

EE 2015

(Partial) Differential Equations and Complex Variables

Ray-Kuang Lee*

Institute of Photonics Technologies,
Department of Electrical Engineering, and Department of Physics,
National Tsing-Hua University

*rkleee@ee.nthu.edu.tw



Syllabus:

★ Course description and Introduction, 9/14

1. Ordinary Differential Equations: 4 weeks

- ▶ First-order ODEs, Ch. 1: 9/16, 9/21
- ▶ Second-order ODEs, Ch. 2: 9/23, 9/28, 9/30, 10/5, 10/7
- ▶ Higher-order ODEs, Ch. 3: 10/12
- ▶ Systems of ODEs, Ch. 4: 10/14
- ▶ 1st EXAM, 10/15 (Friday night)

2. Transform Methods: 4 weeks

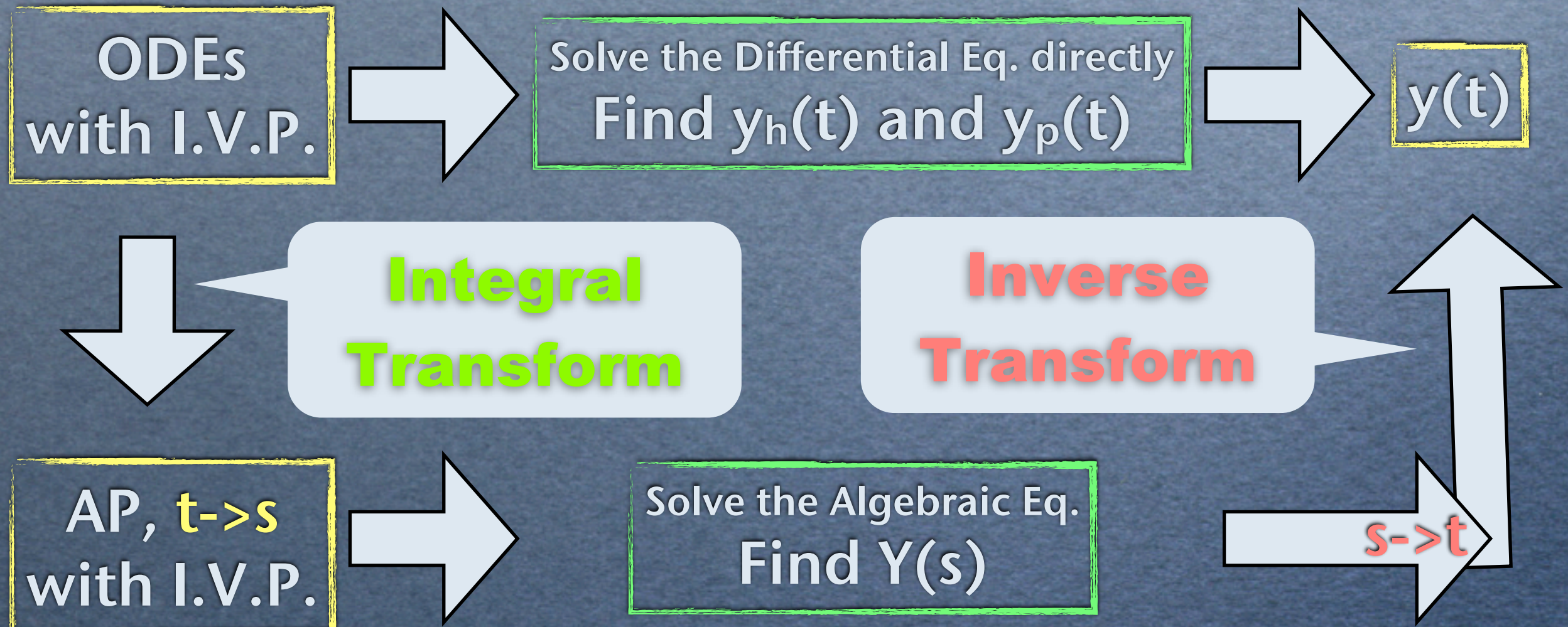
- ▶ Laplace Transforms, Ch. 6: 10/19 - 11/11
- ▶ 2nd EXAM, 11/12 (Friday night)

3. Series and Complex Variables: 9 weeks

- ▶ Power Series, Ch. 5: 11/16, 11/18
- ▶ Fourier Series, Ch. 11: 11/23, 11/25, 11/30, 12/2
- ▶ 3rd EXAM, 12/3 (Friday night)
- ▶ PDE by Fourier Series, Ch. 12, 12/7 - 12/22
- ▶ 4th EXAM, 12/23 (in Class)
- ▶ Taylor and Laurent Series, Ch. 13-16: 12/28, 12/30
- ▶ Complex and Residue Integrations, Ch. 16: 1/4 - 1/13
- ▶ 5th EXAM, 1/14 (Friday night)



Integral Transform: Initial Value Problem



QUIZ: Which way is easier?

Integral Transform:

- An input function $f(t)$ is transformed into another output function $F(s)$ by a particular kind of mathematical operator:

$$F(s) = \mathcal{T}(f) \equiv \int_{t_1}^{t_2} K(t, s) f(t) dt.$$

- The function $K(t, s)$ of two variables, is called the the **kernel function** or nucleus of the transform.
- There are numerous useful integral transforms.
- Some kernels have an associated inverse kernel $K^{-1}(t, s)$, which yields an inverse transform:

$$f(t) = \mathcal{T}^{-1}(F) \equiv \int_{s_1}^{s_2} K^{-1}(t, s) F(s) ds.$$



Integral Transform:

Table of integral transforms

Transform	Symbol	K	t_1	t_2	K^{-1}	u_1	u_2
Fourier transform	$\mathcal{F}$	$\frac{e^{-iut}}{\sqrt{2\pi}}$	$-\infty$	∞	$\frac{e^{+iut}}{\sqrt{2\pi}}$	$-\infty$	∞
Hartley transform	$\mathcal{H}$	$\frac{\cos(ut) + \sin(ut)}{\sqrt{2\pi}}$	$-\infty$	∞	$\frac{\cos(ut) + \sin(ut)}{\sqrt{2\pi}}$	$-\infty$	∞
Mellin transform	$\mathcal{M}$	t^{u-1}	0	∞	$\frac{t^{-u}}{2\pi i}$	$c-i\infty$	$c+i\infty$
Two-sided Laplace transform	$\mathcal{B}$	e^{-ut}	$-\infty$	∞	$\frac{e^{+ut}}{2\pi i}$	$c-i\infty$	$c+i\infty$
Laplace transform	$\mathcal{L}$	e^{-ut}	0	∞	$\frac{e^{+ut}}{2\pi i}$	$c-i\infty$	$c+i\infty$
Weierstrass transform	$\mathcal{W}$	$\frac{e^{-(u-t)^2/4}}{\sqrt{4\pi}}$	$-\infty$	∞	$\frac{e^{+(u-t)^2/4}}{i\sqrt{4\pi}}$	$c-i\infty$	$c+i\infty$
Hankel transform		$t J_\nu(ut)$	0	∞	$u J_\nu(ut)$	0	∞
Abel transform		$\frac{2t}{\sqrt{t^2 - u^2}}$	u	∞	$\frac{-1}{\pi\sqrt{u^2 - t^2}} \frac{d}{du}$	t	∞
Hilbert transform	$\mathcal{Hil}$	$\frac{1}{\pi} \frac{1}{u - t}$	$-\infty$	∞	$\frac{1}{\pi} \frac{1}{u - t}$	$-\infty$	∞
Poisson kernel		$\frac{1 - r^2}{1 - 2r \cos \theta + r^2}$	0	2π			
Identity transform		$\delta(u - t)$	$t_1 < u$	$t_2 > u$	$\delta(t - u)$	$u_1 < t$	$u_2 > t$



Laplace Transform: Definition

- If $f(t)$ is a function defined for all $t \geq 0$, its Laplace transform is defined as

$$F(s) = \mathcal{L}(f) \equiv \int_0^{\infty} e^{-st} f(t) dt.$$

- Here we must assume that $f(t)$ is such that the integral exists (that is, has some finite value).
- This assumption is usually satisfied in applications.



Laplace Transform: Inverse

- The **inverse transform** of $F(s)$ is denoted by

$$f(t) = \mathcal{L}^{-1} F(s).$$

- Any transform and its inverse transform should satisfy:

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}[\mathcal{L}f(t)], \\ F(s) &= \mathcal{L}[\mathcal{L}^{-1}F(s)]. \end{aligned}$$

- Only after the introduction of "**Complex Variables**", we would learn the inverse Laplace transform, i.e.,

$$f(t) = \mathcal{L}^{-1}(F)(s) \equiv \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) e^{st} ds.$$



Laplace Transform: Examples

Example 1:

$$f(t) = 1, \quad t \geq 0.$$

Solution:

$$F(s) = \int_0^{\infty} 1 \cdot e^{-st} dt = \frac{1}{s}.$$

Example 2:

$$f(t) = e^{at}, \quad t \geq 0.$$

Solution:

$$F(s) = \int_0^{\infty} e^{-st} \cdot e^{at} dt = \frac{1}{s - a}.$$



Laplace Transform: Linearity

- Laplace transform is a *linear operation*, that is
- For any functions $f(t)$ and $g(t)$ whose transforms exist and any constants a and b the transform of $af(t) + bg(t)$, then

$$\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)].$$



Laplace Transform: Linearity, Examples

Example 1:

$$\mathcal{L}[\cosh at] = \mathcal{L}\left[\frac{1}{2}(e^{at} + e^{-at})\right].$$

Solution:

$$\mathcal{L}[\cosh at] = \frac{1}{2}\mathcal{L}[e^{at}] + \frac{1}{2}\mathcal{L}[e^{-at}] = \frac{s}{s^2 - a^2}.$$

Example 2:

$$\mathcal{L}[\cos \omega t] = \mathcal{L}\left[\frac{1}{2}(e^{i\omega t} + e^{-i\omega t})\right].$$

Solution:

$$\mathcal{L}[\cos \omega t] = \frac{1}{2}\mathcal{L}[e^{i\omega t}] + \frac{1}{2}\mathcal{L}[e^{-i\omega t}] = \frac{s}{s^2 + \omega^2}.$$



Laplace Transform: common used

	$f(t)$	$\mathcal{L}(f)$		$f(t)$	$\mathcal{L}(f)$
1	1	$1/s$	7	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
2	t	$1/s^2$	8	$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
3	t^2	$2!/s^3$	9	$\cosh at$	$\frac{s}{s^2 - a^2}$
4	t^n ($n = 0, 1, \dots$)	$\frac{n!}{s^{n+1}}$	10	$\sinh at$	$\frac{a}{s^2 - a^2}$
5	t^a (a positive)	$\frac{\Gamma(a + 1)}{s^{a+1}}$	11	$e^{at} \cos \omega t$	$\frac{s - a}{(s - a)^2 + \omega^2}$
6	e^{at}	$\frac{1}{s - a}$	12	$e^{at} \sin \omega t$	$\frac{\omega}{(s - a)^2 + \omega^2}$



Laplace Transform: Examples

Example 1:

$$f(t) = t, \quad t \geq 0.$$

Solution:

$$F(s) = -\frac{1}{s}te^{-st}\Big|_0^\infty + \frac{1}{s}\int_0^\infty 1 \cdot e^{-st} dt = \frac{1}{s^2}.$$

Example 2:

$$f(t) = t^n, \quad t \geq 0, \quad n \in N.$$

Solution:

$$F(s) = \frac{n!}{s^{n+1}}.$$



Laplace Transform: Gamma Function

Example:

$$f(t) = t^a, \quad t \geq 0, \quad a \text{ positive.}$$

Solution:

$$F(s) = \frac{\Gamma(a+1)}{s^{a+1}}.$$

- Γ function is an extension of **factorial function**, defined as

$$\Gamma(z) \equiv \int_0^{\infty} t^{z-1} e^{-t} dt.$$

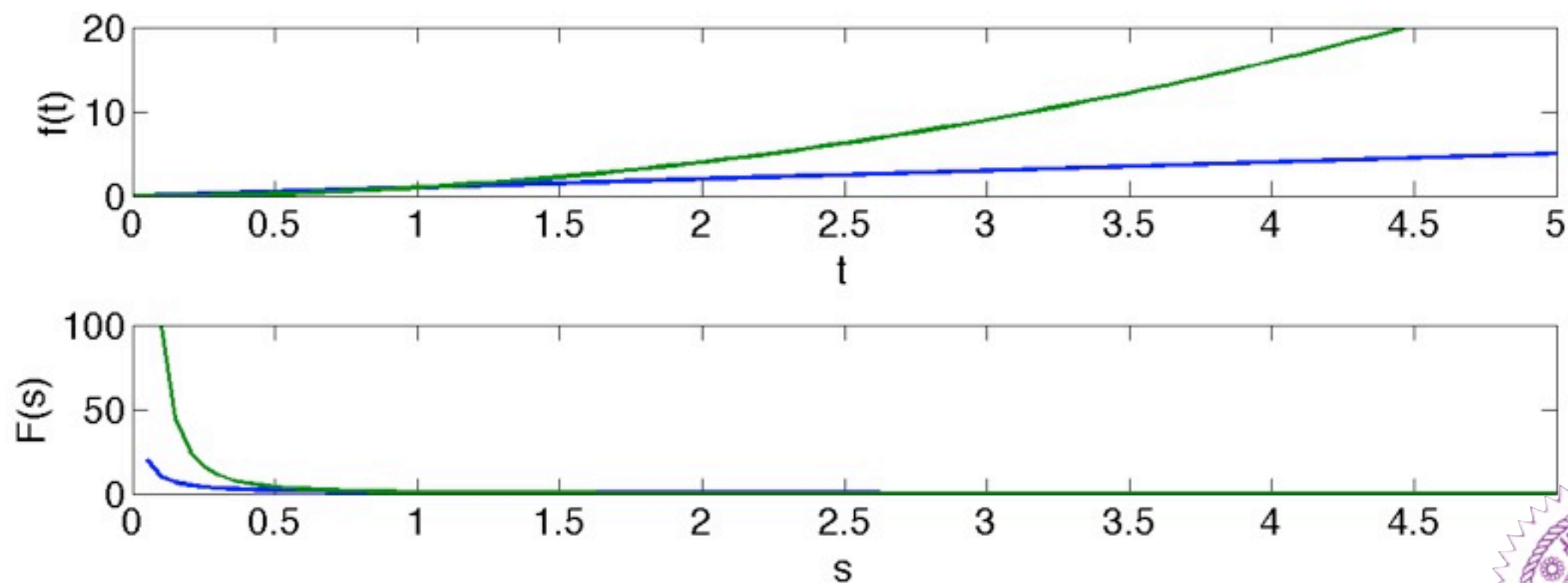
- If z is a positive integer, $z = n$, then

