EE 2015

(Partial) Differential Equations and Complex Variables

Ray-Kuang Lee*

Institute of Photonics Technologies,
Department of Electrical Engineering, and Department of Physics,
National Tsing-Hua University

*rklee@ee.nthu.edu.tw

Syllabus:

★ **Course description and Introduction, 9/14**

1. Ordinary Differential Equations: 4 weeks

- **First-order ODEs, Ch. 1: 9/16, 9/21**
- **Second-order ODEs, Ch. 2: 9/23, ~~9/28~~, 9/30, 10/5, 10/7**
- **Higher-order ODEs, Ch. 3: 10/12**
- **Systems of ODES, Ch. 4: 10/14**
- **1st EXAM, 10/15 (Friday night)**

2. Transform Methods: 4 weeks

- **Laplace Transforms, Ch. 6: 10/19 - 11/11**
- **2nd EXAM, 11/12 (Friday night)**

3. Series and Complex Variables: 9 weeks

- **Power Series, Ch. 5: 11/16, 11/18**
- **Fourier Series, Ch. 11: 11/23, 11/25, 11/30, 12/2**
- **3rd EXAM, 12/3 (Friday night)**
- **PDE by Fourier Series, Ch. 12, 12/7 - 12/22**
- **4th EXAM, 12/23 (in Class)**
- **Taylor and Laurent Series, Ch. 13-16: 12/28, 12/30**
- **Complex and Residue Integrations, Ch. 16: 1/4 - 1/13**
- **5th EXAM, 1/14 (Friday night)**

Higher-order ODEs: Scope

ODEs

- 1st-order
- 2nd-order
  - ☒ Homogeneous & Linear, Ch. 2.1
  - ☒ Constant Coefficients, Ch. 2.2
  - ☐ Differential Operator Ch. 2.3
  - ☒ Mass-Spring system, Ch. 2.4
  - ☒ Euler-Cauchy Eq., Ch. 2.5
  - ☐ Existence and Uniqueness, Ch. 2.6
  - ☒ Wronskian, Ch. 2.6
  - ☒ Non-homogeneous, Ch. 2.7
  - ☒ Forced Oscillation, Ch. 2.8
  - ☒ RLC circuits, Ch. 2.9
  - ☒ Variation of Parameters, Ch. 2.10
- Higher-order
  - ☐ Linear ODEs
    - ☐ Linearly independent
    - ☐ Homogeneous Eq.
    - ☐ Non-homogeneous Eq.
    - ☐ Constant coefficients
  - ☐ Nonlienar ODEs
- Systems of ODEs

Higher-order ODEs: Homogeneous & Linear

- Homogeneous Linear ODE:

$$\boxed{y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0,}$$

- Superposition principle applied.
- General solution:

$$y(x) = c_1 y_1(x) + \cdots + c_n y_n(x),$$

- $y_1, \ldots, y_n$ are the bases (or the fundamental system), and these solutions are *linearly independent*.

Higher-order ODEs: Linearly independent

- Linear independence:

$$y_1(x), y_2(x) \ldots, y_n(x),$$

are called *linearly independent* on some interval I where they are defined if the equation

$$\boxed{k_1 y_1(x) + \cdots + k_n y_n(x) = 0, \qquad \text{on} \quad I,}$$

implies that all $k_1, \ldots, k_n$ are zero.

- These functions are called *linearly dependent* on I if this equation also holds for some $k_1, \ldots, k_n$ not all zero.

Higher-order ODEs: Wroskian

- Homogeneous Linear ODE:

$$\boxed{y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0,}$$

with continuous coefficients $p_0(x), \ldots, p_{n-1}(x)$.

- $y_1, \ldots, y_n$ on I are linearly dependent if and only if their Wronskian is zero for some $x = x_0$ in I, i.e.

$$W(y_1, \cdots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \cdot & \cdot & \cdots & \cdot \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}.$$

Higher-order ODEs: Wronskian, cont.

Example:

4th-order ODE:

$$\boxed{y^{(iv)} - 5y'' + 4y = 0,}$$

Hints:

- Substitute $y = e^{\lambda x}$.

