Voltage Drop
Ohm's resistor		R Ohm's resistance	ohms (Ω)	RI
Inductor		L Inductance	henrys (H)	$L \frac{dI}{dt}$
Capacitor		C Capacitance	farads (F)	Q/C

Electric and Mechanical Analogy

- RLC -circuit:

$$LI'' + RI' + \frac{1}{C}I = E'(t) = E_0 \omega \cos(\omega t),$$

- mass-spring system:

$$my'' + cy' + ky = F(t) = F_0 \cos(\omega t),$$

Electrical System	Mechanical System
Inductance L	Mass m
Resistance R	Damping constant c
Reciprocal $1/C$ of capacitance	Spring modulus k
Derivative $E_0 \omega \cos \omega t$ of electromotive force	Driving force $F_0 \cos \omega t$
Current $I(t)$	Displacement $y(t)$



2nd-order ODEs: Variation of Parameters

For a *Non-homogeneous linear 2nd-order ODE*,

$$y'' + p(x)y' + q(x)y = r(x),$$

How to find a particular solution?

Hints:

1. For the **homogeneous** ODE, the general solution is constructed from the bases,

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x).$$

2. From the same idea of basis, the particular solution may be in the form,

$$y_p(x) = u(x)y_1(x) + v(x)y_2(x),$$

which must satisfy the non-homogeneous ODE.



2nd-order ODEs: Variation of Parameters, cont.

Hints:

1. **Condition 1:** From the same idea of basis, the particular solution may be in the form,

$$y_p(x) = u(x)y_1(x) + v(x)y_2(x),$$

which must satisfy the non-homogeneous ODE.

2. **Condition 2:** The derivative of y_p ,

$$y'_p = u' y_1 + u y'_1 + v' y_2 + v y'_2 = u y'_1 + v y'_2,$$

is reduced by requiring $u' y_1 + v' y_2 = 0$.

The Particular Solution:

$$y_p(x) = -y_1 \int \left[\frac{y_2 r}{W} \right] dx + y_2 \int \left[\frac{y_1 r}{W} \right] dx, \quad \text{where} \quad W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}.$$



Second-order ODEs: Summary

ODEs

- 1st-order

- 2nd-order

- ☒ Homogeneous & Linear, Ch. 2.1
- ☒ Constant Coefficients, Ch. 2.2
- ☐ Differential Operator Ch. 2.3
- ☒ Mass-Spring system, Ch. 2.4
- ☒ Euler-Cauchy Eq., Ch. 2.5
- ☐ Existence and Uniqueness, Ch. 2.6
- ☒ Wronskian, Ch. 2.6
- ☒ Non-homogeneous, Ch. 2.7
- ☒ Forced Oscillation, Ch. 2.8
- ☒ RLC circuits, Ch. 2.9
- ☒ Variation of Parameters, Ch. 2.10

- Higher-order
- Systems of ODEs



Linear Algebra and Homogeneous Linear ODEs

Linear Algebra	Geometry	ODE	Quantum Mechanics
Linear Space	Dimension	Order	Hilbert space
Element	Vector	Function	State
Superposition	$y = c_1 v_1 + c_2 v_2$	$y(x) = c_1 y_1(x) + c_2 y_2(x)$	$ \psi\rangle = c_1 \phi_1\rangle + c_2 \phi_2\rangle$
Basis	$v_1 = (1,0)$ $v_2 = (0,1)$	e^x , for const. coef. Eq. x^m , for Euler-Cauchy Eq.	
Independent basis	Determinant, $A \neq 0$	Wronskian, $W \neq 0$	
Matrix			



Homework #6:

1. (25%) Find the solutions for the given ODE, and identify which part is the *transient* solution and which part is the *steady-state* solution.

$$4y'' + 12y' + 9y = 75 \sin 3t. \quad (1)$$

Problem 11 in the [Textbook], at p.p. 90.

2. (25%) Find the solutions for a *RLC*-circuit, Fig. 60, p.p. 92, for the given data,

$$R = 0.1\Omega, \quad (2)$$

$$L = 0.5\text{H}, \quad (3)$$

$$C = 100/13\text{F}, \quad (4)$$

$$E = e^{-4t}(1.932 \cos 0.5t + 0.246 \sin 0.5t)\text{V}. \quad (5)$$

Problem 15 in the [Textbook], at p.p. 97.

3. (25%) Solve the non-homogeneous ODE by *variation of parameters*

$$x^2 y'' - 2xy' + 2y = x^3 \cos x.$$

Problem 3 in the [Textbook], at p.p. 101.