$$\Gamma(n) = (n-1)!.$$



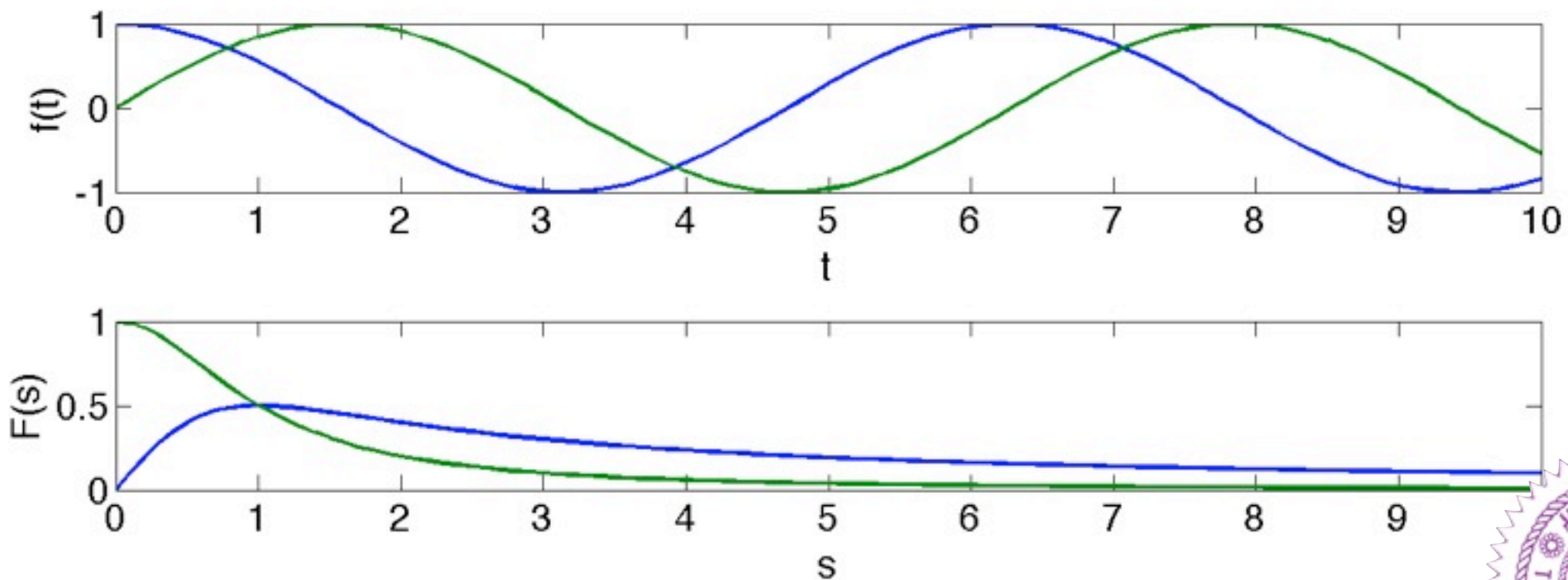
Laplace Transform: polynomial

$$\mathcal{L}[f(t) = t] = \frac{1}{s^2},$$
$$\mathcal{L}[f(t) = t^2] = \frac{2!}{s^3}.$$



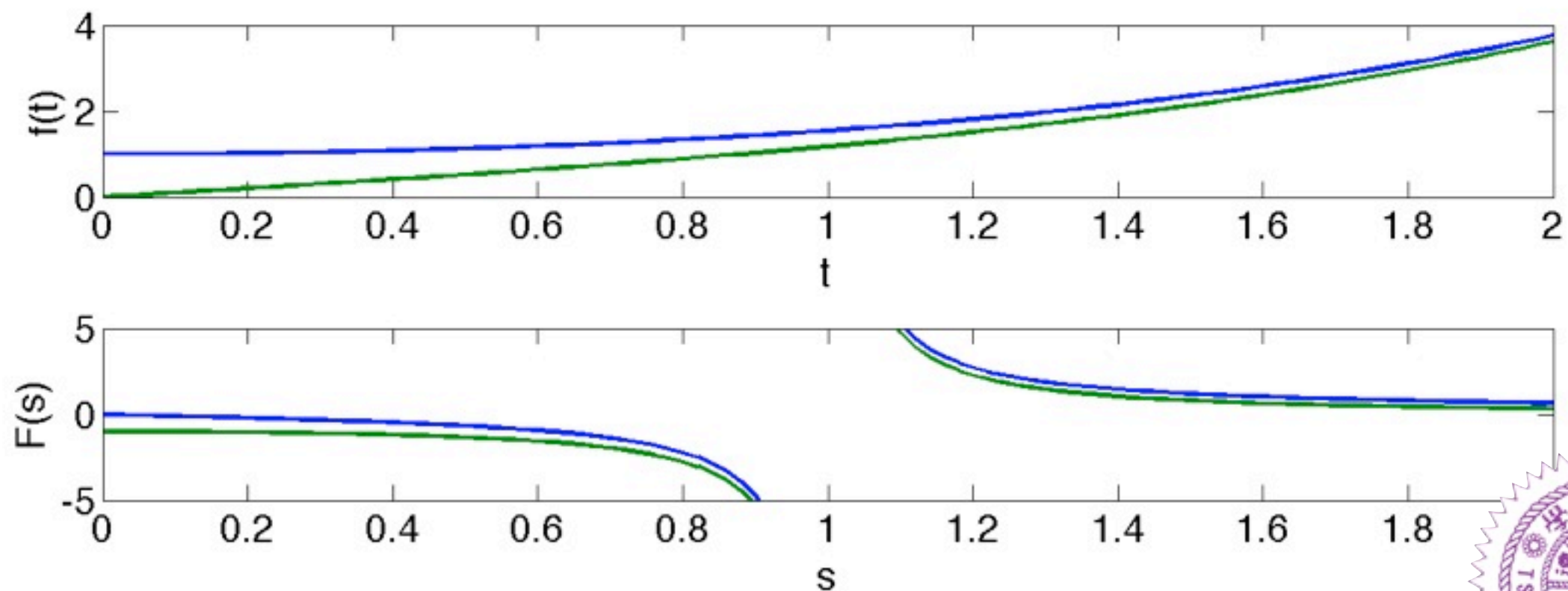
Laplace Transform: Sinusoidal

$$\mathcal{L}[f(t) = \cos t] = \frac{s}{s^2 + 1},$$
$$\mathcal{L}[f(t) = \sin t] = \frac{1}{s^2 + 1}.$$



Laplace Transform: Hyper-sinusoidal

$$\mathcal{L}[f(t) = \cosh t] = \frac{s}{s^2 - 1},$$
$$\mathcal{L}[f(t) = \sinh t] = \frac{1}{s^2 - 1}.$$



Laplace Transform: s -Shifting

- If $f(t)$ has the transform $F(s)$,

$$\mathcal{L}[f(t)] = F(s),$$

then

$$\mathcal{L}[e^{at}f(t)] = F(s - a),$$

- or, if we take the inverse on both sides,

$$\mathcal{L}^{-1}[F(s - a)] = e^{at}f(t).$$



Laplace Transform: s -Shifting, Examples

Example 1:

$$\mathcal{L}[e^{at} \cos \omega t]$$

Solution:

$$F(s - a) = \frac{s - a}{(s - a)^2 + \omega^2}.$$

Example 2:

$$\mathcal{L}[f(t)] = F(s) = \frac{3s - 137}{s^2 + 2s + 401}.$$

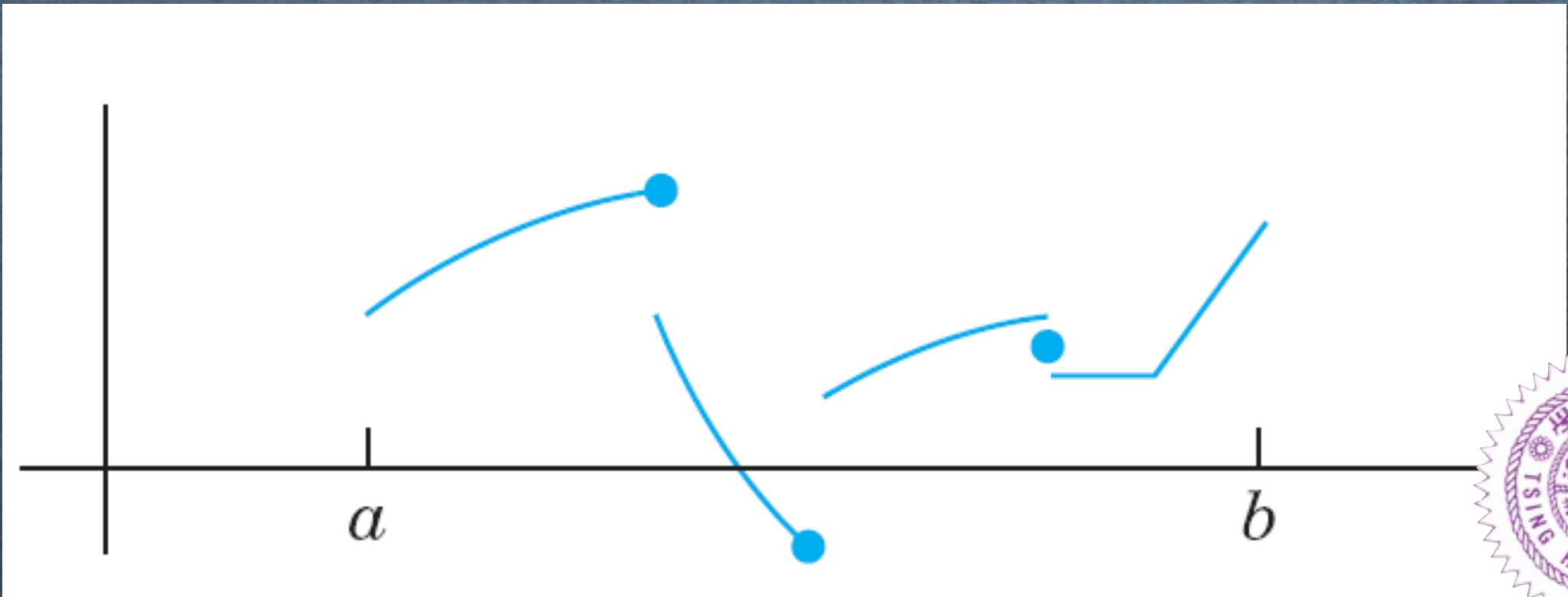
Solution:

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}[F(s)] = 3\mathcal{L}^{-1}\left[\frac{s + 1}{(s + 1)^2 + 20^2}\right] - 7\mathcal{L}^{-1}\left[\frac{20}{(s + 1)^2 + 20^2}\right] \\ &= e^{-t}(3 \cos 20t - 7 \sin 20t). \end{aligned}$$



Laplace Transform: Existence and Uniqueness

- $f(t)$ needs not be **continuous**, but it should not be too bad.
- The technical term is **piecewise continuity**
- $f(t)$ is *piecewise continuous* on a finite interval $a \leq t \leq b$ where f is defined, if this interval can be divided into *finitely* many subintervals in each of which f is continuous and has finite limits as t approaches either endpoint of such a subinterval from the interior.



Laplace Transform: Derivatives

- The transform of first derivative of f satisfies

$$\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0).$$

- The transform of second derivative of f satisfies

$$\mathcal{L}[f''(t)] = s^2\mathcal{L}[f(t)] - sf(0) - f'(0).$$

- The transform of n th-derivative of f satisfies

$$\mathcal{L}[f^{(n)}(t)] = s^n\mathcal{L}[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0).$$



Laplace Transform: Derivatives, Example

- Let

$$f(t) = \cos \omega t.$$

- Then $f(0) = 1$, $f'(0) = 0$, and $f''(t) = -\omega^2 \cos \omega t$.
- We can derive the Laplace transform of $\cos \omega t$ by

$$\begin{aligned}\mathcal{L}[f''(t)] &= s^2 \mathcal{L}[f(t)] - s = -\omega^2 \mathcal{L}[f(t)], \\ \rightarrow \mathcal{L}[\cos \omega t] &= \frac{s}{s^2 + \omega^2}.\end{aligned}$$



Laplace Transform: Integrals

- Let $F(s)$ denote the Laplace transform of a function $f(t)$, which is *piecewise continuous* for $t \geq 0$ and satisfies a growth restriction, then
- For $s > 0$, $s > k$, and $t > 0$,

$$\mathcal{L}\left[\int_0^t f(\tau) d\tau\right] = \frac{1}{s}F(s).$$

- The inverse Laplace transform has

$$\int_0^t f(\tau) d\tau = \mathcal{L}^{-1}\left[\frac{1}{s}F(s)\right].$$



Laplace Transform: Integrals, Example 1

Example:

Find the inverse Laplace transform of

$$\frac{1}{s(s^2 + \omega^2)}.$$

Hints:

$$\mathcal{L}^{-1}\left[\frac{1}{s^2 + \omega^2}\right] = \frac{\sin \omega t}{\omega}.$$

Solution:

$$\mathcal{L}^{-1}\left[\frac{1}{s(s^2 + \omega^2)}\right] = \int_0^t \frac{\sin \omega \tau}{\omega} d\tau = \frac{1}{\omega^2}(1 - \cos \omega t).$$



Laplace Transform: Integrals, Example 2

Example:

Find the inverse Laplace transform of

$$\frac{1}{s^2(s^2 + \omega^2)}.$$

Solution:

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{1}{s^2(s^2 + \omega^2)}\right] &= \frac{1}{\omega^2} \int_0^t (1 - \cos \omega\tau) d\tau, \\ &= \frac{t}{\omega^2} - \frac{\sin \omega t}{\omega^3}.\end{aligned}$$



Homework #8:

1. (20%) Calculate the value of $\Gamma(\frac{1}{2})$.
2. (20%) Find the Laplace transform of a *triangle function*, defined as:

$$f(t) = \begin{cases} t & ; \text{ for } 0 < t < 1 \\ 2 - t & ; \text{ for } 1 < t < 2 \\ 0 & ; \text{ otherwise} \end{cases} \quad (1)$$

Problem 20 in the [Textbook], at p.p. 227.