Solution:

The general solution:

$$\boxed{y = c_1 e^{-2x} + c_2 e^{-x} + c_3 e^{x} + c_4 e^{2x}.}$$

Test of Linearly Independence:

$$W = \begin{vmatrix} e^{-2x} & e^{-x} & e^{x} & e^{2x} \\ -2e^{-2x} & -e^{-x} & e^{x} & 2e^{2x} \\ 4e^{-2x} & e^{x} & e^{x} & 4e^{x} \\ -8e^{-2x} & -e^{-2x} & e^{x} & 8e^{2x} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 1 & 2 \\ 4 & 1 & 1 & 4 \\ -8 & -1 & 1 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 4 \\ -3 & -3 & 0 \\ 7 & 9 & 16 \end{vmatrix} = 72 \neq 0?$$

Higher-order ODEs: Existence & Uniqueness

- An initial value problem for the ODE:

$$\boxed{\begin{aligned} &y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0, \\ \text{ICs:} \quad & y(x_0) = K_0, y'(x_0) = K_1, \cdots, y^{(n-1)}(x_0) = K_{n-1}, \end{aligned}}$$

  with a given x_0 in the open interval I considered.

- If the coefficients $p_0(x), \ldots, p_{n-1}(x)$ are *continuous* on some open interval I, then the **IVP** has a unique solution (a general solution).

Higher-order ODEs: Constant Coefficients

Example: nth-order homogeneous linear ODE:

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0,$$

Hints:

- Substitute $y = e^{\lambda x}$ and obtain a *characteristic equation*,

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0.$$

Solutions:

1. Distinct Real Roots: all the n roots $\lambda_1, \cdots, \lambda_n$ are real and different, then the corresponding general solution is

$$y = c_1 e^{\lambda_1 x} + \cdots + c_n e^{\lambda_n x}.$$

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Higer-order ODEs: Constant Coeff., cont.

Solutions:

2. Simple Complex Roots: if complex roots occur, they must occur in *conjugate pairs* since the coefficients of the ODE are real, i.e., $\lambda = \gamma \pm i\omega$,

$$y_1 = e^{\gamma x}\cos\omega x, \qquad y_2 = e^{\gamma x}\sin\omega x.$$

3. Multiple Real Roots: if λ is a real root of order m, then the corresponding linearly independent solutions are

$$e^{\lambda x}, \quad xe^{\lambda x}, \quad x^2e^{\lambda x}, \cdots, \quad x^{m-1}e^{\lambda x}.$$

4. Multiple Complex Roots: the real solutions are obtained as for Simple Complex Roots above,

$$e^{\gamma x}\cos\omega x, \qquad e^{\gamma x}\sin\omega x,$$
$$xe^{\gamma x}\cos\omega x, \qquad xe^{\gamma x}\sin\omega x,$$

Higher-order ODEs: Constant Coeff., Eample

Example:

$$\boxed{y^{(v)} - 3y^{(iv)} + 3y^{(3)} - y'' = 0,}$$

Hints:

1. Substitute $y = e^{\lambda x}$ and obtain its *characteristic equation*,

$$\lambda^5 - 3\lambda^4 + 3\lambda^3 - \lambda^2 = 0.$$

2. Roots: $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = \lambda_4 = \lambda_5 = 1$.

Solution:

$$\boxed{y = c_1 + c_2 x + (c_3 + c_4 x + c_5 x^2)e^x.}$$

Higher-order ODEs: Non-homogeneous

Non-homogeneous Linear ODE:

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = r(x),$$

Solution:

$$y(x) = y_h(x) + y_p(x),$$

How to find a particular solution?

- Method of Undetermined Coefficients,
- Method of Variation of Parameters.

