# EE 2015
# (Partial) Differential Equations and Complex Variables


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# Syllabus:

★ Course description and Introduction, 9/14

1. Ordinary Differential Equations: 4 weeks
   - First-order ODEs, Ch. 1: 9/16, 9/21
   - Second-order ODEs, Ch. 2: 9/23, ~~9/28~~, 9/30, 10/5, 10/7
   - Higher-order ODEs, Ch. 3: 10/12
   - Systems of ODES, Ch. 4: 10/14
   - 1st EXAM, 10/15 (Friday night)
2. Transform Methods: 3+1 weeks
   - Laplace Transforms, Ch. 6: 10/19 - 11/11
   - 2nd EXAM, 11/12 (Friday night)
3. Series and Complex Variables: 3+3+3 weeks
   - Power Series, Ch. 5: 11/16, 11/18, 11/23
   - Fourier Series, Ch. 11: 11/25, 11/30, 12/2
   - 3rd EXAM, 12/3 (Friday night)
   - PDE by Fourier Series, Ch. 12, 12/7 - 12/22
   - 4th EXAM, 12/23 (in Class)
   - Taylor and Laurent Series, Ch. 13-16: 12/28, 12/30
   - Complex and Residue Integrations, Ch. 16: 1/4 - 1/13
   - 5th EXAM, 1/14 (Friday night)

# Partial Differential Equations: PDE

ODEs

- 1st-order, Ch. 1
- 2nd-order, Ch. 2
- Higher-order, Ch. 3
- Systems of ODEs, Ch. 4
- Integral Transform
  - ☒ Laplace Trans., Ch. 6
  - ☐ Fourier Trans., Ch. 11
  - ☐ $Z$ – Trans.
- Series Solutions
  - ☒ Power Series, Ch. 5
  - ☒ Fourier Series, Ch. 11
  - ☐ Taylor Series, Ch. 15
  - ☐ Laurent Series, Ch. 16

PDEs

# Terminologies:

ONE independent variable

$$y'(x) \equiv \frac{\mathrm{d}}{\mathrm{d}x} y(x)$$

Ordinary Differential Equation, ODE

Ch. 1 - 5, 11

More than ONE independent variable

$$f_x \equiv \frac{\partial}{\partial x} f(x, y, \ldots)$$

Partial Differential Equation, PDE

Ch. 12

# PDEs: Definition

- An equation containing partial derivative(s) of an unknown function $u$ with two or more independent variables. E.g.

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial u(t,x)}{\partial x^2},$$

or written in short $u_t = u_{xx}$.

## Why PDEs ?

•People sense the real world via four (or multiple) dimensions (x, y, z, t), therefore, physical quantities (e.g. electrical field, temperature, electron distribution in an atom) are fully described by four variables.

•Electrostatics (Poisson theory),
•EM waves (Maxwell's equations),
•quantum mechanics (Schrodinger's equation),
•heat transfer (heat equation).

# PDEs: Classification

1. Order of PDE: the order of the *highest partial derivative*. E.g.

$$u_t = u_{xx}, \qquad \text{(2nd order)};$$
$$u_t = u\, u_{xxx} + \text{Sin}\, x, \qquad \text{(third order)}.$$

2. Number of variables: the number of independent variables. E.g.

$$u_t = u_{xx}, \qquad \text{(2nd order, two variables: } x \text{ and } t);$$
$$u_t = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}, \qquad \text{(2nd order, three variables: } r,\ \theta, \text{ and } t);$$

3. Linearity: PDEs are either *linear* or *nonlinear*,

- nonlinear ODE: e.g. time-independent nonlinear Schrödinger equation,

$$\frac{-1}{2m}\frac{\mathrm{d}^2}{\mathrm{d}\,x^2}\Psi(x) + V(x)\Psi(x) + |\Psi(x)|^2\Psi(x) = 0,$$

# PDEs: Classification

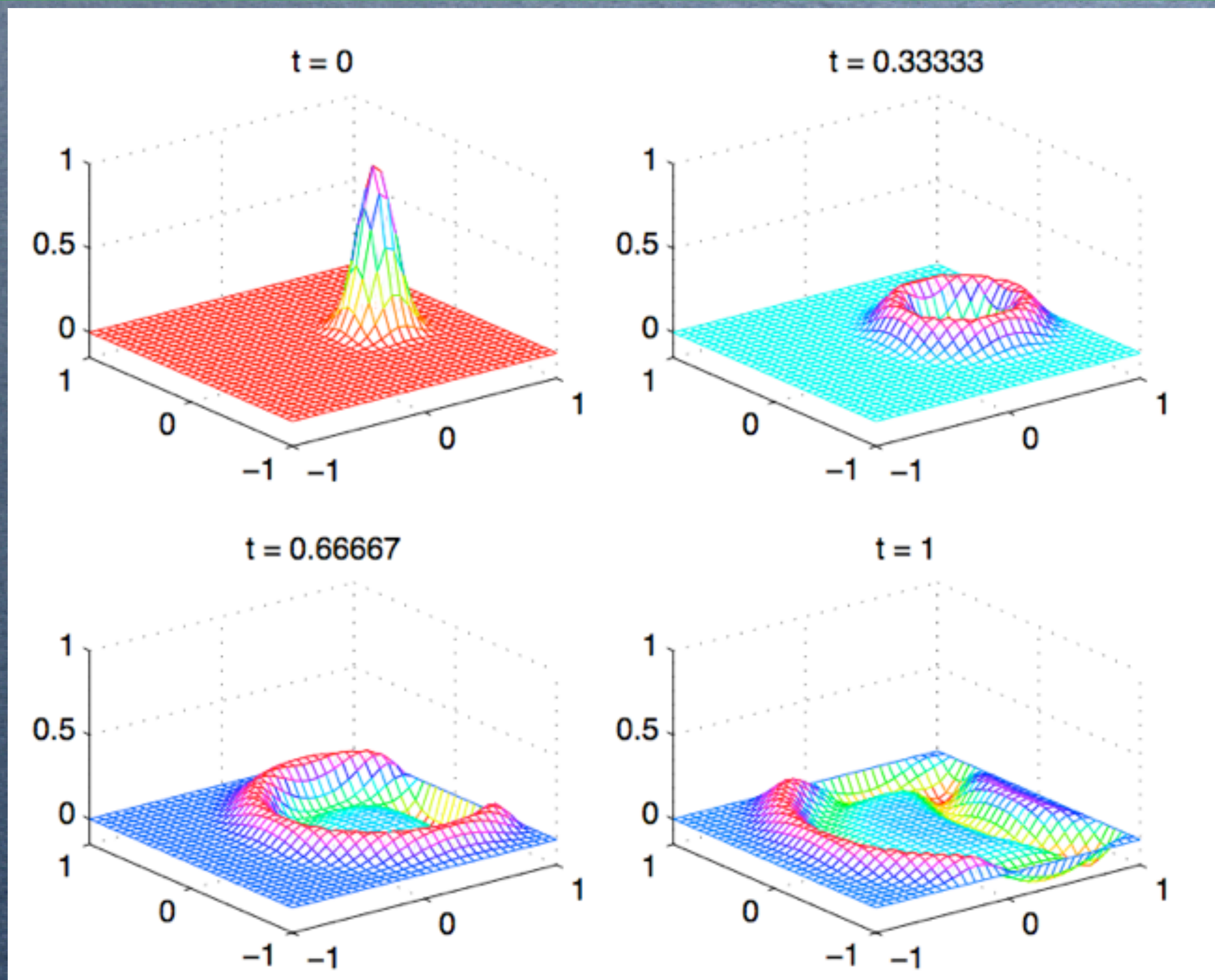
4. Homogeneity: an equation only containing unknown function $u$ and its derivative(s) is homogeneous. E.g.

$$\begin{aligned} u_t &= u_{xx}, && \text{(homogeneous);} \\ u_x + x\,u_y &= e^x u, && \text{(homogeneous);} \\ u_x + x\,u_y &= e^x, && \text{(non-homogeneous).} \end{aligned}$$

5. Kinds of Coefficients: if the coefficients $a_{i,j}(x, y)$ are constants, then the PDE is said to have *constant coefficients* (otherwise, variable coefficients).