3. (20%) Find the Laplace transform by *differentiation*:

$$f(t) = \sin^4 t. \quad (2)$$

Problem 8 in the [Textbook], at p.p. 232.

4. (20%) Find the Laplace transform by *integration*:

$$f(t) = \frac{5}{s^3 - 5s} \quad (3)$$

Problem 31 in the [Textbook], at p.p. 233.

5. (20%) Solve the *initial value problem* by Laplace transform:

$$y'' + 3y' - 4y = 6e^{2t-2}, \quad y(1) = 4, \quad \text{and} \quad y'(1) = 5. \quad (4)$$

Problem 23 in the [Textbook], at p.p. 232.



Laplace Transform: General Formulas

Formula	Name, Comments	Sec.
$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$ $f(t) = \mathcal{L}^{-1}\{F(s)\}$	Definition of Transform Inverse Transform	6.1
$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$	Linearity	6.1
$\mathcal{L}\{e^{at}f(t)\} = F(s - a)$ $\mathcal{L}^{-1}\{F(s - a)\} = e^{at}f(t)$	s -Shifting (First Shifting Theorem)	6.1



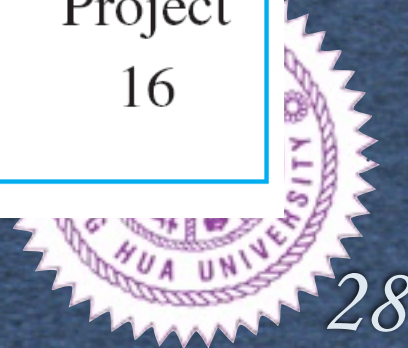
Laplace Transform: General Formulas

Formula	Name, Comments	Sec.
$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$ $\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$ $\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{(n-1)}f(0) - \dots$ $\dots - f^{(n-1)}(0)$	Differentiation of Function	6.2
$\mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\} = \frac{1}{s} \mathcal{L}(f)$	Integration of Function	
$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$ $= \int_0^t f(t - \tau)g(\tau) d\tau$ $\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$	Convolution	6.5



Laplace Transform: General Formulas

Formula	Name, Comments	Sec.
$\mathcal{L}\{f(t-a) u(t-a)\} = e^{-as} F(s)$ $\mathcal{L}^{-1}\{e^{-as} F(s)\} = f(t-a) u(t-a)$	t -Shifting (Second Shifting Theorem)	6.3
$\mathcal{L}\{tf(t)\} = -F'(s)$ $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\tilde{s}) d\tilde{s}$	Differentiation of Transform Integration of Transform	6.6
$\mathcal{L}(f) = \frac{1}{1 - e^{-ps}} \int_0^p e^{-st} f(t) dt$	f Periodic with Period p	6.4 Project 16



Laplace Transform: Table

	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$	Sec.
1	$1/s$	1	} 6.1
2	$1/s^2$	t	
3	$1/s^n \quad (n = 1, 2, \dots)$	$t^{n-1}/(n-1)!$	
4	$1/\sqrt{s}$	$1/\sqrt{\pi t}$	
5	$1/s^{3/2}$	$2\sqrt{t/\pi}$	
6	$1/s^a \quad (a > 0)$	$t^{a-1}/\Gamma(a)$	
7	$\frac{1}{s-a}$	e^{at}	} 6.1
8	$\frac{1}{(s-a)^2}$	te^{at}	
9	$\frac{1}{(s-a)^n} \quad (n = 1, 2, \dots)$	$\frac{1}{(n-1)!} t^{n-1} e^{at}$	
10	$\frac{1}{(s-a)^k} \quad (k > 0)$	$\frac{1}{\Gamma(k)} t^{k-1} e^{at}$	
11	$\frac{1}{(s-a)(s-b)} \quad (a \neq b)$	$\frac{1}{(a-b)} (e^{at} - e^{bt})$	
12	$\frac{s}{(s-a)(s-b)} \quad (a \neq b)$	$\frac{1}{(a-b)} (ae^{at} - be^{bt})$	



Laplace Transform: Table

	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$	Sec.
13	$\frac{1}{s^2 + \omega^2}$	$\frac{1}{\omega} \sin \omega t$	6.1
14	$\frac{s}{s^2 + \omega^2}$	$\cos \omega t$	
15	$\frac{1}{s^2 - a^2}$	$\frac{1}{a} \sinh at$	
16	$\frac{s}{s^2 - a^2}$	$\cosh at$	
17	$\frac{1}{(s - a)^2 + \omega^2}$	$\frac{1}{\omega} e^{at} \sin \omega t$	
18	$\frac{s - a}{(s - a)^2 + \omega^2}$	$e^{at} \cos \omega t$	



Laplace Transform: Table

	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$	Sec.
19	$\frac{1}{s(s^2 + \omega^2)}$	$\frac{1}{\omega^2} (1 - \cos \omega t)$	} 6.2
20	$\frac{1}{s^2(s^2 + \omega^2)}$	$\frac{1}{\omega^3} (\omega t - \sin \omega t)$	
21	$\frac{1}{(s^2 + \omega^2)^2}$	$\frac{1}{2\omega^3} (\sin \omega t - \omega t \cos \omega t)$	6.6
22	$\frac{s}{(s^2 + \omega^2)^2}$	$\frac{t}{2\omega} \sin \omega t$	} 6.6
23	$\frac{s^2}{(s^2 + \omega^2)^2}$	$\frac{1}{2\omega} (\sin \omega t + \omega t \cos \omega t)$	
24	$\frac{s}{(s^2 + a^2)(s^2 + b^2)} \quad (a^2 \neq b^2)$	$\frac{1}{b^2 - a^2} (\cos at - \cos bt)$	
25	$\frac{1}{s^4 + 4k^4}$	$\frac{1}{4k^3} (\sin kt \cos kt - \cos kt \sinh kt)$	
26	$\frac{s}{s^4 + 4k^4}$	$\frac{1}{2k^2} \sin kt \sinh kt$	



Laplace Transform: ODE with IVP, Example

t-space

Given problem

$$y'' - y = t$$

$$y(0) = 1$$

$$y'(0) = 1$$

s-space

Subsidiary equation

$$(s^2 - 1)Y = s + 1 + 1/s^2$$

Solution of given problem

$$y(t) = e^t + \sinh t - t$$

Solution of subsidiary equation

$$Y = \frac{1}{s-1} + \frac{1}{s^2-1} - \frac{1}{s^2}$$



Laplace Transform: ODE with IVP

- Consider a 2nd-order ODE with the constant coefficients, and a **initial value condition**:

$$y'' + ay' + by = r(t), \quad y(0) = K_0, \quad \text{and} \quad y'(0) = K_1,$$

where a and b are constants.

- Now we have a **non-homogeneous** ODE.
- Here $r(t)$ is the given **input** (driving force) applied to the mechanical or electrical system.
- $y(t)$ is the **output** (response to the input) to be obtained.



Laplace Transform: ODE with IVP, cont.

- Define the Laplace transform $Y(s) \equiv \mathcal{L}[y(t)]$, and obtain an algebraic equation:

Step 1: $[s^2Y - sy(0) - y'(0)] + a[sY - y(0)] + bY = \mathcal{L}[r(t)] \equiv R(s),$

- Solve the subsidiary equation by algebra:

Step 2:
$$Y(s) = \frac{(s+a)y(0) + y'(0)}{s^2 + as + b} + \frac{R(s)}{s^2 + as + b},$$
$$\equiv [(s+a)y(0) + y'(0)]Q(s) + R(s)Q(s),$$

- $Q(s)$ is called the **transfer function**, which depends neither on $r(t)$ nor on the initial condition, but only on a and b .
- If $y(0) = y'(0) = 0$, then $Q = \frac{Y}{R} = \frac{\mathcal{L}(\text{output})}{\mathcal{L}(\text{input})}$.

Step 3: • Inversion of $Y(s)$ to obtain $y(t) = \mathcal{L}^{-1}[Y].$



Laplace Transform: ODE with IVP, Example

Example:

$$y'' - y = t, \quad y(0) = 1 \quad \text{and} \quad y'(0) = 1.$$

Hints:

- The subsidiary equation, with $Y = \mathcal{L}[y]$:

$$\begin{aligned} Y &= \frac{s+1}{(s^2-1)} + \frac{1}{s^2(s^2-1)} \\ &= \frac{1}{s-1} + \left(\frac{1}{s^2-1} - \frac{1}{s^2} \right). \end{aligned}$$

Solution:

$$y(t) = \mathcal{L}^{-1}[Y] = e^t + \sinh t - t.$$



Laplace Transform: ODE with IVP, Advantages

- ✓ Solving a **non-homogeneous** ODE does not require first solving the homogeneous ODE.
- ✓ **Initial values** are automatically taken care of.
- ✓ Complicated inputs **$r(t)$** (right sides of linear ODEs) can be handled very efficiently.

Any Disadvantages?



Laplace Transform: ODE with IVP, Shifted Data

Example:

$$y'' + y = 2t, \quad y\left(\frac{1}{4}\pi\right) = \frac{1}{2}\pi \quad \text{and} \quad y'\left(\frac{1}{4}\pi\right) = 2 - \sqrt{2}.$$

- Define $\tau = t - \frac{1}{4}\pi$, then the original equation becomes

Hints:

$$y'' + y = 2\left(\tau + \frac{1}{4}\pi\right), \quad y(0) = \frac{1}{2}\pi \quad \text{and} \quad y'(0) = 2 - \sqrt{2}.$$

- The subsidiary equation, with $Y(s) = \mathcal{L}[y(\tau)]$:

$$Y = \frac{2}{s^2(s^2 + 1)} + \frac{\pi/2}{s(s^2 + 1)} + \frac{\pi s/2}{s^2 + 1} + \frac{2 - \sqrt{2}}{s^2 + 1}.$$

$$y(\tau) = 2\tau + \frac{1}{2}\pi - \sqrt{2}\sin \tau$$

Solution:

$$\Rightarrow y(t) = 2t - \sin t + \cos t.$$



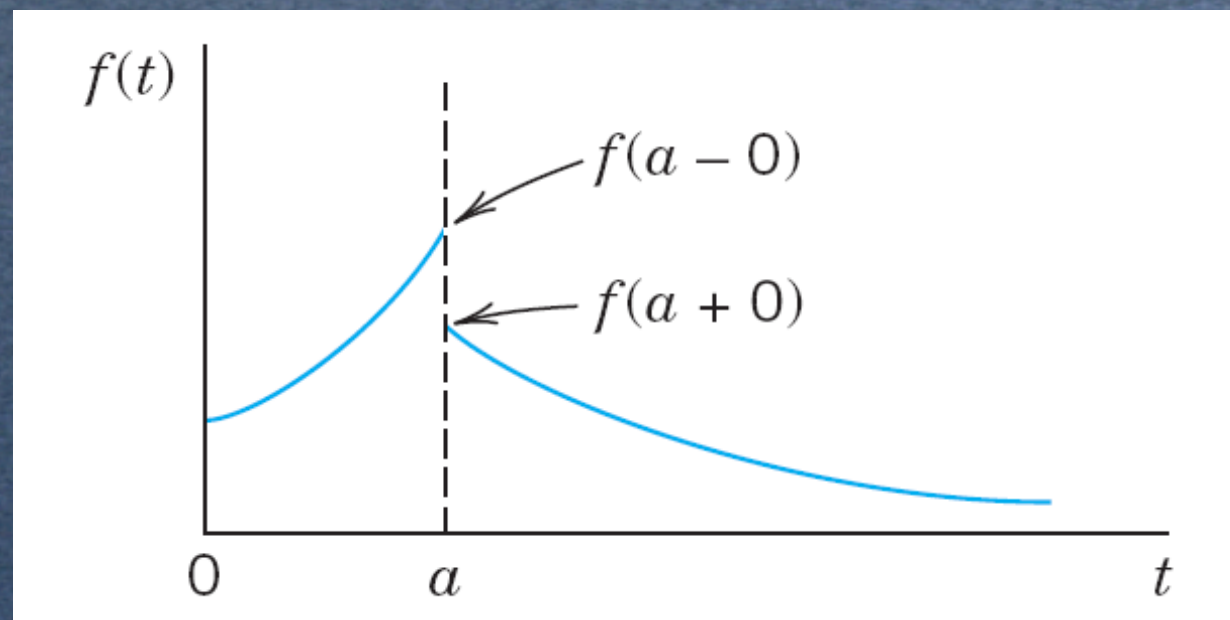
Laplace Transform: Derivatives, Discontinuity