Non-homogeneous, Method of Undetermined coefficients

Example:

$$y''' + 3y'' + 3y' + y = 30e^{-x}, \qquad y(0) = 3, y'(0) = -3, y''(0) = -47,$$

Hints:

1. Substitute $y = e^{\lambda x}$ and obtain its *characteristic equation*,

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = (\lambda + 1)^3 = 0.$$

2. Triple Roots: $\lambda = -1$.

Solution:

1. General solution: $y_h = (c_1 + c_2 x + c_3 x^2)e^{-x}$.

2. Particular solution: $y_p = 5x^3 e^{-x}$ by trying $y_p = cx^3 e^{-x}$.

3. Special solution:

$$\boxed{y = (3 - 25x^2)e^{-x} + 5x^3 e^{-x}.}$$

Non-homogeneous, Method of Variation of Parameters

For 2nd-order ODEs, the particular solution can be found by

$$y_p(x) = -y_1 \int [\frac{y_2\, r}{W}]\, \mathrm{d}x + y_2 \int [\frac{y_1\, r}{W}]\, \mathrm{d}x, \qquad \text{where} \quad W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

Extend the idea to an arbitrary order n, the *particular solution* is

$$y_p(x) = \sum_{k=1}^{n} y_k(x) \int \frac{W_k(x)}{W(x)} r(x)\, \mathrm{d}x, = y_1(x) \int \frac{W_1(x)}{W} r(x)\, \mathrm{d}x + \cdots + y_n(x) \int \frac{W_n(x)}{W} r(x)\, \mathrm{d}x,$$

where W is the Wronskian, and W_j is obtained from W (replacing the jth column by $[00\cdots 1]^T$.

Non-homogeneous, Method of Variation of Parameters, Example

Example:

Non-homogeneous Euler-Cauchy equation:

$$x^3y''' - 3x^2y'' + 6xy' - 6y = x^4 \ln x, \qquad x > 0,$$

Hints:

1. Substitute $y = x^m$ and obtain its *characteristic equation*:

$$m(m-1)(m-2) - 3m(m-1) + 6m - 6 = 0.$$

2. Roots: $m = 1, 2, 3$.

Solution:

1. General solution: $y_h = c_1x + c_2x^2 + c_3x^3$.

2. Particular solution: the corresponding determinants are $W = 2x^3$, $W_1 = x^4$, $W_2 = -2x^3$, and $W_3 = x^2$,

$$y_p = x\int \frac{x}{2} x \ln x \, \mathrm{d}x - x^2 \int x \ln x \, \mathrm{d}x + x^3 \int \frac{1}{2x} x \ln x \, \mathrm{d}x, \\ = \frac{1}{6}x^4(\ln x - \frac{11}{16}).$$

Higher-order ODEs: Summary

ODEs
- 1st-order
- 2nd-order
- Higher-order
  - ☒ Linear ODEs
    - ☒ Linearly independent
    - ☒ Homogeneous Eq., Ch. 3.1
    - ☒ Constant coefficients, Ch. 3.2
    - ☒ Non-homogeneous Eq., Ch. 3.3
      - ☒ Undetermined Coefficients
      - ☒ Variation of Parameters
    - □ Elastic Beams

      $y^{(iv)} = f(x),$
  - □ Nonlienar ODEs
- Systems of ODEs

Systems of ODEs: Scope

ODEs
- 1st-order
- 2nd-order
- Higher-order
  - ☒ Linear ODEs
    - ☒ Linearly independent
    - ☒ Homogeneous Eq., Ch. 3.1
    - ☒ Constant coefficients, Ch. 3.2
    - ☒ Non-homogeneous Eq., Ch. 3.3
      - ☒ Undetermined Coefficients
      - ☒ Variation of Parameters
    - ☐ Elastic Beams
    - $y^{(iv)} = f(x),$
  - ☐ Nonlienar ODEs
- Systems of ODEs
  - ☐ Linear ODEs
    - ☐ Matrices and Vectors
    - ☐ Models
    - ☐ Basic Theory
    - ☐ Constant-Coefficient Systems
    - ☐ Phase Plane method
    - ☐ Stability
    - ☐ Non-homogeneous Linear Sys
  - ☐ Nonlienar ODEs

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Systems of ODEs: Matrix and Vector