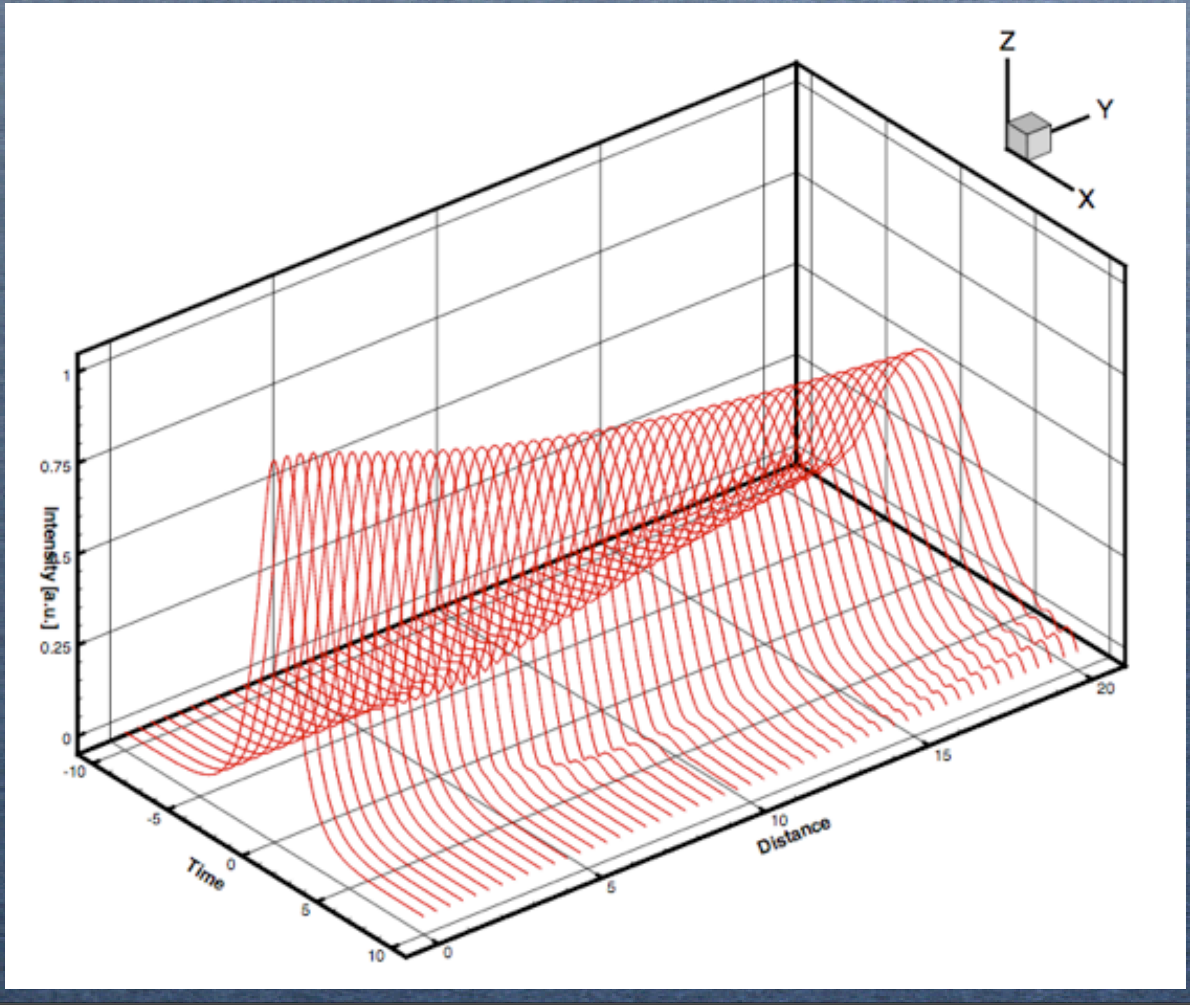
# PDEs: Example, Diffusion

$$u_t = u_{xx} + u_{yy}, \qquad -1 < x, y < 1, \quad t > 0, \quad u = 0 \quad \text{on the boundary}$$



# PDEs: Example, Wave propagation

$$\frac{\partial}{\partial z}U(z,t) = \frac{i}{2}\frac{\partial^2}{\partial t^2}U(z,t)$$



# Maxwell’s equations :


(1831-1879)

- Gauss’s law for the electric field:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \Longleftrightarrow \oint_S \mathbf{E} \cdot \mathrm{d}A = \frac{q}{\epsilon_0},$$

- Gauss’s law for magnetism:

$$\nabla \cdot \mathbf{B} = 0 \Longleftrightarrow \oint_S \mathbf{B} \cdot \mathrm{d}A = 0,$$

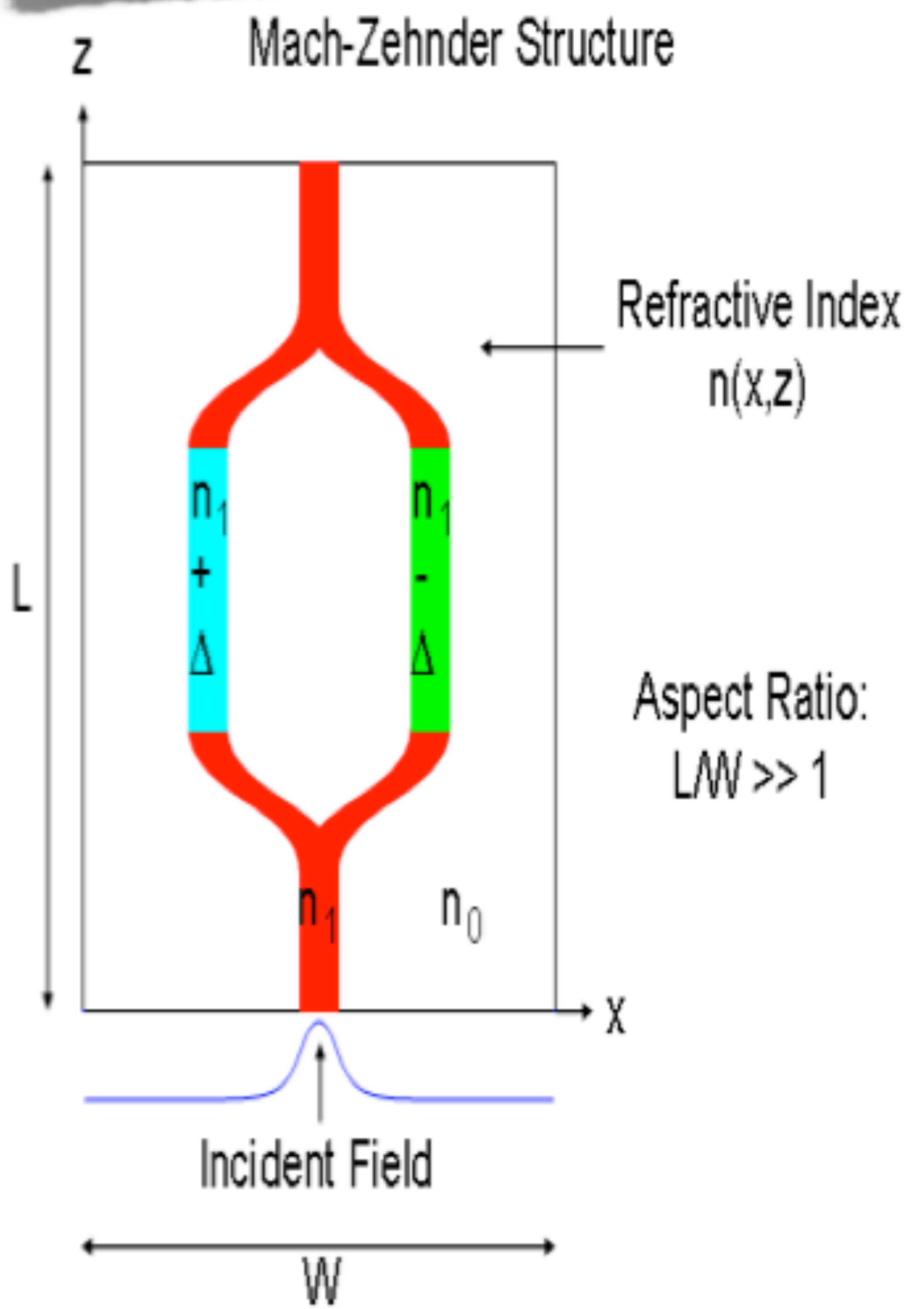
- Faraday’s law of induction:

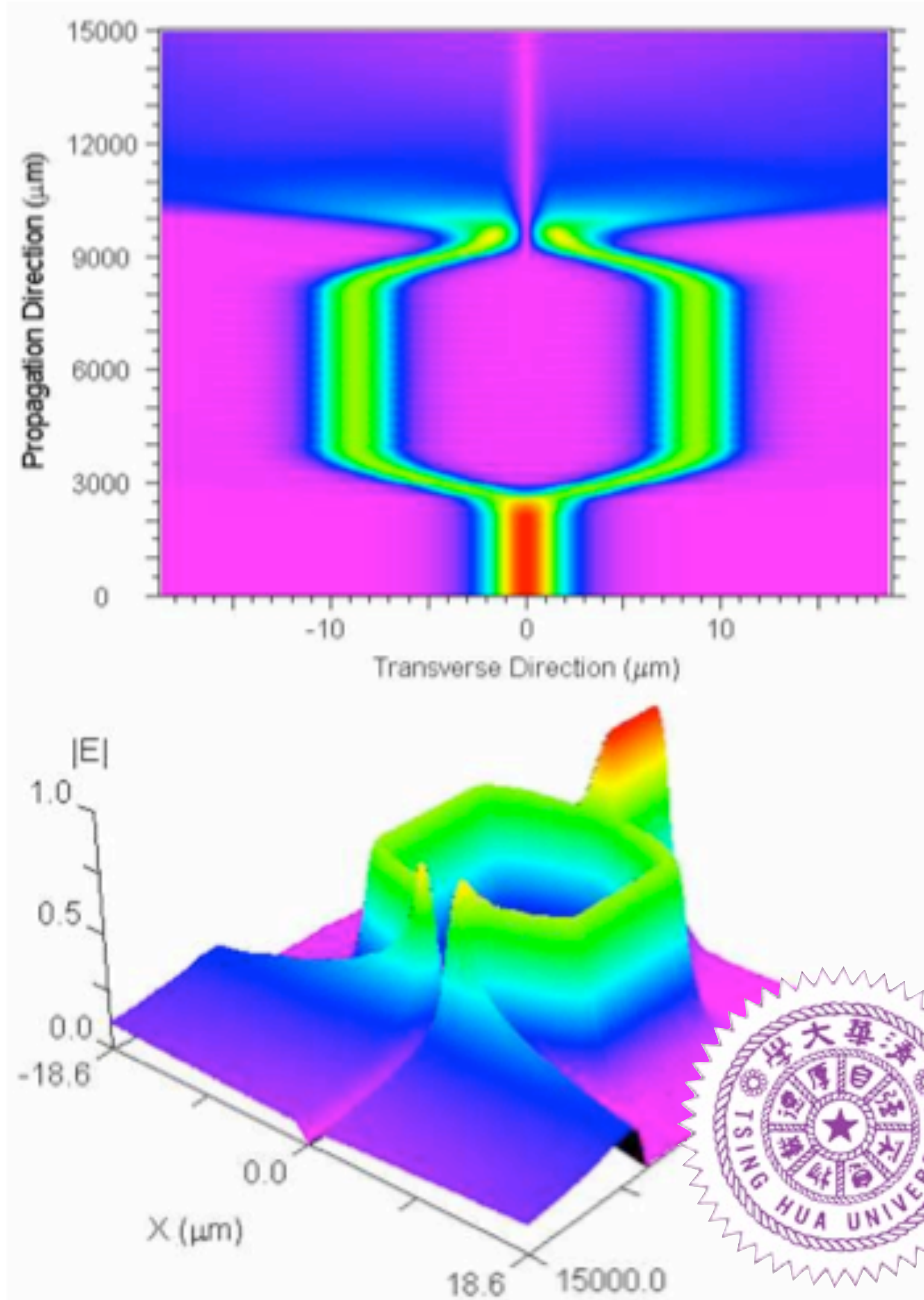
$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B} \Longleftrightarrow \oint_C \mathbf{E} \cdot \mathrm{d}l = -\frac{\partial}{\partial t}\Phi_B,$$