- The transform of first derivative of f satisfies

$$\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0).$$

- The transform of first derivative of f satisfies, but with a discontinuity at $t = a$,

$$\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0) - [f(a+0) - f(a-0)]e^{-as}.$$

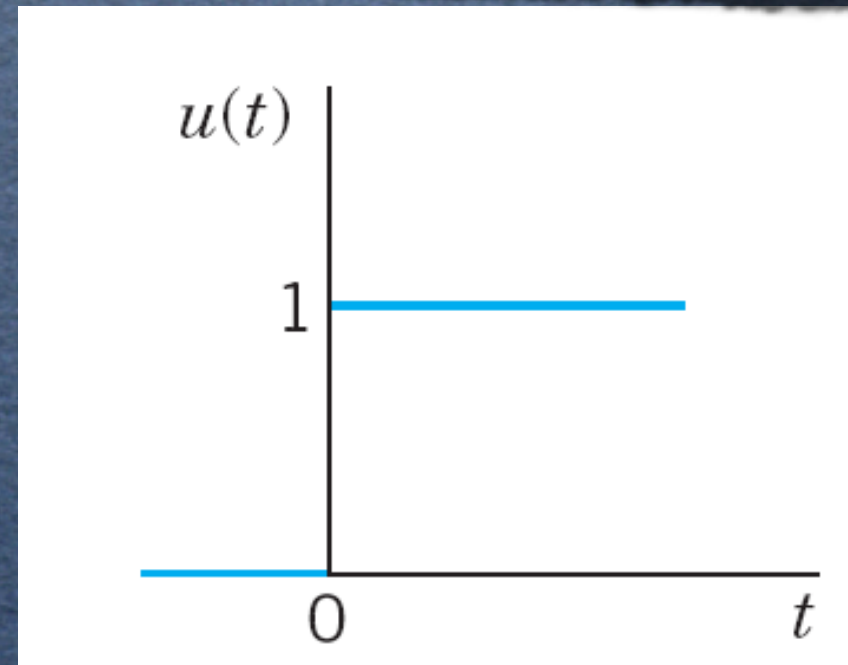
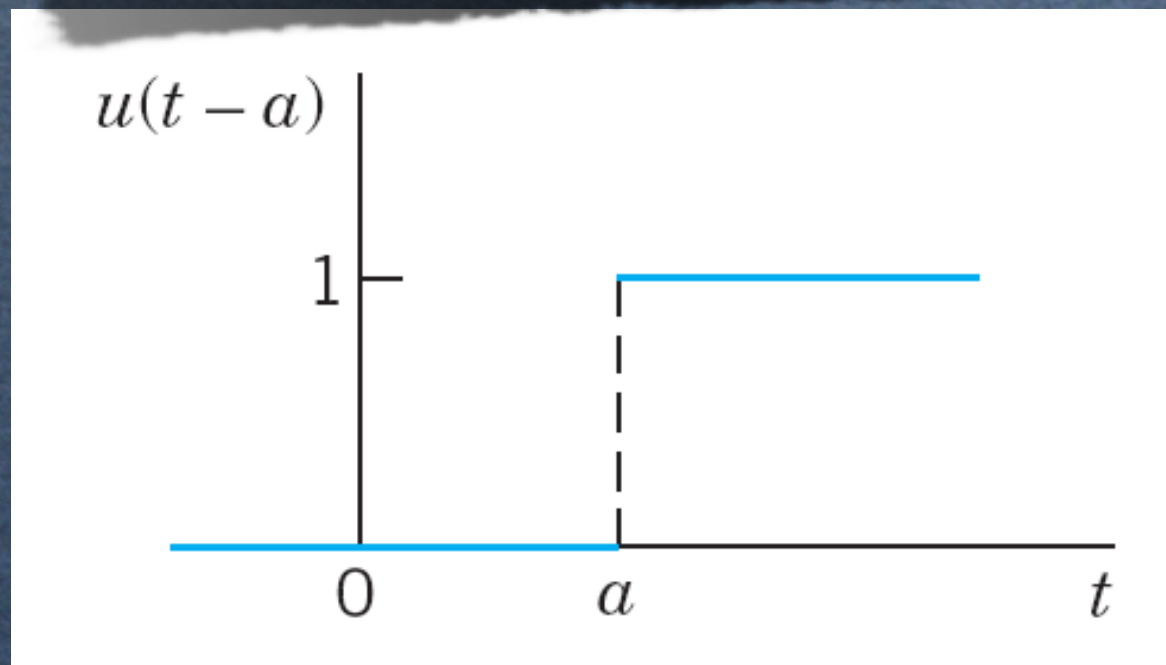


Laplace Transform: Unit-Step Function

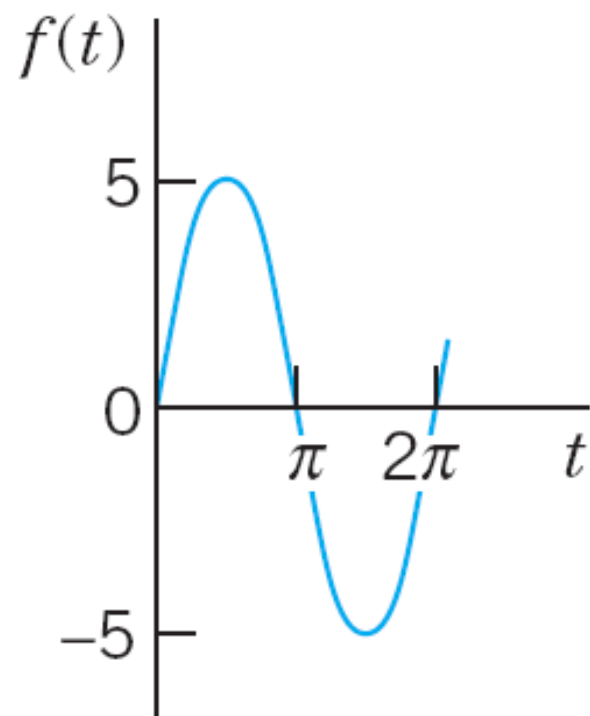
- The unit-step function or Heaviside function is defined as

$$u(t - a) = \begin{cases} 0 & ; \text{ for } t < a \\ 1 & ; \text{ for } t > a \end{cases}$$

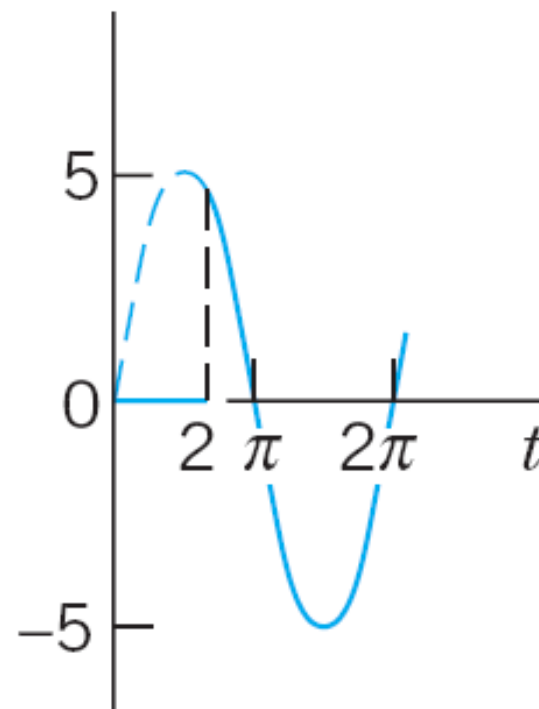
The unit step function is a typical “engineering function”, for the mechanical or electrical driving forces to “off” or “on”.



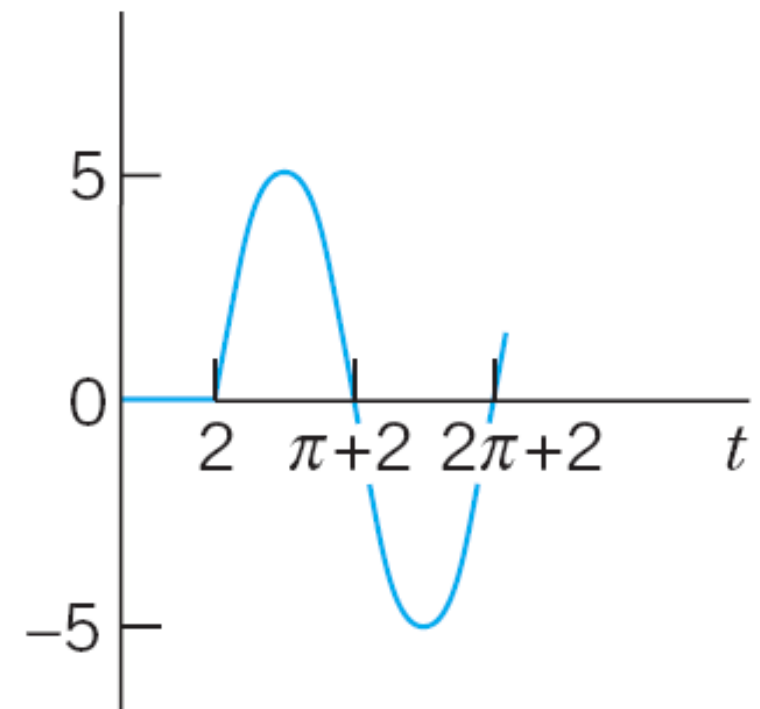
Laplace Transform: Unit-Step Function, on-off



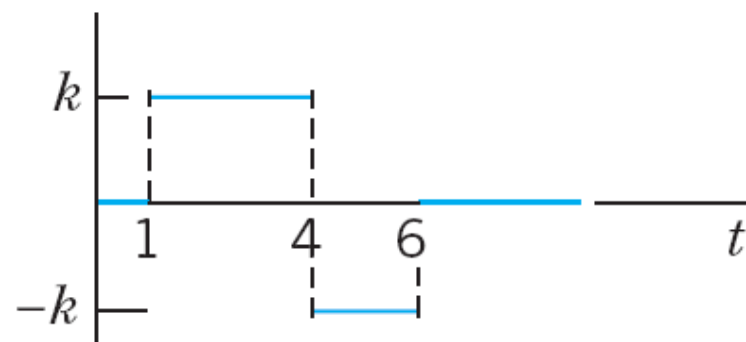
(A) $f(t) = 5 \sin t$



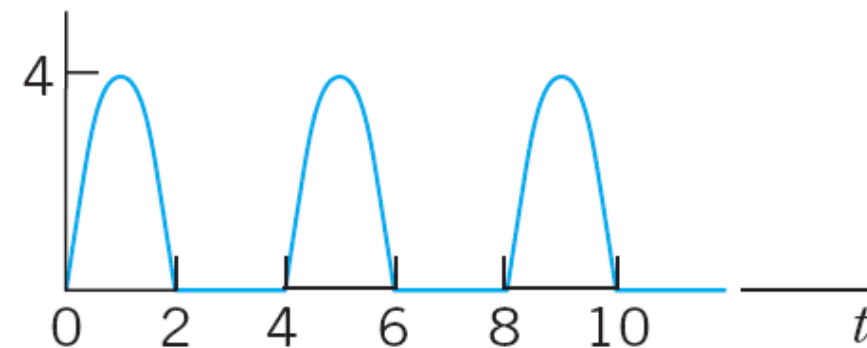
(B) $f(t)u(t-2)$



(C) $f(t-2)u(t-2)$



(A) $k[u(t-1) - 2u(t-4) + u(t-6)]$



(B) $4 \sin(\frac{1}{2}\pi t)[u(t) - u(t-2) + u(t-4) - + \dots]$



Laplace Transform: t -Shifting

- If $f(t)$ has the Laplace transform $F(s) \equiv \mathcal{L}[f(t)]$, then the "shifted function"

$$\tilde{f}(t) \equiv f(t-a)u(t-a) = \begin{cases} 0 & ; \text{ for } t < a \\ f(t-a) & ; \text{ for } t > a \end{cases}$$

has the transform:

$$\mathcal{L}[f(t-a)u(t-a)] = e^{-as}F(s).$$

- Or, if we take the inverse,

$$\mathcal{L}^{-1}[e^{-as}F(s)] = f(t-a)u(t-a).$$

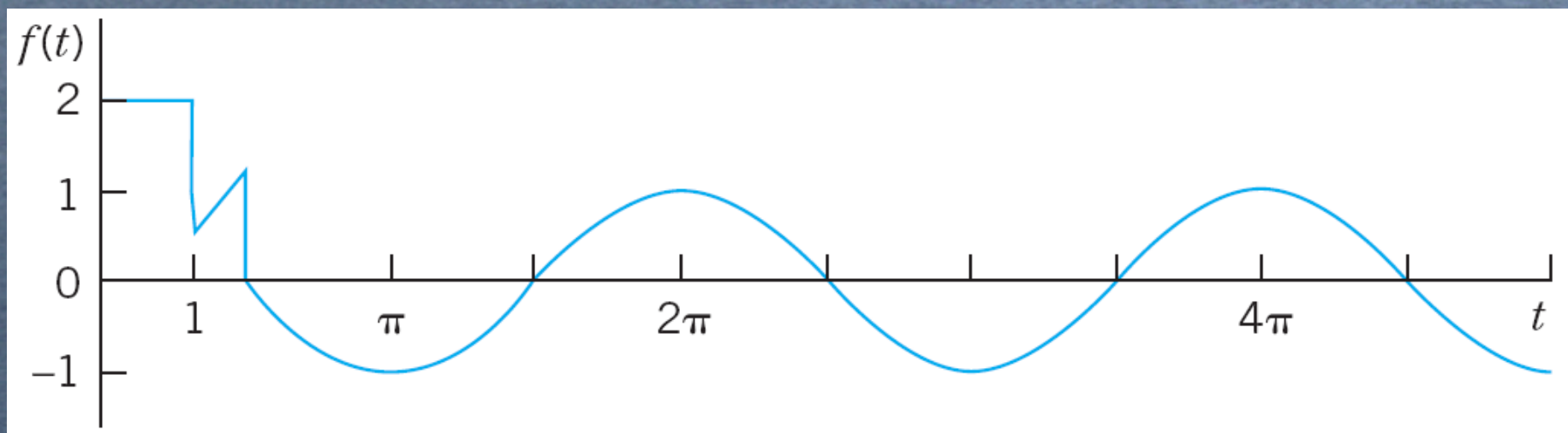


Laplace Transform: t -Shifting, Example 1

Example:

$$f(t) = \begin{cases} 2 & ; \text{ for } 0 < t < 1 \\ \frac{1}{2}t^2 & ; \text{ for } 1 < t < \frac{1}{2}\pi \\ \cos t & ; \text{ for } t > \frac{1}{2}\pi \end{cases}$$