- A linear system of n first-order ODEs in n unknown functions $y_1(x), \cdots, y_n(x)$ is of the form:

$$
\begin{array}{lcl}
y_1' & = & a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n + r_1(x) \\
y_2' & = & a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n + r_2(x) \\
\cdots & & \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
y_n' & = & a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n + r_n(x),
\end{array}
$$

- Matrix form:

$$
\vec{Y}'(x) \equiv \begin{bmatrix} y_1' \\ y_2' \\ \cdot \\ y_n' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ y_n \end{bmatrix} + \begin{bmatrix} r_1 \\ r_2 \\ \cdot \\ r_n \end{bmatrix} \equiv \overline{\overline{M}} \cdot \vec{Y}(x) + \vec{R}(x).
$$

Systems of ODEs: Matrix operations

Example:

Take $n = 2$ as an example, $\overline{\overline{A}} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $\overline{\overline{B}} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$:

- Addition:

$$\overline{\overline{A}} + \overline{\overline{B}} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}$$

- Multiplication:

$$\overline{\overline{C}} = \overline{\overline{AB}} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

i.e. $c_{jk} = \sum_{m=1}^{n} a_{jm}b_{mk}$.

Systems of ODEs: Matrix operations, cont.

Take $n = 2$ as an example, $\overline{\overline{A}} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $\overline{\overline{B}} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$:

- Transposition:

$$\overline{\overline{A}}^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$$

- Inverse: $\overline{\overline{A}}\,\overline{\overline{A}}^{-1} = \overline{\overline{A}}^{-1}\overline{\overline{A}} = \overline{\overline{I}}$, i.e.,

$$\overline{\overline{A}}^{-1} = \frac{1}{\det(\overline{\overline{A}})} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix},$$

where

$$\det(\overline{\overline{A}}) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

Systems of ODEs: Vectors

Take $n = 3$ as an example, column vectors $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$:

- Transposition: row vector

$$\vec{v}^T = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$$

- Dot Product: Scalar-product, Inner product,

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3,$$

two vectors are orthogonal if $\vec{u} \cdot \vec{v} = 0$.

- Cross Product: Vector-product

$$\vec{u} \times \vec{v} = \det \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Systems of ODEs: Eigenvalue and Eigenvector

- Eigen equation:

$$\boxed{\overline{\overline{A}}\vec{x} = \lambda\vec{x}, \qquad \lambda \quad \text{is a scalar,}}$$

- If some vectors $\vec{x} \neq 0$ hold with a scalar λ, then $\vec{x}$ is called the eigen-vector and λ is called the eigen-value of the matrix $\overline{\overline{A}}$.
- Rewrite the Eigen equation into,

$$(\overline{\overline{A}} - \lambda\overline{\overline{I}})\vec{x} = 0,$$

- then we have a characteristic equation:

$$\det\left|\overline{\overline{A}} - \lambda\overline{\overline{I}}\right| = 0.$$

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Systems of ODEs: Eigen Problem

Example: Find the eigenvalues and eigenvectors of the matrix,

$$\overline{\overline{A}} = \begin{bmatrix} -4.0 & 4.0 \\ -1.6 & 1.2 \end{bmatrix}$$

Solution:

1. The characteristic equation is,

$$\det\left|\overline{\overline{A}} - \lambda \overline{\overline{I}}\right| = \begin{bmatrix} -4.0-\lambda & 4.0 \\ -1.6 & 1.2-\lambda \end{bmatrix} = \lambda^2 + 2.8\lambda + 1.6 = 0,$$

   with the solutions $\lambda_1 = -2$ and $\lambda_2 = -0.8$, which are the eigenvalues.