- Ampére’s circuital law:

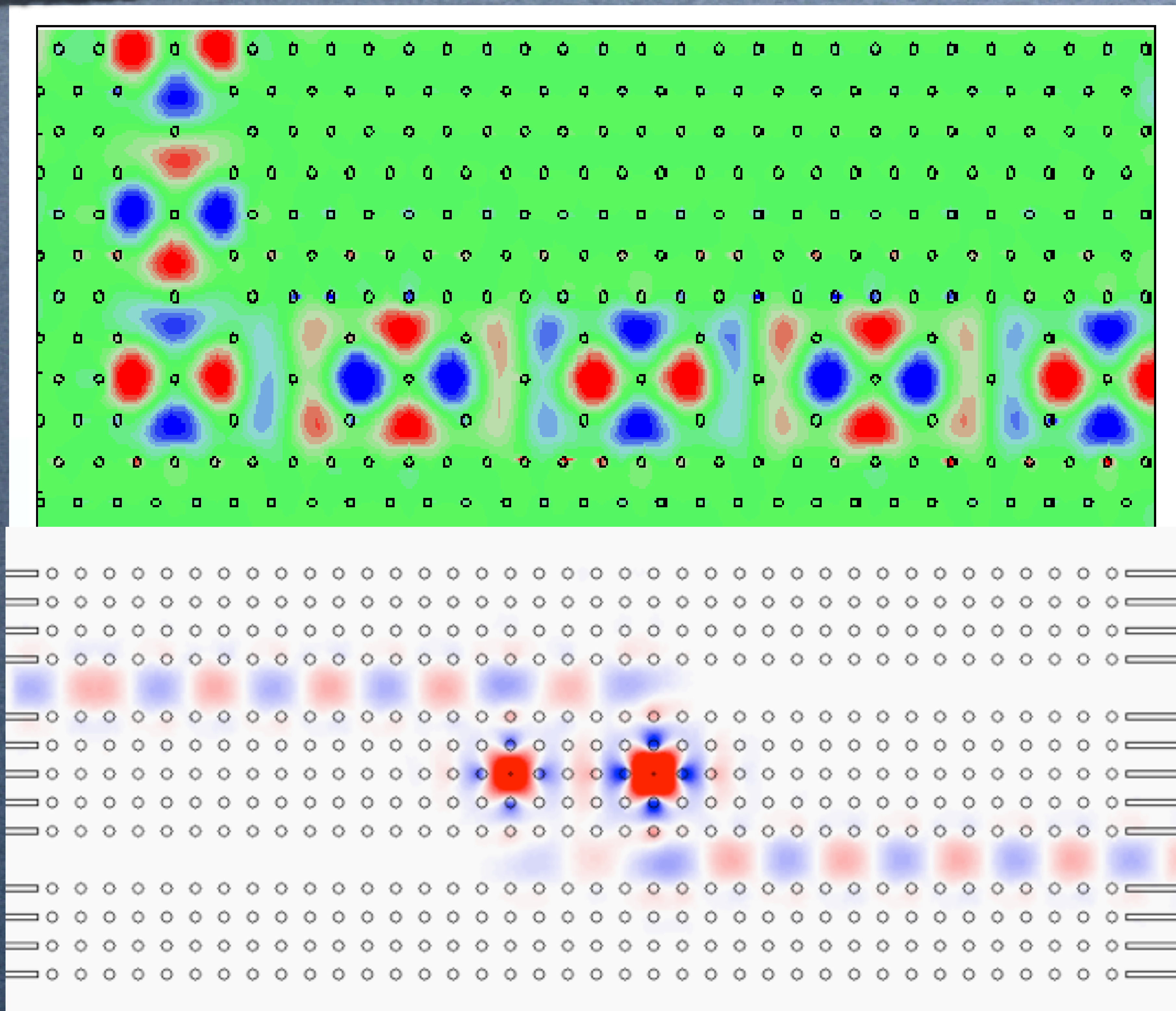
$$\nabla \times \mathbf{B} = \mu_0(\mathbf{J} + \frac{\partial}{\partial t}\mathbf{D}) \Longleftrightarrow \oint_C \mathbf{B} \cdot \mathrm{d}l = -\mu_0(\mathbf{I} + \frac{\partial}{\partial t}\Phi_D)$$

# PDEs: Example, Mach-Zehnder Interferometer



# Photonic Crystal Waveguides



# PDEs: For this course, again

1. Formulate the PDE from the physical problem,
   (constructing the mathematical model.)
   *You should know what you are working for first (physical picture).*

2. Solve the PDE,
   (along with initial and boundary conditions.)
   *The techniques you learn to attack the problems (tools at hand).*

# Engineering (Applied) Mathematics

Modeling → Solving → Interpretation

- **Analytical approach**
- **Numerical approach**

# PDEs: Methods to solve PDEs

1. Separation of Variables: reduce a PDE in $n$ variables to $n$ ODEs.
2. Integral Transforms: reduce a PDE in $n$ variables to one in $n-1$ variables.
3. Change of Coordinates: change the original PDE to an ODE or else another PDE (an easier one).
4. Transformation of the Dependent Variable: transform the unknown of a PDE into a new unknown that is easier to find.
5. Eigenfunction Expansion: find the solutions of a PDE as an infinite sum of *eigenfunctions*.

# PDEs: Methods to solve PDEs, cont.

6. Impulse-response Methods (Green's function): decompose initial and boundary conditions of the problem into *simple impulse* and finds the response to each impulse.

7. Integral Equations: changes a PDE to an *integral equation* where the unknown is inside the integral.

8. Calculus of Variations Methods: reformulate the equation as a *minimization problem*.

9. Numerical Methods: change a PDE to a system of *difference equations* on a computer.

10. Perturbation Methods: change a *nonlinear* problem into a sequence of linear ones that approximates the nonlinear one.

11. other methods $\cdots$

# PDEs: Scope

ODEs
- 1st, 2nd, higher-orders, and Systems, Ch. 1-4
- Integral Transform
  - ☒ Laplace Trans., Ch. 6
  - ☐ Fourier Trans., Ch. 11
- Series Solutions
  - ☒ Power Series, Ch. 5
  - ☒ Fourier Series, Ch. 11

PDEs
- ☒ Basic concepts, Ch. 12.1
- ☐ Modeling, Ch. 12.2, Ch. 12.7
- ☒ Separating Variables, Ch. 12.3
- ☒ Solved by Fourier Series, Ch. 12.5
- ☒ Heat Eq., Ch. 12.5
- ☐ D'Alember's solutions for Wave Eq., Ch. 12.4
- ☐ Solved by Fourier Integrals, Ch. 12.6
- ☒ 2D Cartesian, Ch. 12.8
- ☐ Laplace Eq., Ch. 12.9
- ☐ Cylindrical Coordinates, Ch. 12.10
- ☐ Spherical Coordinates, Ch. 12.10
- ☒ Solved by Laplace Transforms, Ch. 12.11

# PDEs: Big-three PDEs, 2nd-order and linear

Second-order linear PDE with two variables:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + F\,u(x,y) = G,$$

where $A$, $B$, $C$, $D$, $E$, $F$, and $G$ can be *constants* or given *functions* of $x$ and $y$.

1. Parabolic PDEs: $B^2 - 4AC = 0$, describe heat flow and diffusion processes, i.e.

$$\frac{\partial}{\partial t}u = \alpha^2 \nabla^2 u = \alpha^2 (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})u,$$

2. Hyperbolic: $B^2 - 4AC > 0$, describe vibrating systems and wave motion, i.e.

$$\nabla^2 E(x,y,z,t) - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} E = 0$$

3. Elliptic: $B^2 - 4AC < 0$, describe *steady-state phenomena*, i.e. eigenmodes of Laplacian equations,

$$[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}]u(x,y) = f(x,y).$$

# PDEs: First-order PDEs

For a first order PDE:

$$\frac{\partial}{\partial z}U(z,t) - \frac{1}{v_g}\frac{\partial}{\partial t}U(z,t) = 0,$$

**Hint:**

- Define the new coordinates:

$$\eta \equiv z + v_g t, \qquad \tau \equiv t.$$

- Use:

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial \eta}, \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} + v_g \frac{\partial}{\partial \eta}.$$

**Solution:**

$$-\frac{1}{v_g}\frac{\partial}{\partial \tau}U(\eta,\tau) = 0.$$

# PDEs: Big-three PDEs, canonical form

Second-order linear PDE with two variables:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + F\,u(x,y) = G,$$

- Define new coordinates, $\xi = \xi(x,y)$ and $\eta = \eta(x,y)$
- Substitute into the original PDE,

$$\boxed{\bar{A}\,u_{\xi\xi} + \bar{B}\,u_{\xi\eta} + \bar{C}\,u_{\eta\eta} + \bar{D}\,u_{\xi} + \bar{E}\,u_{\eta} + F\,u(\xi,\eta) + G = 0,}$$

where

$$\begin{aligned}
\bar{A} &= A\,\xi_x^2 + B\,\xi_x\xi_y + C\,\xi_y^2 \\
\bar{B} &= 2A\,\xi_x\eta_x + B\,(\xi_x\eta_y + \xi_y\eta_x) + C\,\xi_y\eta_y \\
\bar{C} &= A\,\eta_x^2 + B\,\eta_x\eta_y + C\,\eta_y^2 \\
\bar{D} &= A\,\xi_{xx} + B\,\xi_{xy} + C\,\xi_{yy} + D\,\xi_x + E\,\xi_y \\
\bar{E} &= A\,\eta_{xx} + B\,\eta_{xy} + C\,\eta_{yy} + D\,\eta_x + E\,\eta_y
\end{aligned}$$

# PDEs: Big-three PDEs, canonical form, cont.

- Set the coefficients $\bar{A}$ and $\bar{C}$ equal to zero,

$$\begin{aligned}\bar{A} &= A\,\xi_x^2 + B\,\xi_x\xi_y + C\,\xi_y^2 = 0\\ \bar{C} &= A\,\eta_x^2 + B\,\eta_x\eta_y + C\,\eta_y^2 = 0\end{aligned}$$

or rewrite these two equations in the form

$$A\,[\xi_x/\xi_y]^2 + B\,[\xi_x/\xi_y] + C = 0$$
$$A\,[\eta_x/\eta_y]^2 + B\,[\eta_x/\eta_y] + C = 0$$