Hints:



$$f(t) = 2[1 - u(t - 1)] + \frac{1}{2}t^2[u(t - 1) - u(t - \frac{1}{2}\pi)] + \cos(t)u(t - \frac{1}{2}\pi).$$

Solution:

$$F(s) = \frac{2}{s} - \frac{2}{s}e^{-s} + \left(\frac{1}{s^3} + \frac{1}{s^2} + \frac{1}{2s}\right)e^{-s} \\ - \left(\frac{1}{s^3} + \frac{\pi}{2s^2} + \frac{\pi^2}{8s}\right)e^{-\pi s/2} - \frac{1}{s^2 + 1}e^{-\pi s/2}$$



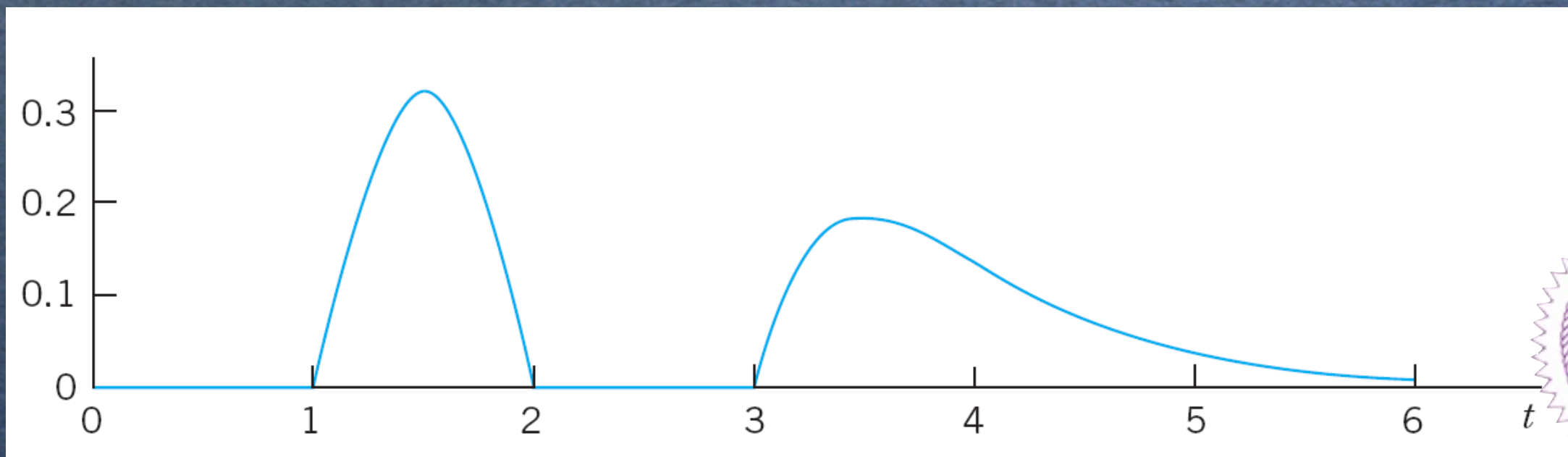
Laplace Transform: *t*-Shifting, Example 2

Example:

$$F(s) = \frac{e^{-s}}{s^2 + \pi^2} + \frac{e^{-2s}}{s^2 + \pi^2} + \frac{e^{-3s}}{(s + 2)^2}.$$

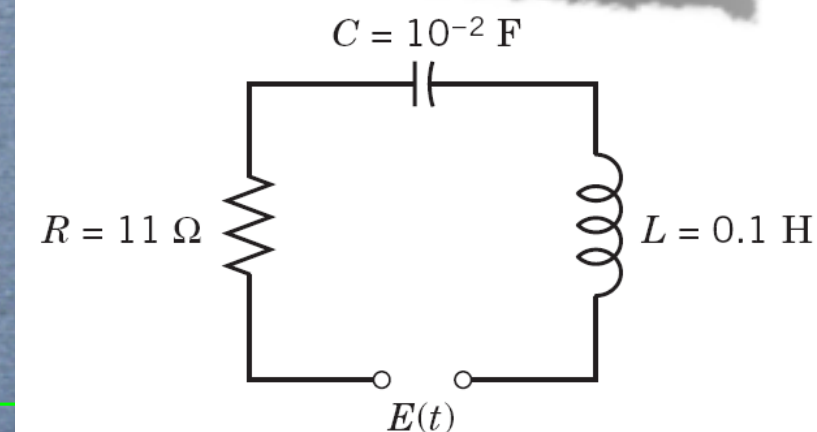
Solution:

$$F(t) = \frac{1}{\pi} \sin[\pi(t - 1)]u(t - 1) + \frac{1}{\pi} \sin[\pi(t - 2)]u(t - 2) + (t - 3)e^{-2(t-3)}u(t - 3).$$



Laplace Transform: t -Shifting, Example 3

Example:



$$0.1i' + 11i + 100 \int_0^t i(\tau) d\tau = E(t),$$

$$E(t) = \begin{cases} 100 \sin 400t & ; \text{ for } 0 < t < 2\pi \\ 0 & ; \text{ for } t > 2\pi \end{cases} = [100 \sin 400t][1 - u(t - 2\pi)],$$

$$\text{I.C.: } i(0) = 0 \quad \text{and} \quad i'(0) = 0.$$

Solution:

$$i(t) = -0.2776[e^{-10t} - e^{-10(t-2\pi)}] + 2.6144[e^{-100t} - e^{-100t(t-2\pi)}].$$



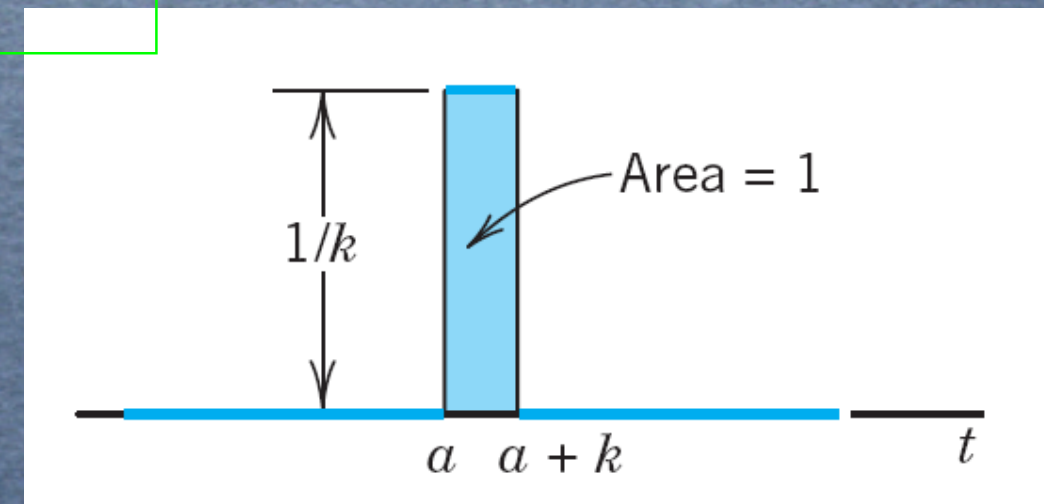
Laplace Transform: Dirac's Delta Function

- The **impulse function** is defined as:

$$f_k(t - a) = \begin{cases} 1/k & ; \text{ for } a \leq t \leq a + k \\ 0 & ; \text{ otherwise} \end{cases}$$

- The area of the impulse function is a unit,

$$I_k = \int_{-\infty}^{\infty} f_k(t - a) dt = 1.$$



- As $k \rightarrow 0$, we have **Dirac δ function**, or the *unite impulse function*,

$$\delta(t - a) \equiv \lim_{k \rightarrow 0} f_k(t - a) = \begin{cases} \infty & ; \text{ for } t = a \\ 0 & ; \text{ otherwise} \end{cases} \quad \text{but} \quad \int_{-\infty}^{\infty} \delta(t - a) dt = 1.$$

The Delta function is not a function in the ordinary sense as used in Calculus, but a so-called **generalized function.**



Laplace Transform: Dirac's Delta Function

- Differentiable:

$$\delta_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2 x^2},$$

- *Sinc* function:

$$\delta_n(x) = \frac{\sin nx}{\pi x},$$

- Sequence:

$$\delta_n(x) = \frac{n}{\pi} \frac{1}{(1 + n^2 x^2)},$$



Laplace Transform: Delta Function, cont.

- Shifting property of $\delta(t - a)$:

$$\int_{-\infty}^{\infty} g(t) \delta(t - a) dt = g(a).$$

- Laplace transform of $\delta(t - a)$:
- The Laplace transform for $f_k(t - a) = \frac{1}{k}[u(t - a) - u(t - a - k)]$ is,

$$\mathcal{L}[f_k(t - a)] = \frac{e^{-as} - e^{-(a+k)s}}{ks}.$$

- As $k \rightarrow 0$,

$$\mathcal{L}[\delta(t - a)] = e^{-as}.$$



Laplace Transform: Delta Function, Example 1

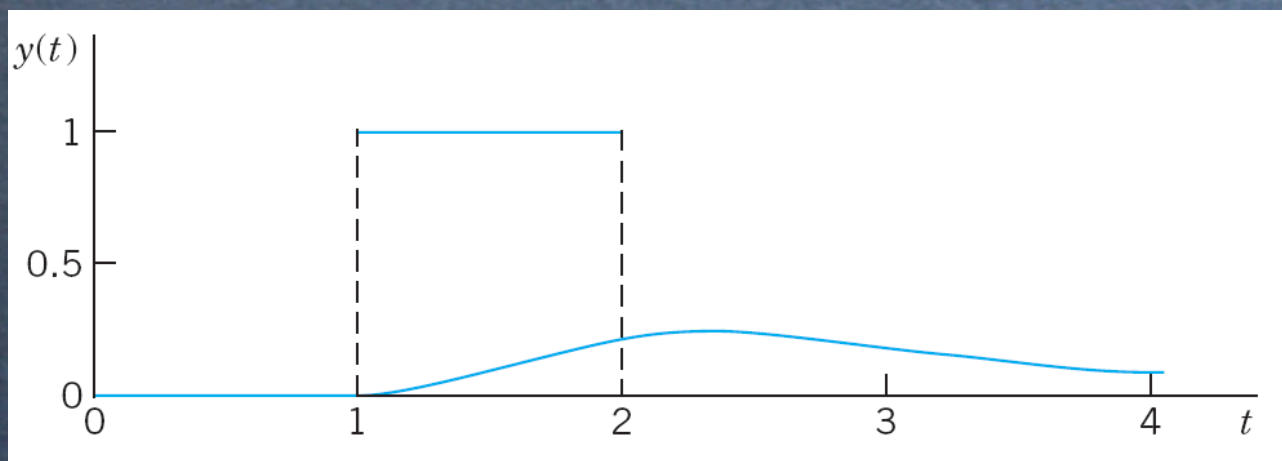
Example:

$$y'' + 3y' + 2y = u(t-1) - u(t-2), \quad y(0) = 0 \quad \text{and} \quad y'(0) = 0.$$

Hints:

- Define the Laplace transform $Y(s) \equiv \mathcal{L}[y(t)]$, and solve

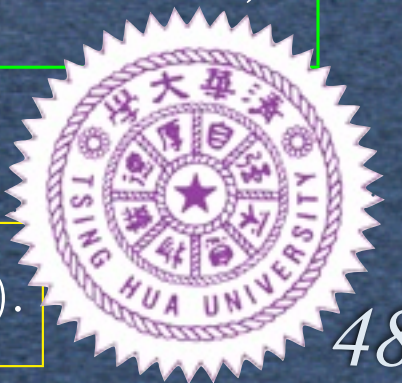
$$s^2 Y + 3sY + 2Y = \frac{1}{s}(e^{-s} - e^{-2s}).$$



$$Y(s) = \left[\frac{1/2}{s} - \frac{1}{s+1} + \frac{1/2}{s+2} \right] (e^{-s} - e^{-2s}).$$

Solution:

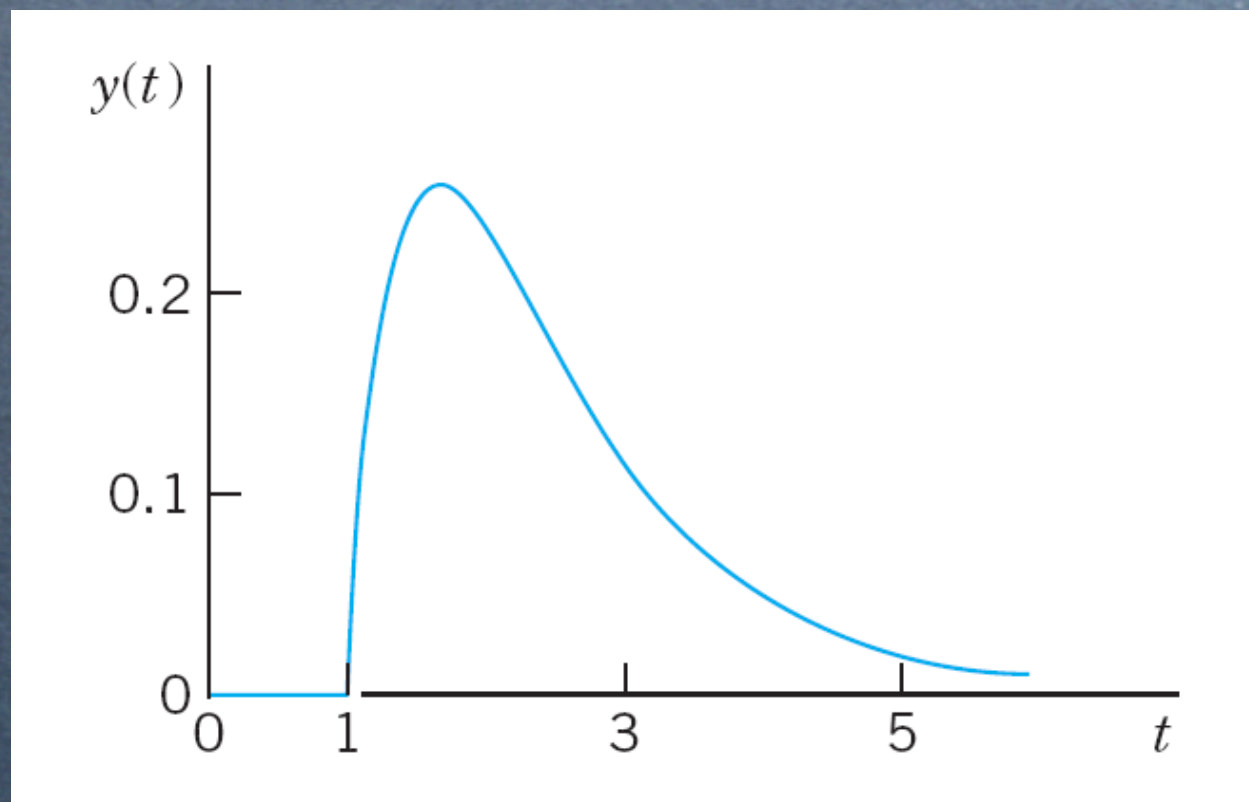
$$y(t) = \mathcal{L}^{-1}[Y(s)] = \left[\frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)} \right] u(t-1) - \left[\frac{1}{2} - e^{-(t-2)} + \frac{1}{2}e^{-2(t-2)} \right] u(t-2).$$



Laplace Transform: Delta Function, Example 2

Example:

$$y'' + 3y' + 2y = \delta(t - 1), \quad y(0) = 0 \quad \text{and} \quad y'(0) = 0.$$



Solution:

$$y(t) = \mathcal{L}^{-1}[Y(s)] = [e^{-(t-1)} - e^{-2(t-1)}]u(t-1).$$

Take the Laplace transform $Y(s) \equiv \mathcal{L}[y(t)]$, and solve

$$s^2 Y + 3sY + 2Y = e^{-s}.$$

Partial fraction decomposition to have

$$Y(s) = \left[\frac{1}{s+1} - \frac{1}{s+2} \right] e^{-s}.$$



Laplace Transform: Delta Function, Example 3

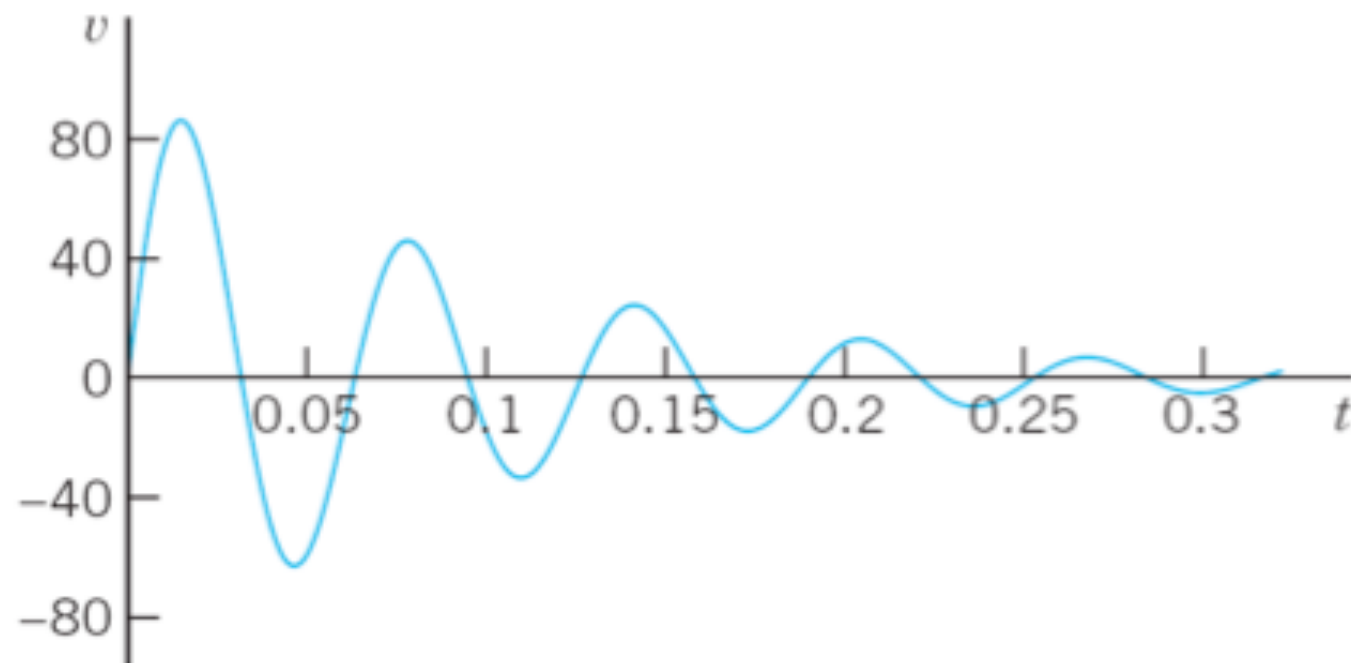
- $R = 20\Omega$, $L = 1H$, and $C = 10^{-4}F$:

Example:

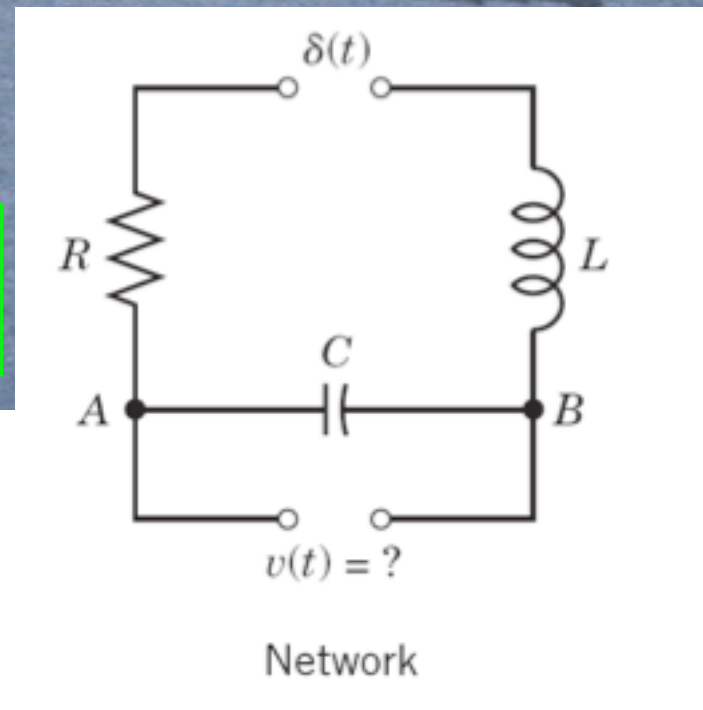
$$Li' + Ri + \frac{q}{C} = Lq'' + Rq' + \frac{q}{C} = \delta(t).$$

Hints:

- Define t
- Solve the



Voltage on the capacitor

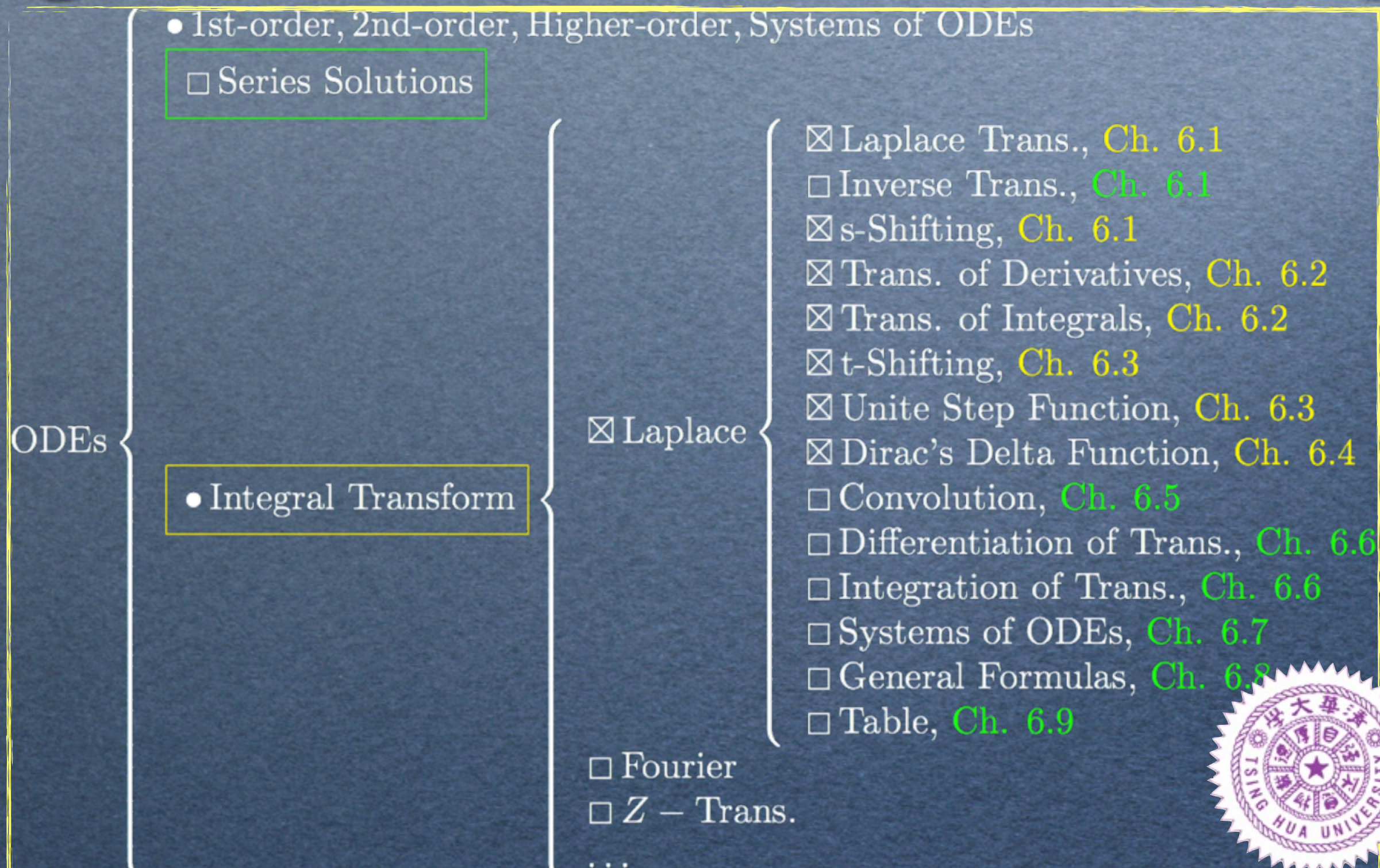


Solution:

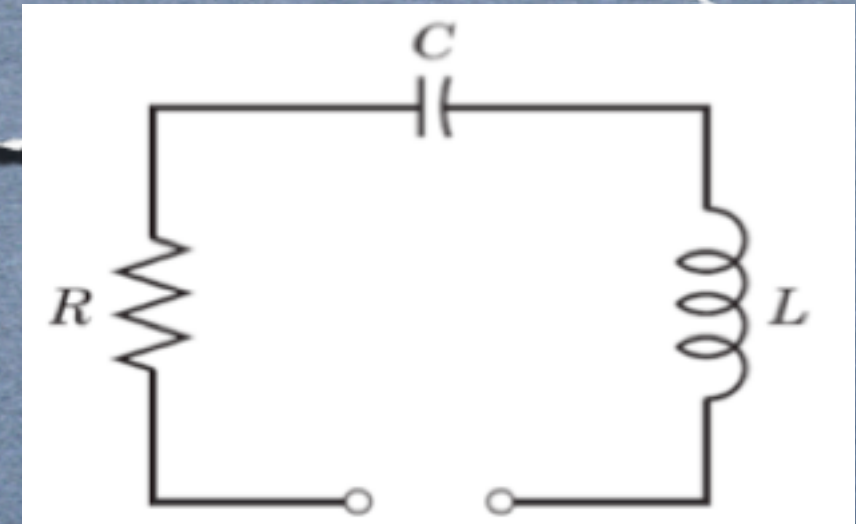
$$v(t) = \frac{q(t)}{C} = 100.5e^{-10t} \sin 99.5t.$$



Systems of ODEs: Summary



Homework #9:



1. (25%) Using the Laplace transform to find the current in a RLC circuit with

$$R = 2 \quad (\Omega), \quad L = 1 \quad (\text{H}), \quad C = 0.1 \quad (\text{F}), \quad i(t) = 0 \quad q(0) = 0,$$

and

$$v(t) = \begin{cases} 255 \sin t & (\text{V}) \quad ; \quad \text{for } 0 < t < 2\pi \\ 0 & ; \quad \text{for } t > 2\pi \end{cases} \quad (1)$$

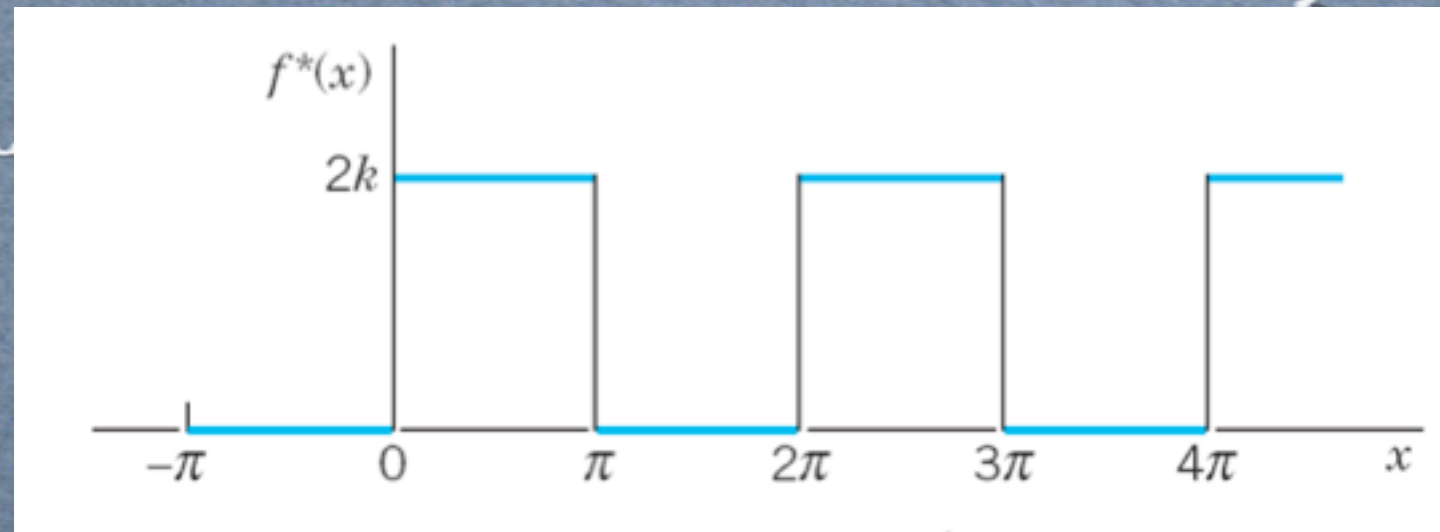
Problem 47 in the [Textbook], at p.p. 241.

2. (25%) Solve the 2nd-order non-homogeneous equation:

$$y'' + 3y' - 4y = 2e^t - 8e^2\delta(t - 2), \quad y(0) = 2 \quad \text{and} \quad y'(0) = 0.$$



Homework #9:



3. (50%) Laplace transform of a *periodic function*:

(a) Find the Laplace transform of a piecewise continuous function $f(t)$ with period p

$$f^*(t + n * p) = f^*(t), \quad \text{where } n \text{ is an integer} \quad (3)$$

(b) Find the Laplace transform of the *Half-Wave Rectifier*:

$$f^*(t) = \begin{cases} 2k & ; \text{ for } 0 < t < \pi \\ 0 & ; \text{ for } \pi < t < 2\pi \\ f^*(t + 2 * n * \pi) & ; \text{ where } n \text{ is an integer} \end{cases} \quad (4)$$

where k is a constant.

You can find a similar Problem in the [Textbook], at p.p. 248.



Homework #9:

1. Please do the homework Yourself !!
2. Homework is designed for your PRACTICE, Take It Easy ^.^
3. If you have any questions, please write an email to me or come to my office.
5. Please return the Homework by the Deadline:

**Deadline Oct. 28 (this Thursday), 1:00PM
before the class!!**



Laplace Transform: Convolution

- If two functions have the corresponding Laplace transforms, $F \equiv \mathcal{L}[f(t)]$ and $G \equiv \mathcal{L}[g(t)]$, then

$$H(s) \equiv F(s) \cdot G(s).$$

- The inverse Laplace transform of $H(s)$ is called the **convolution** of f and g ,

$$h(t) = \mathcal{L}^{-1}[F(s) \cdot G(s)] \equiv f * g(t) = \int_0^t f(\tau) g(t - \tau) d\tau.$$



Laplace Transform: Convolution, Examples

Example 1:

$$H(s) = \frac{1}{(s-a)s}.$$

Solution:

$$h(t) = e^{at} * 1 = \int_0^t e^{a\tau} \cdot 1 \, d\tau = \frac{1}{a} [e^{at} - 1].$$

Example 2:

$$H(s) = \frac{1}{(s^2 + \omega^2)^2}.$$

Solution:

$$h(t) = \frac{\sin \omega t}{\omega} * \frac{\sin \omega t}{\omega} = \frac{1}{2\omega^2} \left[-t \cos \omega t + \frac{\sin \omega t}{\omega} \right].$$



Laplace Transform: Convolution, Example 2

- In general, $f * 1 \neq f$. For example

$$t * 1 = \int_0^t \tau \cdot 1 \, d\tau = \frac{1}{2}t^2 \neq t.$$

- Also, $f * f \geq 0$ may not hold. For example,

$$\sin t * \sin t = -\frac{1}{2}t \cos t + \frac{1}{2} \sin t.$$



Laplace Transform: Convolution, Example 3

In an undamped mass-spring system, **resonance** occurs if the frequency of the driving force equals the natural frequency of the system.

$$y'' + \omega_0^2 y = K \sin \omega_0 t, \quad \text{with } y(0) = 0 \quad \text{and} \quad y'(0) = 0.$$

Hints:

- Define the Laplace transform $Y(s) \equiv \mathcal{L}[y(t)]$, and solve

$$s^2 Y + \omega_0^2 Y = \frac{K \omega_0}{s^2 + \omega_0^2}.$$

- Solve the algebraical equation to have

The first term grows without bound.

Solution:

$$y(t) = \frac{K}{2\omega_0^2} [-\omega_0 t \cos \omega_0 t + \sin \omega_0 t].$$



Laplace Transform: Non-homogeneous ODEs

- Consider a non-homogeneous linear ODE with constant coefficients:

$$y'' + ay' + by = r(t).$$

- By Laplace transform, $Y(s) \equiv \mathcal{L}[y(t)]$, we have

$$Y(s) = [(s + a)y(0) + y'(0)]Q(s) + R(s)Q(s).$$

where $Q(s) = \frac{1}{s^2 + as + b}$ and $R(s) \equiv \mathcal{L}[r(t)]$.

- If $y(0) = 0$ and $y'(0) = 0$, then $Y(s) = R(s)Q(s)$, and the convolution theorem gives the solution,

$$y(t) = \int_0^t q(t - \tau)r(\tau) d\tau.$$



Laplace Transform: Non-homogeneous, Example

Example:

- Determine the response of the damped mass-spring system:

$$y'' + 3y' + 2y = r(t), \quad \text{where } r(t) = \begin{cases} 1 & ; \text{ for } 1 < t < 2 \\ 0 & ; \text{ otherwise} \end{cases}$$

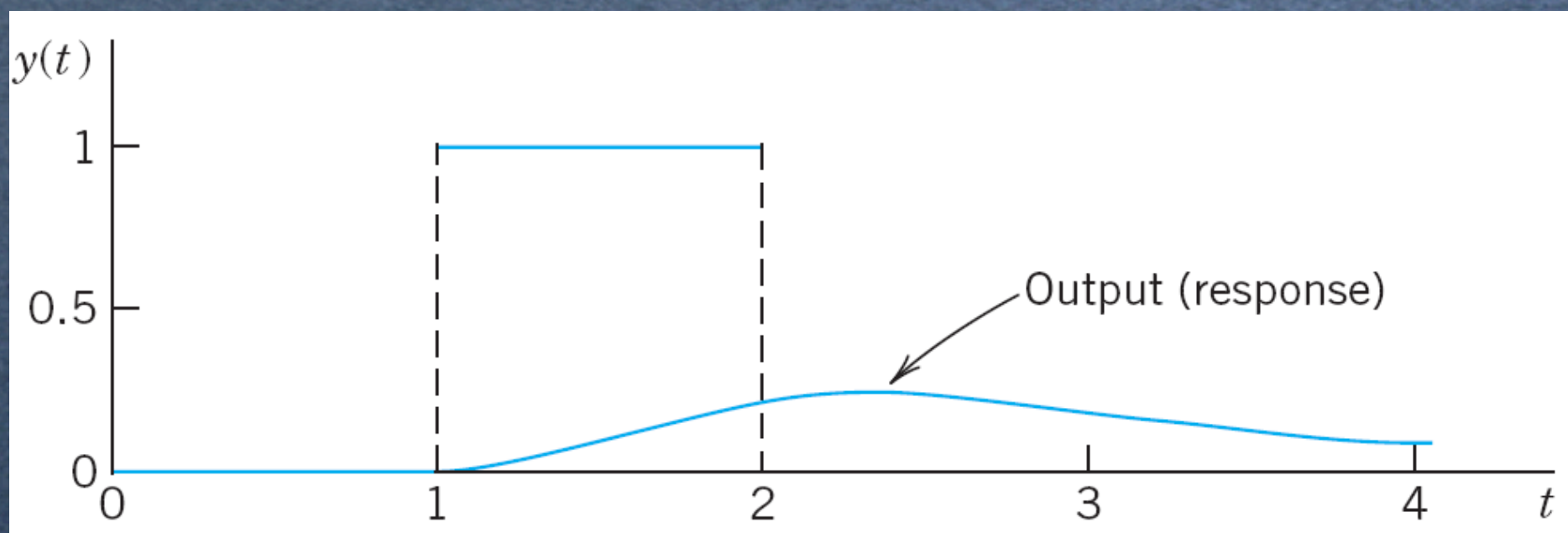
with the initial conditions $y(0) = y'(0) = 0$.



Laplace Transform: Non-homogeneous, Example

Solution:

$$y(t) = \begin{cases} 0 & ; \text{ for } t < 1 \\ \int_1^t [e^{-(t-\tau)} - e^{-2(t-\tau)}] \cdot 1 \, d\tau & ; \text{ for } 1 < t < 2 \\ \int_1^2 [e^{-(t-\tau)} - e^{-2(t-\tau)}] \cdot 1 \, d\tau & ; \text{ for } t > 2 \end{cases}$$
$$= \begin{cases} 0 & ; \text{ for } t < 1 \\ \frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)} & ; \text{ for } 1 < t < 2 \\ e^{-(t-2)} - \frac{1}{2}e^{-2(t-2)} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)} & ; \text{ for } t > 2 \end{cases}$$



Laplace Transform: Integral Equations

Example:

- Solve the Volterra integral equation of the second kind:

$$y(t) - \int_0^t y(\tau) \sin(t - \tau) d\tau = t.$$

Solution:

- The solution is

$$Y(s) = \frac{s^2 + 1}{s^4},$$

- and gives the answer

$$y(t) = t + \frac{t^3}{6}.$$



Laplace Transform: Differentiation and Integration

- If the Laplace transform for $f(t)$ is defined as $F(s) \equiv \mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t) dt$, then

$$F'(s) = - \int_0^\infty e^{-st} t f(t) dt.$$

- Differentiation of Transforms:

$$\mathcal{L}[tf(t)] = -F'(s), \quad \text{or} \quad \mathcal{L}^{-1}[F'] = -tf(t).$$

- Integration of Transforms:

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(\tilde{s}) d\tilde{s}, \quad \text{or} \quad \mathcal{L}^{-1}\left[\int_s^\infty F(\tilde{s}) d\tilde{s}\right] = \frac{f(t)}{t}.$$



Laplace Transform: Differentiation, Example

Example:

$\mathcal{L}(f)$	$f(t)$
$\frac{1}{(s^2 + \beta^2)^2}$	$\frac{1}{2\beta^3} (\sin \beta t - \beta t \cos \beta t)$
$\frac{s}{(s^2 + \beta^2)^2}$	$\frac{t}{2\beta} \sin \beta t$
$\frac{s^2}{(s^2 + \beta^2)^2}$	$\frac{1}{2\beta} (\sin \beta t + \beta t \cos \beta t)$

Hints:

- Try to find the Laplace transform of

$t \sin \beta t$, and $t \cos \beta t$.



Laplace Transform: Integration, Example

Example:

Find the inverse Laplace transform of

$$\ln\left(1 + \frac{\omega^2}{s^2}\right).$$

Hints:

- Denote that the give transform by $F(s)$, and its derivative is

$$F'(s) = \frac{2s}{s^2 + \omega^2} - \frac{2s}{s^2}.$$

- Taking the inverse Laplace transform, we obtain

$$\mathcal{L}^{-1}[F'] = -tf(t) = 2\cos\omega t - 2.$$

Solution:

$$f(t) = \frac{2(1 - \cos\omega t)}{t}.$$



Laplace Transform: ODEs with Variable Coef.