2. The corresponding eigenvectors are,

$$\vec{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad \text{for} \quad \lambda = \lambda_1 = -2;$$

$$\vec{x}_2 = \begin{bmatrix} 1 \\ 0.8 \end{bmatrix}, \qquad \text{for} \quad \lambda = \lambda_2 = -0.8.$$

Homework #7:

1. (25%) Solve the non-homogeneous ODE:

$$y''' - 4y' = x + 3\cos x + e^{-2x}. \tag{1}$$

2. (25%) Solve the non-homogeneous linear system:

$$\frac{\mathrm{d}}{\mathrm{d}t} y_1 = -2y_1 + y_2 + 2e^{-t}, \tag{2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} y_2 = -2y_2 + y_1 + 3t. \tag{3}$$

Homework #7:

1. Please do the homework Yourself !!

2. Homework is designed for your PRACTICE, Take It Easy ^.^

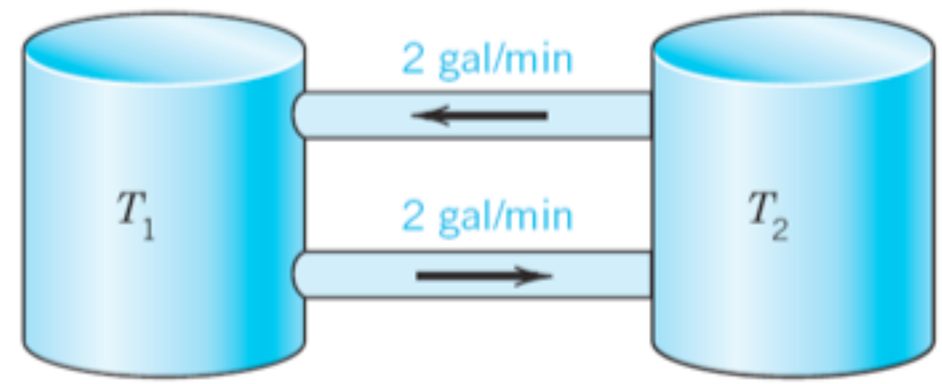
3. If you have any questions, please write an email to me or come to my office.

5. Please return the Homework by the Deadline:

Deadline Oct. 14 (this Thursday),1:00PM before the class!!

Systems of ODEs: Model

Example:



System of tanks

Mixing problem involving two tanks,

$$\begin{aligned}
&\text{Tank} \quad T_1: \quad \text{time rate of change} = -\text{Outflow/min} + \text{Inflow/min} \Rightarrow y_1' = -0.02y_1 + 0.02y_2,\\
&\text{Tank} \quad T_2: \quad \text{time rate of change} = -\text{Outflow/min} + \text{Inflow/min} \Rightarrow y_2' = +0.02y_1 - 0.02y_2,
\end{aligned}$$

Hints:

$$\boxed{y_1' = -0.02y_1 + 0.02y_2,}$$

$$\boxed{y_2' = +0.02y_1 - 0.02y_2,}$$

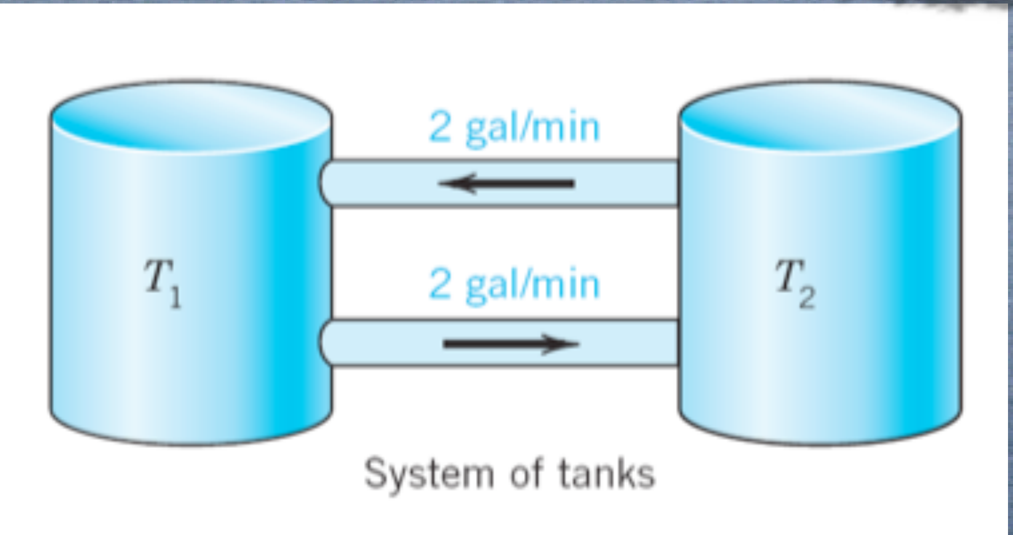
1. Matrix form:

$$\vec{y}'(t) = \overline{\overline{A}} \cdot \vec{y}(t),$$

2. Try an exponential function of t, i.e. $\vec{y} = \vec{x}e^{\lambda t}$.

Systems of ODEs: Model, cont.

$$
\boxed{\begin{aligned} y_1' &= -0.02y_1 + 0.02y_2, \\ y_2' &= +0.02y_1 - 0.02y_2, \end{aligned}}
$$



Solution:

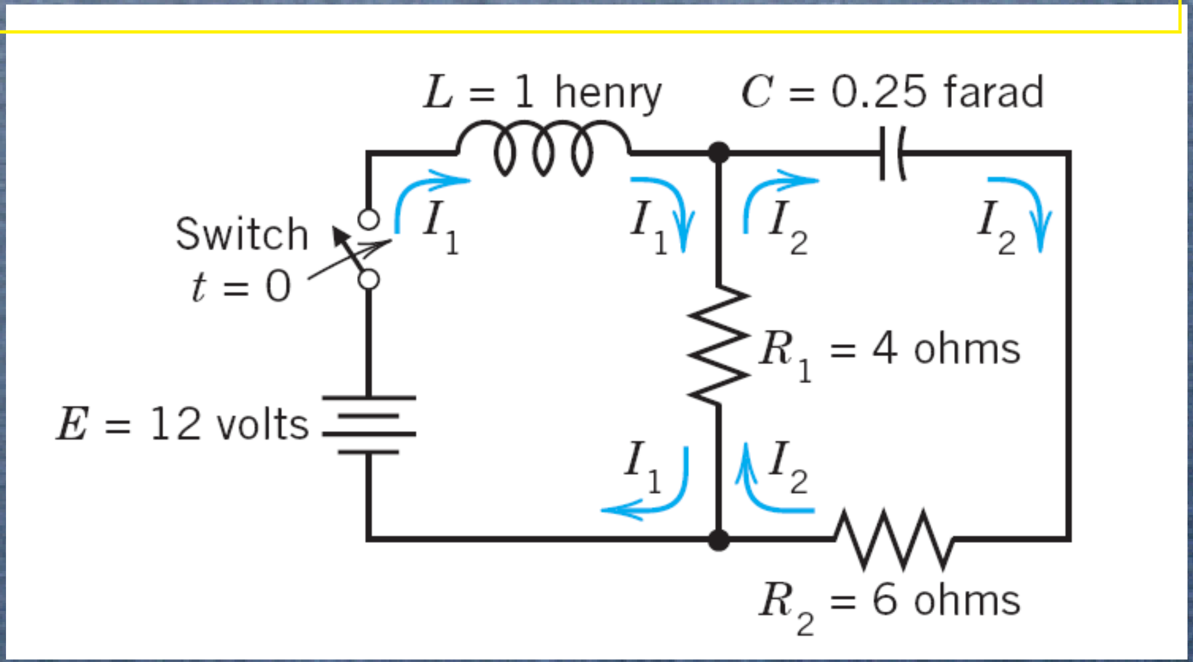
The general solutions are:

$$
\vec{y} = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = c_1\vec{x}_1 e^{\lambda_1 t} + c_2\vec{x}_2 e^{\lambda_2 t} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-0.04t}.
$$

Systems of ODEs: Electrical Network

Example: Find the currents $I_1(t)$ and $I_2(t)$ in the network,

$$\boxed{\begin{aligned} I_1' &= -4I_1 + 4I_2 + 12, & I_1(0) &= 0, \\ I_2' &= -1.6I_1 + 1.2I_2 + 4.8, & I_2(0) &= 0, \end{aligned}}$$



Hints:

1. Matrix form:

$$\vec{y}'(t) = \overline{\overline{A}} \cdot \vec{y}(t) + \vec{R},$$

2. Find the *homogeneous solution* first by trying $\vec{y}_h = \vec{x}e^{\lambda t}$.

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Systems of ODEs: Electrical Network, cont.