- Solving these equations for $[\xi_x/\xi_y]$ and $[\eta_x/\eta_y]$, we have two *characteristic equations*

$$[\xi_x/\xi_y] = \frac{-B + \sqrt{B^2 - 4AC}}{2A},$$

$$[\eta_x/\eta_y] = \frac{-B - \sqrt{B^2 - 4AC}}{2A},$$

# PDEs: Big-three PDEs, canonical form, Example

- For example,

$$u_{xx} - 4\,u_{yy} + u_x = 0, \qquad B^2 - 4AC > 0$$

- the characteristic equations are

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = -[\xi_x/\xi_y] = -2,$$

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = -[\eta_x/\eta_y] = 2,$$

- To find $\xi$ and $\eta$, we have

$$\begin{aligned} y &= -2x + c_1, \qquad c_1 = y + 2x \equiv \xi, \\ y &= 2x + c_2, \qquad c_1 = y - 2x \equiv \eta, \end{aligned}$$

# PDEs: Big-three PDEs, canonical form, cont.

- Hyperbolic equation: $B^2 - 4AC > 0$

$$u_{\xi\eta} = \Phi(\xi, \eta, u, u_\xi, u_\eta)$$

- By the transformation, $\alpha = \xi + \eta$ and $\beta = \xi - \eta$,

$$u_{\alpha\alpha} - u_{\beta\beta} = \Phi(\alpha, \beta, u, u_\alpha, u_\beta)$$

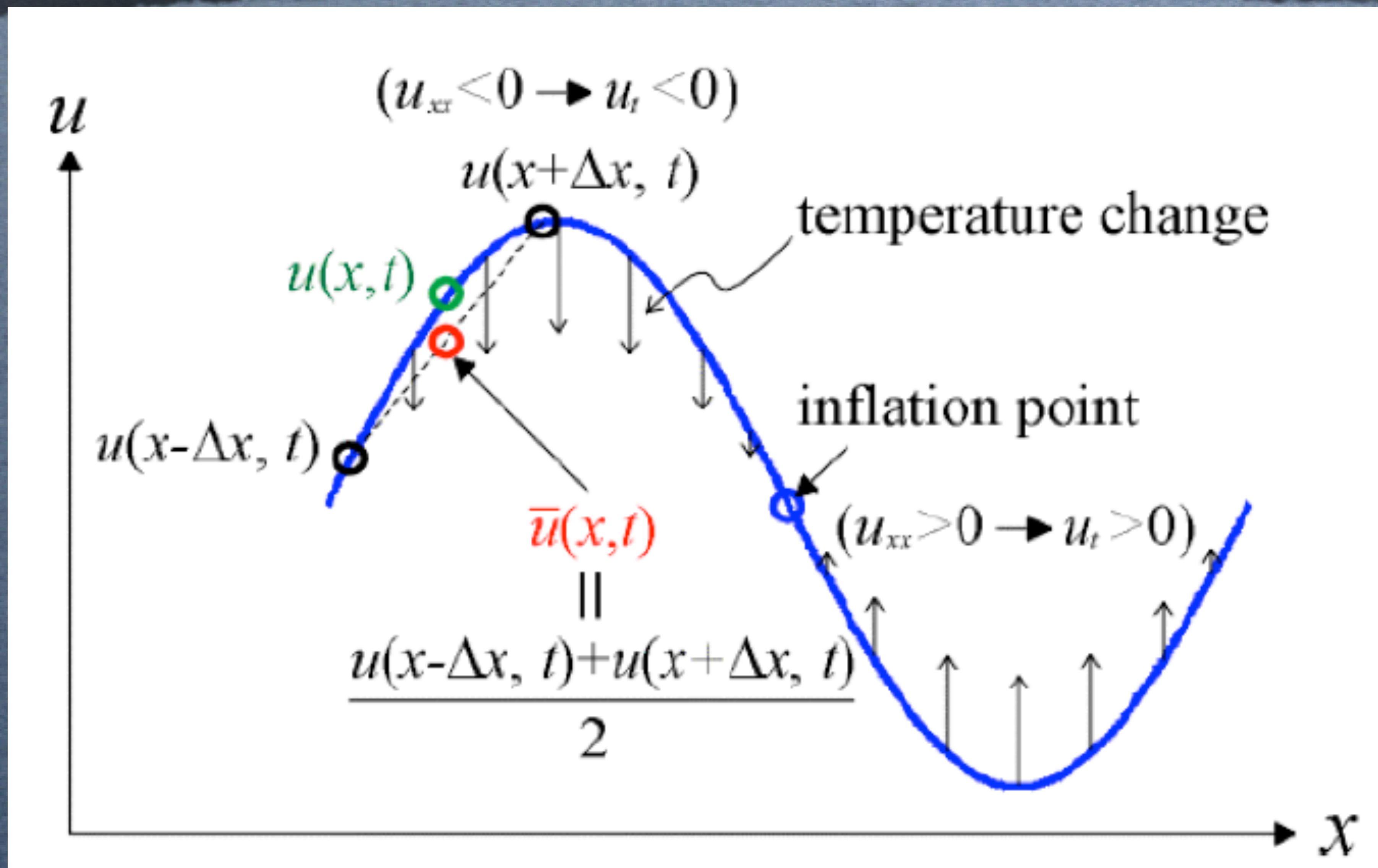
- Parabolic equation: $B^2 - 4AC = 0$

$$u_{\eta\eta} = \Phi(\xi, \eta, u, u_\xi, u_\eta)$$

- Elliptic equation: $B^2 - 4AC < 0$, with the transformation, $\alpha = (\xi + \eta)/2$ and $\beta = (\xi - \eta)/2i$,

$$u_{\alpha\alpha} + u_{\beta\beta} = \Phi(\alpha, \beta, u, u_\alpha, u_\beta)$$

# PDEs: Heat Equation, 1D, Ch. 12.5




- The heat equation means that the time-rate of change in temperature ($u_t$) is proportional to the concavity ($u_{xx}$) of the temperature distribution, i.e.

$$u_t \propto -[u(x,t) - \bar{u}(x,t)] \approx \frac{-1}{\Delta x^2}[u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)].$$

# PDEs: Finite-Difference Approximation

$$
\begin{aligned}
u(x_{j+1}) &= u(x_j) + u'(x_j)\Delta x + \frac{u''(x_j)}{2}(\Delta x)^2 + \frac{u'''(x_j)}{3!}(\Delta x)^3 + \dots, \\
u(x_{j-1}) &= u(x_j) - u'(x_j)\Delta x + \frac{u''(x_j)}{2}(\Delta x)^2 - \frac{u'''(x_j)}{3!}(\Delta x)^3 + \dots,
\end{aligned}
$$

one can approximate $u'(x_j)$ and $u''(x_j)$ by

$$
\begin{aligned}
u'(x_j) &= \frac{u(x_{j+1}) - u(x_{j-1})}{2\Delta x} - \frac{u'''(x_j)}{2 * 3!}(\Delta x)^2 + \dots, \\
&\approx \frac{u(x_{j+1}) - u(x_{j-1})}{2\Delta x} + \mathbf{O}(\Delta x^2), \\
u''(x_j) &= \frac{u(x_{j+1}) + u(x_{j-1}) - 2u(x_j)}{\Delta x^2} - 2\frac{u''''(x_j)}{4!}(\Delta x)^2 + \dots, \\
&\approx \frac{u(x_{j+1}) + u(x_{j-1}) - 2u(x_j)}{\Delta x^2} + \mathbf{O}(\Delta x^2),
\end{aligned}
$$

# PDEs: Heat Equation, 1D

$$u_t = \alpha^2 u_{xx},$$

- Heat always flows from high-temperature to low-temperature regions.
- The flow rate is proportional to the temperature gradient, i.e. $\nabla_t u$.
  - If the temperature at $x$ is higher than its surrounding average,

    $$u > \bar{u}, \quad \text{then} \quad u_{xx} < 0 \quad \text{and} \quad u_t < 0,$$

    the net flow of heat into $x$ is negative.
  - If the temperature at $x$ is lower than its surrounding average,

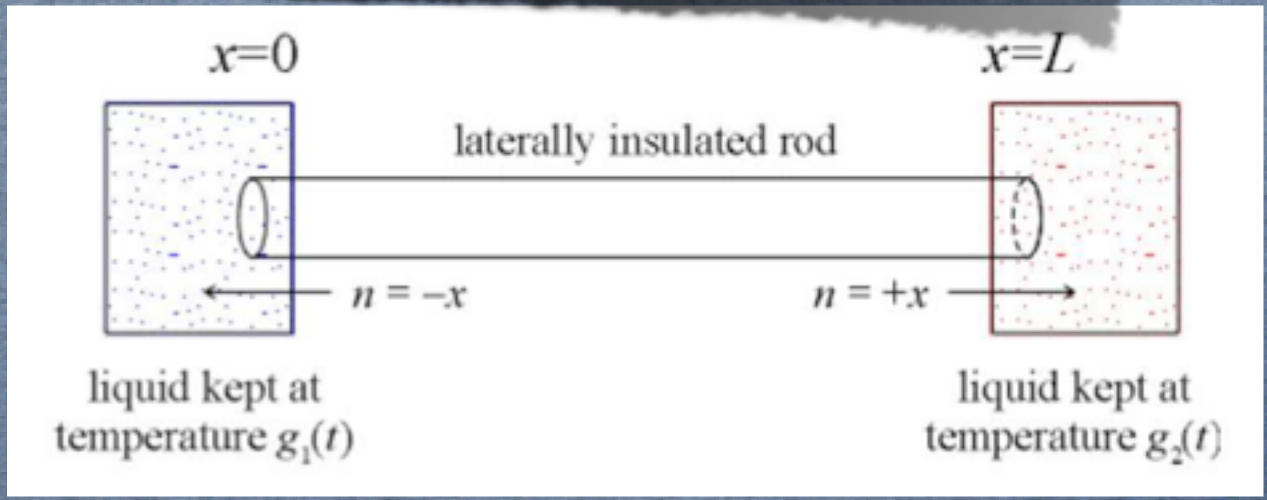
    $$u < \bar{u}, \quad \text{then} \quad u_{xx} > 0 \quad \text{and} \quad u_t > 0,$$

    the net flow of heat into $x$ is positive.
  - If the temperature at $x$ is equal to its surrounding average,

    $$u = \bar{u}, \quad \text{then} \quad u_{xx} = 0 \quad \text{and} \quad u_t = 0,$$

    the net flow of heat into $x$ is zero.