Laguerre's ODE: $ty'' + (1 - t)y' + ny = 0.$ with $n = 0, 1, 2, \dots$

Solution:

- In the s -domain, we have

$$(s - s^2) \frac{d}{ds} Y + (n + 1 - s)Y(s) = 0.$$

- By separating variables, one solve $Y(s)$

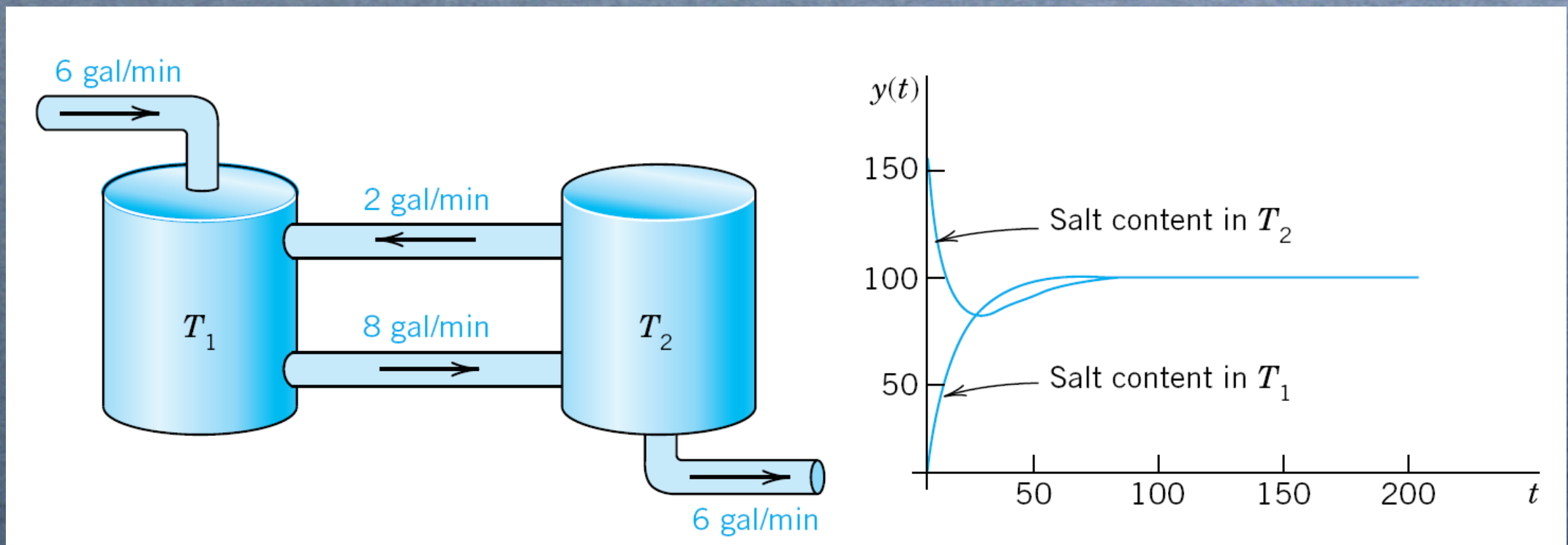
$$Y(s) = \frac{(s - 1)^n}{s^{n+1}}.$$

- By the inverse Laplace transform, through *Rodrigues' formula*, we have the **Laguerre's polynomials**,

$$I_n \equiv \mathcal{L}^{-1}[Y(s)] = \begin{cases} 1 & ; \text{ for } n = 0 \\ \frac{e^t}{n!} \frac{d^n}{dt^n} (t^n e^{-t}) & ; \text{ for } n = 1, 2, \dots \end{cases}$$



Laplace Transform: Systems of ODEs, Example 1



Example:

$$\begin{aligned} y_1' &= -\frac{8}{100}y_1 + \frac{2}{100}y_2 + y, \\ y_2' &= +\frac{8}{100}y_1 - \frac{2}{100}y_2, \end{aligned}$$

with the initial conditions $y_1(0) = 0$ and $y_2(0) = 150$.



Laplace Transform: Systems of ODEs, Example 1

- Define the corresponding Laplace transforms, and the subsidiary system becomes

Hints:

$$\begin{aligned}(-0.08 - s)Y_1 + 0.02Y_2 &= -\frac{6}{s}, \\ 0.08Y_1 + (-0.08 - s)Y_2 &= -150.\end{aligned}$$

- Solve the algebraical equations for Y_1 and Y_2 to get

Solution:

$$\begin{aligned}Y_1 &= \frac{100}{s} - \frac{62.5}{s + 0.12} - \frac{37.5}{s + 0.04}, \\ Y_2 &= \frac{100}{s} + \frac{125}{s + 0.12} - \frac{75}{s + 0.04}.\end{aligned}$$

- Take the inverse Laplace transform,

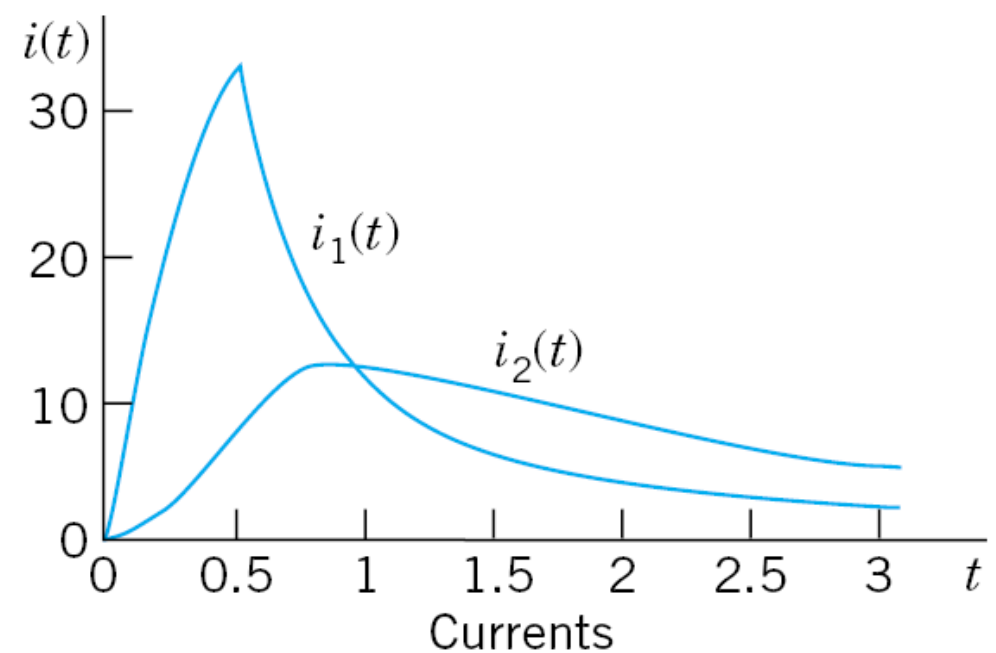
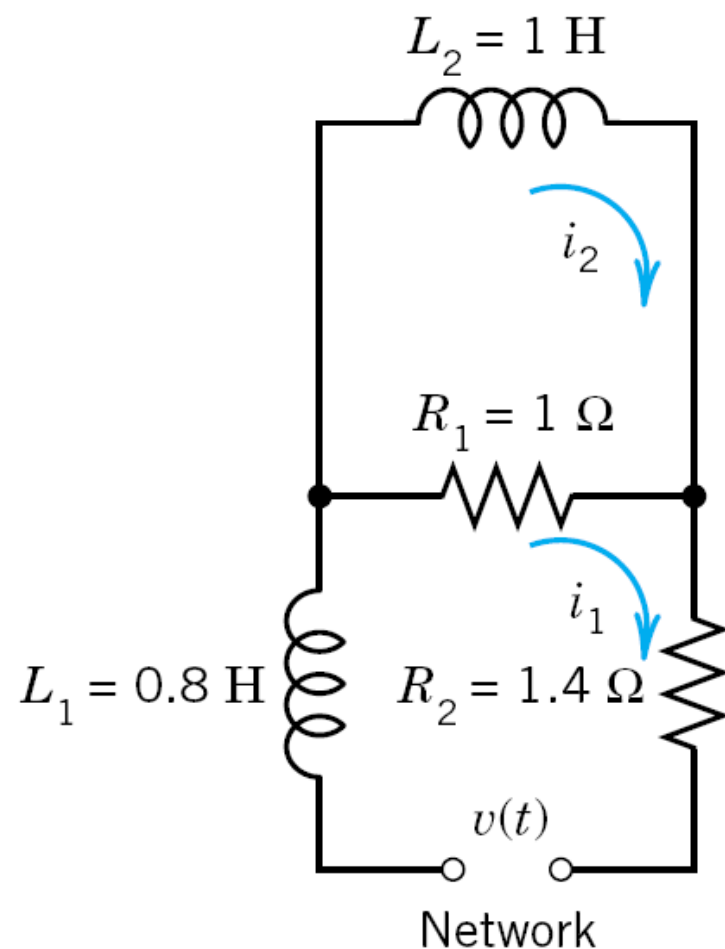
$$\begin{aligned}y_1(t) &= 100 - 62.5e^{-0.12t} - 37.5e^{-0.04t}, \\ y_2(t) &= 100 + 125e^{-0.12t} - 75e^{-0.04t}.\end{aligned}$$



Laplace Transform: Systems of ODEs, Example 2

Find the currents $i_1(t)$ and $i_2(t)$ in the electrical network, with $v(t) = 100$ volts if $0 \leq t \leq 0.5$ sec and 0 thereafter, and $i(0) = i'(0) = 0$.

Example:



$$\begin{aligned} 0.8i_1' + 1 \cdot (i_1 - i_2) + 1.4i_1 &= 100[1 - u(t - \frac{1}{2})], \\ 1 \cdot i_2' + 1 \cdot (i_2 - i_1) &= 0. \end{aligned}$$



Laplace Transform: Systems of ODEs, Example 2

Hints:

- Define the corresponding Laplace transforms, and the subsidiary system becomes

$$\begin{aligned}(s+3)I_1 - 1.25I_2 &= 125\left(\frac{1}{s} - \frac{e^{-s/2}}{s}\right), \\ -I_1 + (s+1)I_2 &= 0.\end{aligned}$$

- Solve the algebraical equations for I_1 and I_2 to get

$$\begin{aligned}I_1 &= \frac{125(s+1)}{s(s+\frac{1}{2})(s+\frac{7}{2})}(1 - e^{-s/2}), \\ I_2 &= \frac{125}{s(s+\frac{1}{2})(s+\frac{7}{2})}(1 - e^{-s/2}).\end{aligned}$$



Laplace Transform: Systems of ODEs, Example 2

Solution:

- For $0 \leq t \leq \frac{1}{2}$

$$\begin{aligned}i_1(t) &= -\frac{125}{3}e^{-t/2} - \frac{625}{21}e^{-7t/2} + \frac{500}{7}, \\i_2(t) &= -\frac{250}{3}e^{-t/2} + \frac{250}{21}e^{-7t/2} + \frac{500}{7}.\end{aligned}$$

- For $t > \frac{1}{2}$

$$\begin{aligned}i_1(t) &= -\frac{125}{3}(1 - e^{1/4})e^{-t/2} - \frac{625}{21}(1 - e^{7/4})e^{-7t/2}, \\i_2(t) &= -\frac{250}{3}(1 - e^{1/4})e^{-t/2} + \frac{250}{21}(1 - e^{7/4})e^{-7t/2}.\end{aligned}$$



Laplace Transform: Table

	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$	Sec.
27	$\frac{1}{s^4 - k^4}$	$\frac{1}{2k^3} (\sinh kt - \sin kt)$	
28	$\frac{s}{s^4 - k^4}$	$\frac{1}{2k^2} (\cosh kt - \cos kt)$	
29	$\sqrt{s-a} - \sqrt{s-b}$	$\frac{1}{2\sqrt{\pi t^3}} (e^{bt} - e^{at})$	
30	$\frac{1}{\sqrt{s+a} \sqrt{s+b}}$	$e^{-(a+b)t/2} I_0\left(\frac{a-b}{2} t\right)$	5.6
31	$\frac{1}{\sqrt{s^2 + a^2}}$	$J_0(at)$	5.5
32	$\frac{s}{(s-a)^{3/2}}$	$\frac{1}{\sqrt{\pi t}} e^{at}(1 + 2at)$	
33	$\frac{1}{(s^2 - a^2)^k} \quad (k > 0)$	$\frac{\sqrt{\pi}}{\Gamma(k)} \left(\frac{t}{2a}\right)^{k-1/2} I_{k-1/2}(at)$	5.6
34	e^{-as}/s	$u(t-a)$	6.3
35	e^{-as}	$\delta(t-a)$	6.4



Laplace Transform: Table

	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$	Sec.
36	$\frac{1}{s} e^{-k/s}$	$J_0(2\sqrt{kt})$	5.5
37	$\frac{1}{\sqrt{s}} e^{-k/s}$	$\frac{1}{\sqrt{\pi t}} \cos 2\sqrt{kt}$	
38	$\frac{1}{s^{3/2}} e^{k/s}$	$\frac{1}{\sqrt{\pi k}} \sinh 2\sqrt{kt}$	
39	$e^{-k\sqrt{s}} \quad (k > 0)$	$\frac{k}{2\sqrt{\pi t^3}} e^{-k^2/4t}$	
40	$\frac{1}{s} \ln s$	$-\ln t - \gamma \quad (\gamma \approx 0.5772)$	5.6



Laplace Transform: Table

	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$	Sec.
41	$\ln \frac{s-a}{s-b}$	$\frac{1}{t} (e^{bt} - e^{at})$	
42	$\ln \frac{s^2 + \omega^2}{s^2}$	$\frac{2}{t} (1 - \cos \omega t)$	6.6
43	$\ln \frac{s^2 - a^2}{s^2}$	$\frac{2}{t} (1 - \cosh at)$	
44	$\arctan \frac{\omega}{s}$	$\frac{1}{t} \sin \omega t$	
45	$\frac{1}{s} \operatorname{arccot} s$	$\operatorname{Si}(t)$	App. A3.1