Find the currents $I_1(t)$ and $I_2(t)$ in the network,

Solution:

$$\boxed{\begin{aligned} I_1' &= -4I_1 + 4I_2 + 12, & I_1(0) &= 0, \\ I_2' &= -1.6I_1 + 1.2I_2 + 4.8, & I_2(0) &= 0, \end{aligned}}$$

1. The general solutions:

$$\vec{y}_h = \begin{bmatrix} I_1(t) \\ I_2(t) \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 0.8 \end{bmatrix} e^{-0.8t},$$

2. The particular solutions:

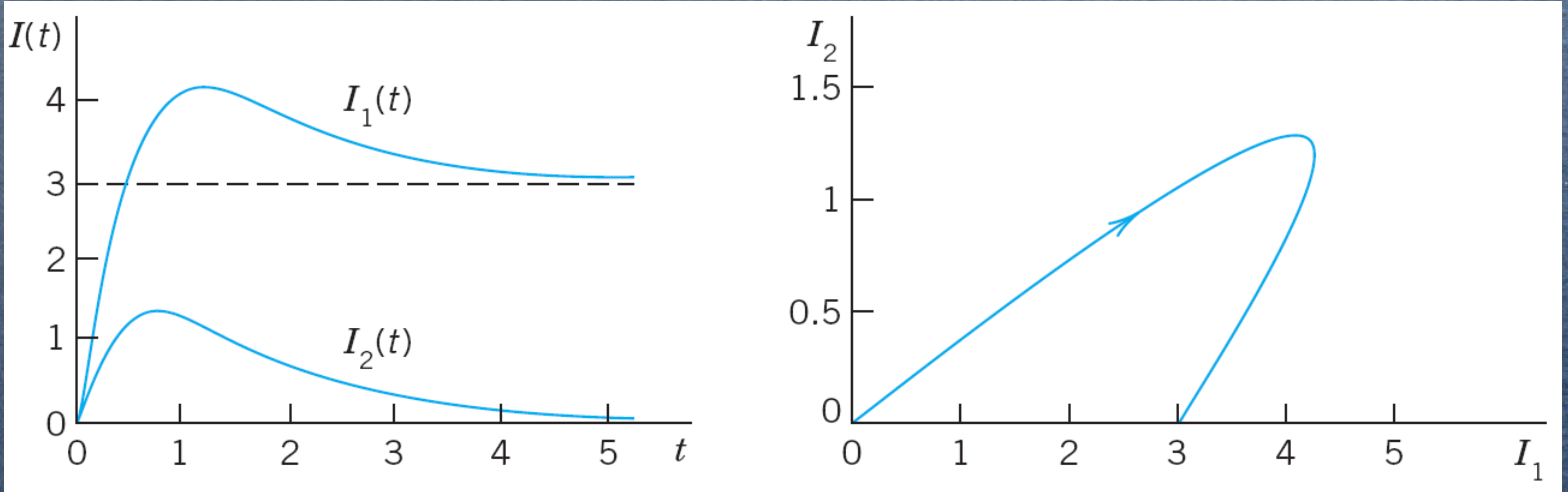
$$\vec{y}_p = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

3. with the initial condition: $c_1 = -4$ and $c_2 = 5$,

$$\vec{y} = -4 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-2t} + 5 \begin{bmatrix} 1 \\ 0.8 \end{bmatrix} e^{-0.8t} + \begin{bmatrix} 3 \\ 0 \end{bmatrix}.$$

Systems of ODEs: Parameter representation

- Show the two separate curves $I_1(t)$ and $I_2(t)$ as a single curve (trajectory) in the $I_1 - I_2$ plane, i.e. phase plane.