# PDEs: Heat Equation, Boundary Conditions




1. Dirichlet BC: temperature is specified on the boundary, i.e.

$$u(x=0,t)=g_1(t) \qquad u(x=L,t)=g_2(t).$$

2. Neumann BC: heat flow across the boundary specified, i.e.

$$\frac{\partial u}{\partial n}|_{x=0}=g_1(t) \qquad \frac{\partial u}{\partial n}|_{x=L}=g_2(t).$$

where $\vec{n}$ is the outward normal direction to the boundary.

3. Mixed BC: temperature of the *surrounding medium* is specified, i.e.

$$\frac{\partial u}{\partial n}|_{x=0}+\lambda u(x=0,t)=g_1(t) \qquad \frac{\partial u}{\partial n}|_{x=L}+\lambda u(x=L,t)=g_2(t).$$

# PDEs: Heat Equation, Typical BCs for 1D heat flow

Initial-boundary-value problem:

- Suppose we have a copper rod 200cm long that is laterally insulated and has an initial temperature of $0^o$C.
- Suppose the top of the rod ($x = 0$) is insulated, while the bottom ($x = 200$) is immersed in moving water that has a constant temperature of $g_2(t) = 20^o$C.
- The mathematical model for this problem:

$$\text{PDE:} \quad u_t = \alpha^2 u_{xx}, \qquad 0 < x < 200, \quad 0 < t <$$

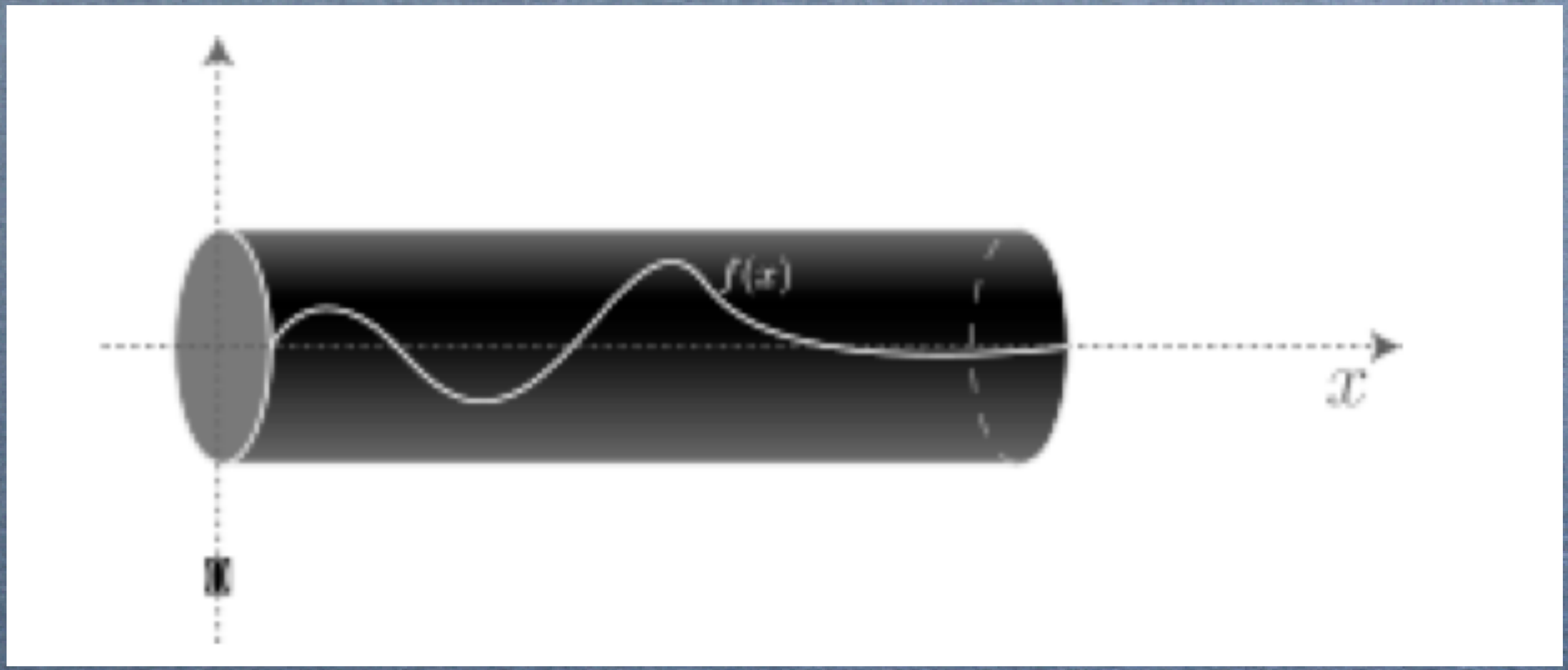
$$\text{BCs:} \quad \begin{cases} u_x(0,t) = 0 \\ u_x(200,t) = -\frac{h}{\kappa}[u(200,t) - 20] \end{cases}, \qquad 0 < t < \infty$$

$$\text{IC:} \quad u(x,0) = 0, \qquad 0 \le x \le 200$$

where

$$\begin{aligned} \alpha^2 &= 1.16\text{cm}^2/\text{sec}, && \text{(diffusivity constant for copper)} \\ \kappa &= 0.93\text{cal/cm-sec}^o\text{C}, && \text{(thermal conductivity of copper)} \\ h &= \text{heat exchange coefficient, to determine by the experiment.} \end{aligned}$$

# PDEs: Heat Equation, Initial-Boundary-Value Problem




Initial-boundary-value problem:

PDE: $$u_t = \alpha^2 u_{xx}, \qquad 0 < x < L, \quad \text{and} \quad 0 < t$$

BCs: $$\left\{ \begin{array}{l} u_x(0,t) = K_1(t) \\ u_x(200,t) = K_2(t) \end{array} \right. , \quad 0 < t$$

IC: $$u(x,0) = F(x), \quad 0 \le x \le L$$

# Divide and Conquer

# PDEs: Heat Equation, Separation of Variables

- The basic idea of separation of variables is to break down the *initial conditions* of the problem into simple components, find the response to each component, and the add up these individual responses.

- *Divide and Conquer*

- Separation of variables applies to problems where

  1. The PDE is linear and homogeneous (not necessarily constant coefficients).
  2. The boundary conditions are of the form

$$\begin{aligned} \alpha\, u_x(0,t) + \beta\, u(0,t) &= 0, \\ \gamma\, u_x(1,t) + \delta\, u(1,t) &= 0, \end{aligned}$$

  where $\alpha$, $\beta$, $\gamma$, and $\delta$ are constants (boundary conditions of this are called linear homogeneous BCs).

# PDEs: Heat Equation, Separation of Variables, Step 1

$$\text{PDE:} \quad u_t = \alpha^2 u_{xx}, \quad 0 < x < 1, \quad 0 < t < \infty$$

$$\text{BCs:} \quad \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases}, \quad 0 < t < \infty$$

$$\text{IC:} \quad u(x,0) = \phi(x), \quad 0 \le x \le 1$$

- Find elementary solutions to the PDE:

$$\boxed{u(x,t) = X(x)\,T(t), \qquad \text{fundamental solutions}}$$

- substitute this trial solution into the PDE,

$$X(x)\,T'(t) = \alpha^2 X''(x)\,T(t),$$

- divide each side of this equation by $\alpha^2 X(x)\,T(t)$,

$$\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)},$$

and obtain what is call *separated variables.*

# PDEs: Heat Equation, Separation of Variables, Step 1

- In this case, $x$ and $t$ are independent of each other, each side must be a fixed constant (say $k$),

$$\boxed{\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = k,}$$

or two ODEs

$$T'(t) - k\,\alpha^2 T(t) = 0,$$
$$X''(x) - k\,X(x) = 0.$$

- To meet the condition as $t \to \infty$, $k$ must to be *negative*, i.e. $k = -\lambda^2$, where $\lambda$ is nonzero.

$$T'(t) + \lambda^2\,\alpha^2 T(t) = 0,$$
$$X''(x) + \lambda^2\,X(x) = 0.$$

# Review of ODEs

## PDEs: Heat Equation, Separation of Variables, Step 2

- For the two ODEs

$$T'(t) + \lambda^2 \alpha^2 T(t) = 0,$$
$$X''(x) + \lambda^2 X(x) = 0.$$

- The corresponding solutions are:

$$\begin{aligned} T(t) &= C_1 \, e^{-\lambda^2 \alpha^2 t}, \qquad (C_1 \text{ an arbitrary constant}) \\ X(x) &= C_2 \, \text{Sin}(\lambda x) + C_3 \, \text{Cos}(\lambda x), \qquad (C_2 \text{ and } C_3 \text{ arbitrary}) \end{aligned}$$

- The total solution for $u(x,t) = X(x)\,T(t)$ is

$$u(x,t) = e^{-\lambda^2 \alpha^2 t}[A \, \text{Sin}(\lambda x) + B \, \text{Cos}(\lambda x)]$$

# PDEs: Heat Equation, Separation of Variables, Step 3

- Find solutions to match the BCs
- The total solution for $u(x,t) = X(x)\,T(t)$

$$u(x,t) = e^{-\lambda^2\alpha^2 t}[A\,\mathrm{Sin}(\lambda x) + B\,\mathrm{Cos}(\lambda x)]$$

to satisfy the boundary conditions

$$\begin{aligned} u(0,t) &= 0, \\ u(1,t) &= 0, \end{aligned}$$

needs to enforce $B = 0$ and $\mathrm{Sin}\lambda = 0$ (or $\lambda = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$).

- We have an infinite number of functions,

$$u_n(x,t) = A_n\, e^{-(n\pi\alpha)^2 t}\,\mathrm{Sin}(n\pi x), \qquad n = 1, 2, \ldots$$

which is called the fundamental solution (an infinite number).

# PDEs: Heat Equation, Separation of Variables, Step 4

- Find the solutions to match the IC
- To add the fundamental solutions

$$u(x,t) = \sum_{n=1}^{\infty} A_n\, e^{-(n\pi\alpha)^2 t}\, \mathrm{Sin}(n\pi x),$$

and meet the initial condition, $u(x,0) = \phi(x)$, i.e.

$$\sum_{n=1}^{\infty} A_n\, \mathrm{Sin}(n\pi x) = \phi(x),$$

- Luckily, $\mathrm{Sin}(n\pi x)$ is **orthogonal** for different $n$, i.e.

$$\int_0^1 \mathrm{Sin}(m\pi x)\mathrm{Sin}(n\pi x)\mathrm{d}\,x = \frac{1}{2}\delta_{mn}, \quad \text{by} \quad \mathrm{Sin}(x)\mathrm{Sin}(y) = \frac{1}{2}[\mathrm{Cos}(x-y) - \mathrm{Cos}(x+y)].$$

- By multiply $\mathrm{Sin}(m\pi x)$, we obtain

$$A_m = 2\int_0^1 \phi(x)\mathrm{Sin}(m\pi x)\mathrm{d}\,x$$

# Power Series: Sturm-Liouville Problem

- Sturm-Liouville Problem:

$$[p(x)y']' + [q(x) + \lambda r(x)]y = 0, \qquad \text{with the boundary conditions}$$
$$\begin{cases} k_1 y(a) + k_2 y'(a) = 0 \\ l_1 y(b) + l_2 y'(b) = 0 \end{cases}$$

- $\lambda$ is a parameter.
- For the interval $a \le x \le b$, $p(x)$, $q(x)$, $r(x)$, and $p'(x)$ are *continuous*.
- $k_1$, $k_2$ are given constants, not both zero, and so are $l_1$, $l_2$ , not both zero.
- If $k_2 = l_2 = 0$, one has Dirichlet B.C.
- If $k_1 = l_1 = 0$, one has Neumann B.C.
- If $k_1 = l_2 = 0$ or $k_2 = l_1 = 0$, one has Mixed (Robin) B.C.

# Power Series: Orthogonality of Eigenfunctions

- For a given number $\lambda$, eigen-value, a solution to satisfy the Sturm-Liouville Problem is called eigen-function.

- Functions $y_1(x), y_2(x), \ldots$ defined on some interval $a \leq x \leq b$ are called orthogonal on this interval with respect to the *weight function* $r(x) > 0$ if

$$(y_m(x), y_n(x)) \equiv \int_a^b r(x) y_m(x) y_n(x)\, \mathrm{d}x = \delta_{mn},$$

where we introduce Kronecker's delta $\delta_{mn}$ which gives the value

$$\delta_{mn} = \begin{cases} 0 & ; \quad \text{if} \quad m \neq n \\ 1 & ; \quad \text{if} \quad m = n \end{cases}$$

- The norm $\| y_m \|$ of $y_m(x)$ is defined by

$$\| y_m(x) \| = \sqrt{\int_a^b r(x) y_m^2(x)\, \mathrm{d}x},$$

# Power Series: Orthogonality, Example 1

- The functions $y_m(x) = \sin mx$, $m = 1, 2, \ldots$, form an orthogonal set on the interval $-\pi \leq x \leq \pi$, for

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, \mathrm{d}x = 0, \qquad m \neq n$$

i.e. $2 \sin\alpha \sin\beta = \cos(\alpha - \beta) - \cos(\alpha + \beta)$.

- The norm $\| y_m \|$ of $\sin mx$ is

$$\| y_m(x) \| = \sqrt{\int_{-\pi}^{\pi} \sin^2 mx \, \mathrm{d}x} = \sqrt{\pi}.$$

# Power Series: Orthogonality, Example 2

- The Legendre Polynomials form an orthogonal set on the interval $-1 \leq x \leq 1$, for

$$\int_{-1}^{1} P_m(x) P_n(x)\, \mathrm{d}x = 0, \qquad m \neq n$$

- The Bessel's functions form an orthogonal set on the interval $0 \leq x \leq R$, for

$$\int_{0}^{R} x J_\nu(k_{\nu,m}x) J_\nu(k_{\nu,n}x)\, \mathrm{d}x = 0, \qquad m \neq n$$

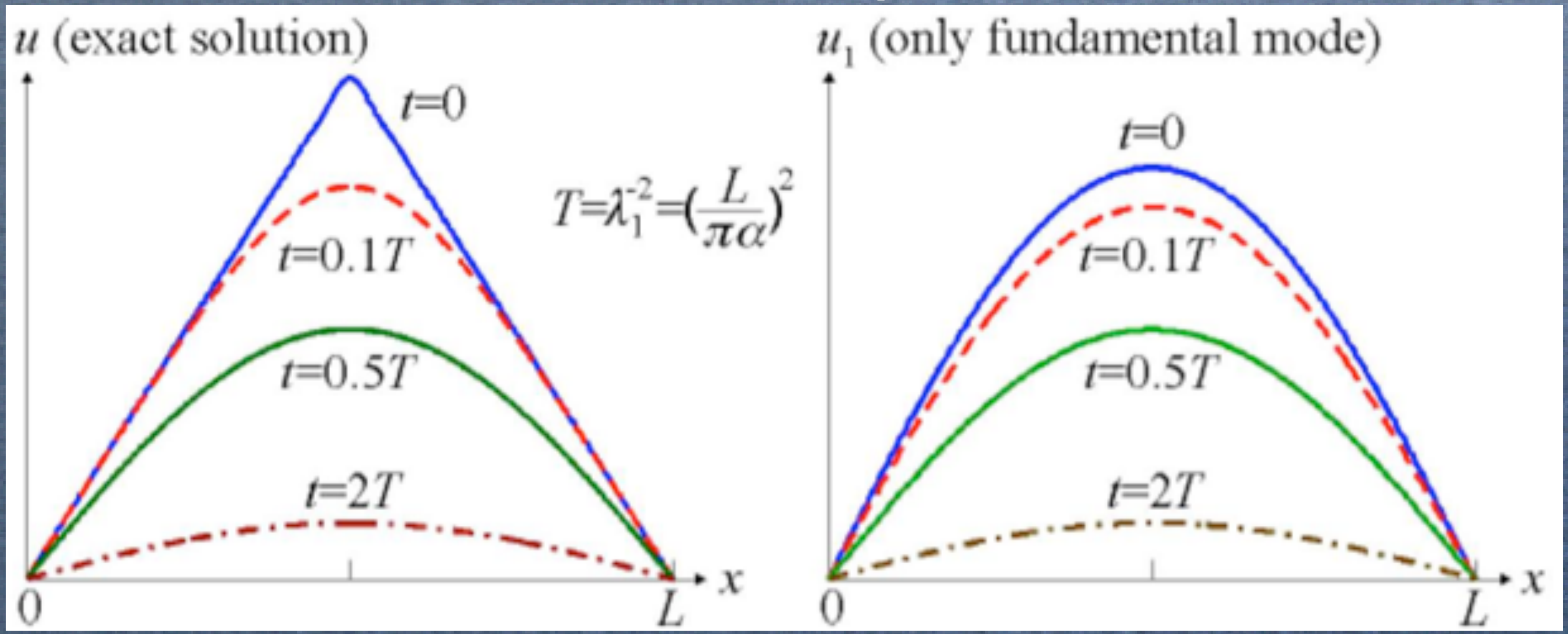
# PDEs: Heat Equation, Separation of Variables, Dirichlet BCs

**Example:**

PDE: $u_t = \alpha^2 u_{xx}, \quad 0 < x < L, \quad 0 < t < \infty$

BCs: $u(0,t) = 0$ and $u(L,t) = 0, \quad 0 < t < \infty$

IC: $$u(x,0) = \begin{cases} x & ; \text{ for } 0 \le x \le L/2 \\ L - x & ; \text{ for } L/2 \le x \le L \end{cases}$$




**Solution:**

$$u(x,t) = \sum_{n=1} A_n e^{-(n\pi/L)^2 \alpha^2 t} \text{Sin}(\frac{n\pi}{L} x),$$

$$A_n = [\frac{4L}{n^2\pi^2}] \text{Sin}(\frac{n\pi}{2}) = \begin{cases} (-1)^{(n-1)/2} [\frac{4L}{n^2\pi^2}] & ; \text{ if } n \text{ is odd} \\ 0 & ; \text{ if } n \text{ is even} \end{cases}$$

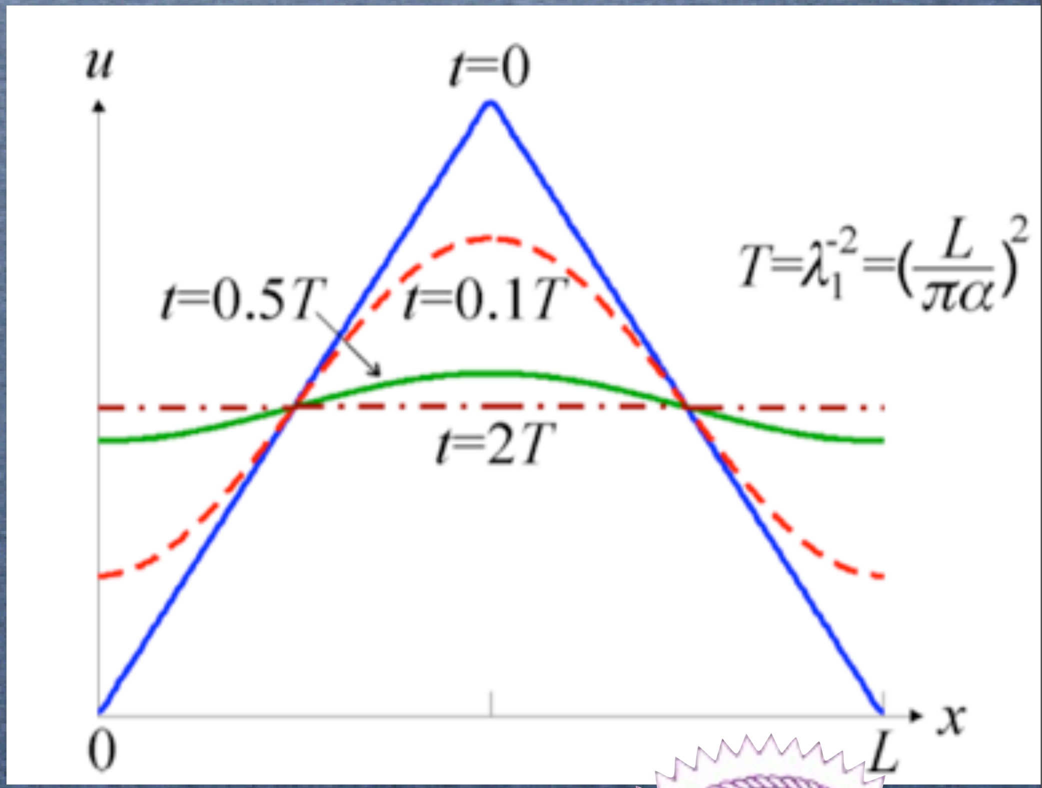
# PDEs: Heat Equation, Separation of Variables, Neumann BCs