(a) Currents I_1 (upper curve) and I_2

(b) Trajectory $[I_1(t), I_2(t)]^T$ in the I_1I_2-plane (the "phase plane")

Reduce a *n*th-order ODE into a system

- An nth-order ODE:

$$\boxed{y^{(n)} = F(t, y, y', \cdots, y^{(n-1)}),}$$

can be converted to a system of n first-order ODEs by setting

$$y_1 = y, \quad y_2 = y', \quad y_3 = y'', \cdots, y_n = y^{(n-1)}.$$

- This system for an nth-order ODE is

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= y_3, \\ &\cdots \\ y_n' &= F(t, y_1, y_2, \cdots, y_n). \end{aligned}$$

Reduce a *n*th-order ODE into a system, Example

Example: Mass on a Spring: $my'' + cy' + ky = 0$

Hints:

1. Convert into a system:

$$\vec{y} = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

2. The characteristic equation with $\vec{y}(t) = \vec{x}e^{\lambda t}$ is

$$\det(\overline{\overline{A}} - \lambda \overline{\overline{I}}) = \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0,$$

with the eigenvalues $\lambda_1 = -0.5$ and $\lambda_2 = -1.5$.

3. The general solution for the system is,

$$\vec{y} = c_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{-0.5t} + c_2 \begin{bmatrix} 1 \\ -1.5 \end{bmatrix} e^{-1.5t}$$

Solution: $y = y_1 = 2c_1e^{-0.5t} + c_2e^{-1.5t}.$

Systems of ODEs: Basic theory

- First-order systems of ODEs:

$$\begin{aligned} y_1' &= f_1(t, y_1, \cdots, y_n), \\ y_2' &= f_2(t, y_1, \cdots, y_n), \\ &\cdots \\ y_n' &= f_n(t, y_1, \cdots, y_n). \end{aligned}$$

- with the initial conditions:

$$y_1(t_0) = K_1, \quad y_2(t_0) = K_2, \quad \cdots \quad y_n(t_0) = K_n.$$

- Existence and Uniqueness Theorem:
Let $f_1, \cdots, f_n$ be *continuous function* having continuous partial derivatives

$$\partial f_1/\partial y_1, \cdots, \partial f_1/\partial y_n, \cdots \partial f_n/\partial y_n,$$

in some domain R of $(t, y_1, \cdots, y_n)$-space containing the points $t_0, K_1, \cdots, K_n$.

- Then there exist a unique solution on some interval $t_0 - \alpha < t < t_0 + \alpha$.

Systems of ODEs: Summary

ODEs
- 1st-order
- 2nd-order
- Higher-order
- Systems of ODEs
  - ☒ Linear ODEs
    - ☒ Matrices and Vectors, Ch. 4.0
    - ☒ Models, Ch. 4.1
    - ☒ Basic Theory, Ch. 4.2
    - ☒ Constant-Coefficient Systems, Ch. 4.3
    - □ Phase Plane method, Ch. 4.3
    - □ Stability, Ch. 4.4
    - ☒ Non-homogeneous Linear Systems, Ch. 4.6
  - □ Nonlienar ODEs, Ch. 4.5

1st EXAM:

1. Topics cover "Ordinary Differential Equations":
   - First-order ODEs: Ch. 1.1, ~~1.2~~, 1.3, 1.4, 1.5, ~~1.6~~, ~~1.7~~
   - Second-order ODEs, Ch. 2.1, 2.2, 2.3(Partial), 2.4, 2.5, 2.6 (Partial), 2.6, 2.7, 2.8, 2.9, 2.10
   - Higher-order ODEs, Ch. 3.1, 3.2, 3.3(Partial)
   - Systems of ODES, Ch. 4.0, 4.1, 4.2, 4.3(Partial), ~~4.4~~, ~~4.5~~, 4.6 (partial)
2. You can bring an ONE-PAGE NOTE.

Deadline Oct. 15 (this Friday),7:00PM at Room105!!