**Example:**

PDE: $u_t = \alpha^2 u_{xx}, \quad 0 < x < L, \quad 0 < t < \infty$

BCs: $u_x(0,t) = 0$ and $u_x(L,t) = 0, \quad 0 < t < \infty$

IC: $$u(x,0) = \begin{cases} x & ; \text{ for } 0 \le x \le L/2 \\ L - x & ; \text{ for } L/2 \le x \le L \end{cases}$$




**Solution:**

$$u(x,t) = \sum_{n=0} A_n e^{-(n\pi/L)^2 \alpha^2 t} \text{Cos}(\frac{n\pi}{L} x),$$

$$A_n = \begin{cases} L/4 & ; \text{ for } n = 0 \\ \frac{2L}{n^2\pi^2}[2\text{Cos}(\frac{n\pi}{2}) - \text{Cos}(n\pi) - 1] & ; \text{ for } n \ne 0 \end{cases}$$

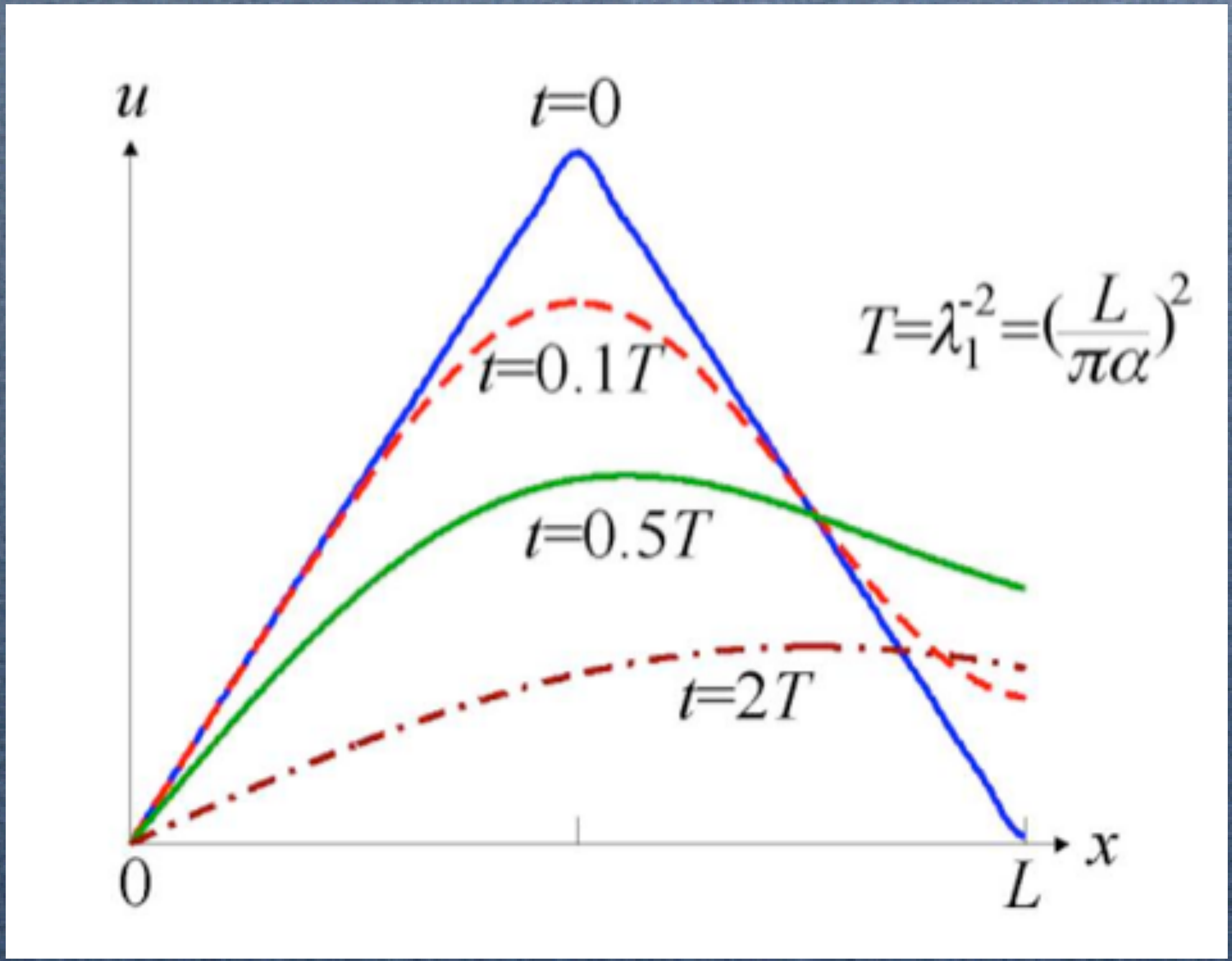
# PDEs: Heat Equation, Separation of Variables, Mixed BCs

**Example:**

PDE: $u_t = \alpha^2 u_{xx}, \quad 0 < x < L, \quad 0 < t < \infty$

BCs: $u(0,t) = 0$ and $u_x(L,t) + h\,u(L,t) = 0, \quad 0 < t < \infty$

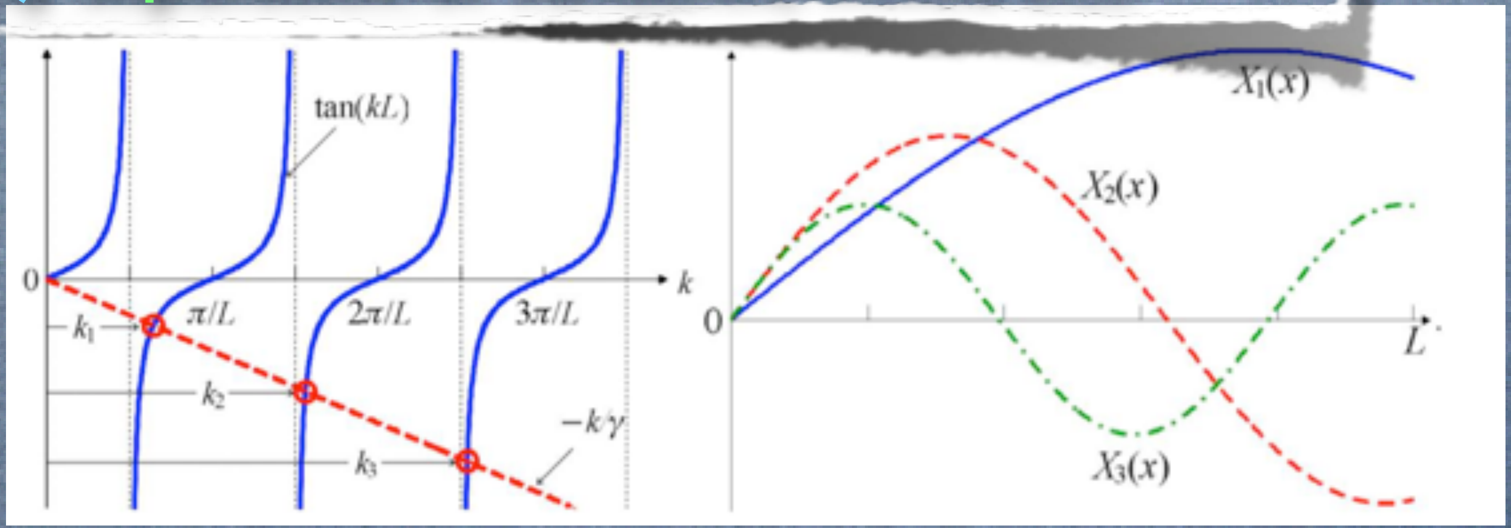
IC: $u(x,0) = \begin{cases} x & ; \text{ for } 0 \le x \le L/2 \\ L - x & ; \text{ for } L/2 \le x \le L \end{cases}$




**Solution:**

# PDEs: Heat Equation, Separation of Variables, Mixed BCs

## Solution:




- From the BCs:

$$X_n(x) = \mathrm{Sin}(k_n x), \quad \text{where} \quad \tan(k_n L) = -\frac{k_n}{h}.$$

- The total solution:

$$u(x,t) = \sum_{n=1}^{\infty} A_n\, e^{-(k_n \alpha)^2 t}\, \mathrm{Sin}(k_n x),$$

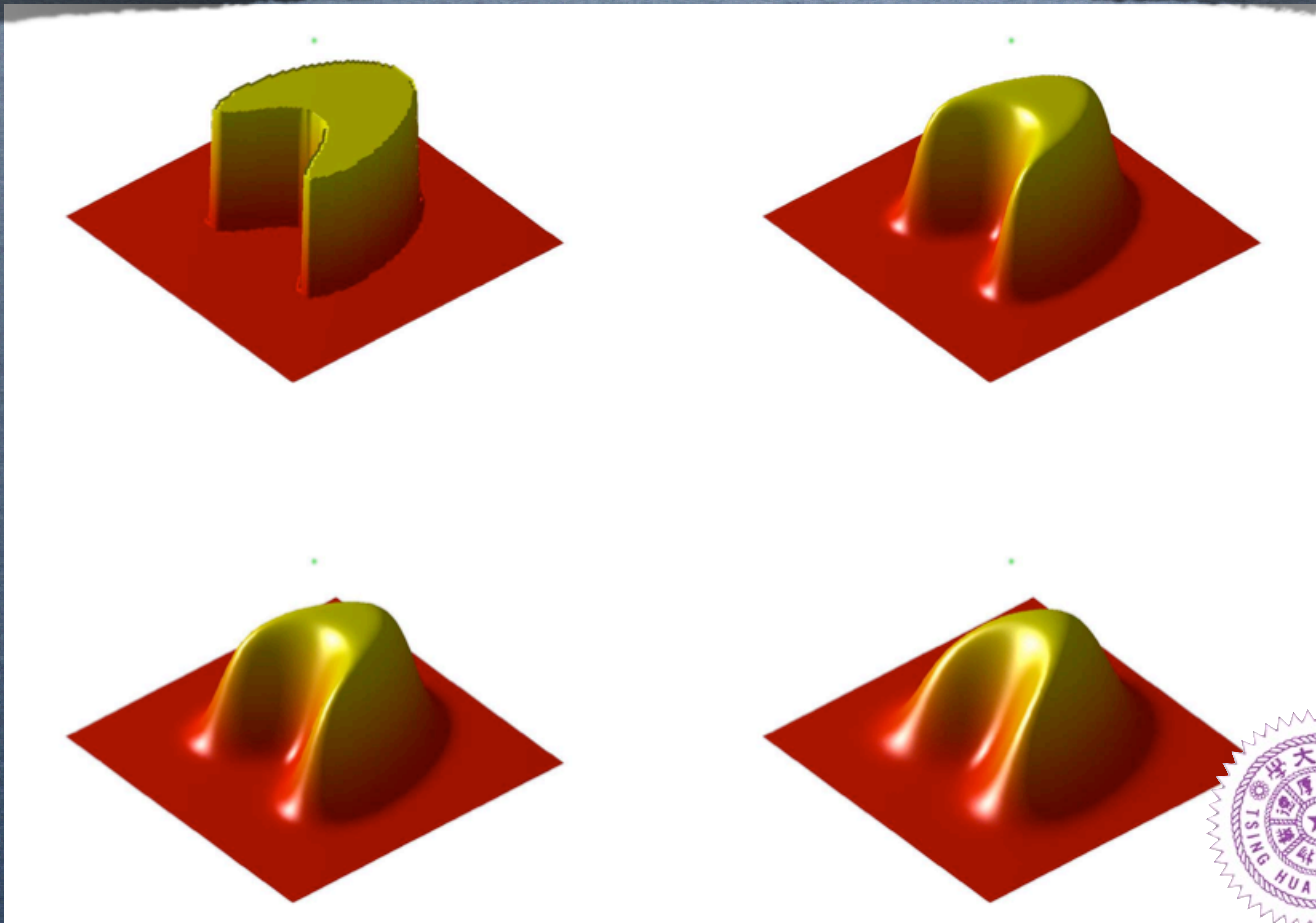
- By the orthogonality of $X_n(x)$ in $[0, L]$ (for the spatial ODE is a Sturm-Liouville problem):

$$\int_0^L \mathrm{Sin}(k_n x)\mathrm{Sin}(k_m x)\mathrm{d}\,x = [\frac{L}{2} - \frac{\mathrm{Sin}(2k_n L)}{4k_n}]\delta_{mn} \equiv \frac{L}{2}[1 - \mathrm{Sinc}(2k_n L)]\delta_{mn},$$

where

$$A_n = \frac{2}{L[1 - \delta_{mn}\mathrm{Sinc}(2k_n L)]} \int_0^L \phi(x)\mathrm{Sin}(k_m x)\mathrm{d}\,x.$$

# PDEs: Heat Equation, Separation of Variables, 2D



# Homework #15:

1. **[Homogeneous Heat Equations]**:
Solve the temperature distribution $u(x,t)$ described by the heat equation,

$$\text{PDE:} \qquad u_t = \alpha^2 u_{xx}, \qquad 0 < x < L, \quad 0 < t < \infty$$

$$\text{IC:} \qquad u(x,0) = \begin{cases} 1 & ; \ \text{for} \quad 0 \le x \le L/2 \\ 0 & ; \ \text{for} \quad L/2 \le x \le L \end{cases}$$

with three different boundary conditions (BCs):

(a) Neumann BCs:

$$u_x(0,t) = 0 \quad \text{and} \quad u_x(L,t) = 0, \quad 0 < t < \infty$$

(b) Mixed BCs:

$$u(0,t) = 0 \quad \text{and} \quad u_x(L,t) + h\,u(L,t) = 0, \quad 0 < t < \infty$$

where $h$ is a constant.

2. **[Homogeneous Heat Equations with Loss]**:
Solve the temperature distribution $u(x,t)$ described by the heat equation,

$$\text{PDE:} \qquad u_t = \alpha^2 u_{xx} - \beta u, \qquad 0 < x < L, \quad 0 < t < \infty$$

$$\text{BCs:} \qquad u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0, \quad 0 < t < \infty$$

$$\text{IC:} \qquad u(x,0) = \text{Sin}(\frac{\pi}{L}x) + 0.5\,\text{Sin}(\frac{3\pi}{L}x), \qquad 0 < x < L.$$

where the constant $\beta$ describes the heat loss.
Hint: try the transformation $u(x,t) = e^{-\beta t}\, w(x,t)$ first.

# Maxwell's equations :

(1831-1879)

- Gauss's law for the electric field:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \Longleftrightarrow \oint_S \mathbf{E} \cdot \mathrm{d}A = \frac{q}{\epsilon_0},$$

- Gauss's law for magnetism:

$$\nabla \cdot \mathbf{B} = 0 \Longleftrightarrow \oint_S \mathbf{B} \cdot \mathrm{d}A = 0,$$

- Faraday's law of induction:

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B} \Longleftrightarrow \oint_C \mathbf{E} \cdot \mathrm{d}l = -\frac{\partial}{\partial t}\Phi_B,$$

- Ampére's circuital law:

$$\nabla \times \mathbf{B} = \mu_0(\mathbf{J} + \frac{\partial}{\partial t}\mathbf{D}) \Longleftrightarrow \oint_C \mathbf{B} \cdot \mathrm{d}l = -\mu_0(\mathbf{I} + \frac{\partial}{\partial t}\Phi_D)$$

# Wave equation

- Divergence of the gradient: Laplacian

$$\nabla \cdot (\nabla \phi) \equiv \nabla^2 \phi \equiv \Delta \phi.$$

- For a source-free medium, $\rho = \mathbf{J} = 0$,

$$\nabla \times (\nabla \times E) = -\mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} E,$$

$$\Rightarrow \qquad \nabla(\nabla \cdot E) - \nabla^2 E = -\mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} E.$$

- When $\nabla \cdot E = 0$, one has wave equation, i.e. $\nabla \cdot D = \nabla \cdot (\epsilon E) = 0$,

$$\nabla^2 E = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}) E = \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} E.$$

# PDEs: Wave Equation, Finite-vibration string

PDE: $$u_{tt} = \alpha^2 u_{xx}, \qquad 0 < x < L, \quad 0 < t < \infty$$

BCs: $$\begin{cases} u(0,t) = 0 \\ u(L,t) = 0 \end{cases}, \qquad 0 < t < \infty$$

ICs: $$\begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}, \qquad 0 \le x \le L$$

**Hints:**

- Applying separation of variables

$$u(x,t) = X(x)\,T(t),$$

- Substituting this expression into the wave equation, we have two ODEs

$$T'' - \alpha^2 \lambda T = 0$$

$$X'' - \lambda X = 0$$

# PDEs: Wave Equation, Finite-vibration string, cont.

## Solutions:

- Only negative values of $\lambda$ give feasible (nonzero and bounded) solutions, i.e., $\lambda \equiv -\beta^2$.

$$u(x,t) = [C\,Sin(\beta x) + D\,Cos(\beta x)][A\,Sin(\alpha\beta t) + B\,Cos(\alpha\beta t)],$$

- With the BCs,

$$\begin{aligned} u(0,t) &= X(0)\,T(t) = 0, \qquad \Rightarrow D = 0, \\ u(L,t) &= X(L)\,T(t) = 0, \qquad \Rightarrow \beta_n = \frac{n\pi}{L}, \quad n = 1, 2, \ldots \end{aligned}$$

- The fundamental solution

$$\begin{aligned} u_n(x,t) = X_n(x)\,T_n(t) &= Sin(\frac{n\pi x}{L})[a_n Sin(\frac{n\pi\alpha t}{L}) + b_n\,Cos(\frac{n\pi\alpha t}{L})] \\ &= R_n Sin(\frac{n\pi x}{L})\,Cos[\frac{n\pi\alpha(t-\delta_n)}{L}] \end{aligned}$$

# PDEs: Wave Equation, Finite-vibration string, cont.

- The solution for wave equation

**Solutions:**

$$u(x,t) = \sum_{n=1}^{\infty} Sin(\frac{n\pi x}{L})[a_n Sin(\frac{n\pi\alpha t}{L}) + b_n Cos(\frac{n\pi\alpha t}{L})]$$

- With ICs,

$$\begin{aligned} u(x,0) &= \sum_{m=1}^{\infty} b_n Sin(\frac{n\pi x}{L}) = f(x), \\ u_t(x,0) &= \sum_{m=1}^{\infty} a_n (\frac{n\pi\alpha}{L}) Sin(\frac{n\pi x}{L}) = g(x), \end{aligned}$$

- Using the orthogonality conditions, $\int_0^L Sin(\frac{m\pi x}{L}) Sin(\frac{n\pi x}{L}) \mathrm{d}\, x = \frac{L}{2}\delta_{mn}$, we have

$$\begin{aligned} a_n &= \frac{2}{n\pi\alpha}\int_0^L g(x) Sin(\frac{n\pi x}{L}) \mathrm{d}\, x, \\ b_n &= \frac{2}{L}\int_0^L f(x) Sin(\frac{n\pi x}{L}) \mathrm{d}\, x. \end{aligned}$$

# PDEs: Heat Equation, with Lateral Heat Loss

**Example:**

$$\begin{aligned}
&\text{PDE:} && u_t = \alpha^2 u_{xx} - \beta u, && 0 < x < L, \quad 0 < t < \infty \\
&\text{BCs:} && u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0, && 0 < t < \infty \\
&\text{IC:} && u(x,0) = \phi(x), && 0 \le x \le L
\end{aligned}$$

**Hits:** where $-\beta u$ represents heat flow across the lateral boundary.

By means of the transformation

$$\boxed{u(x,t) = e^{-\beta t} w(x,t),}$$

then the original heat equation with lateral loss becomes,

$$\begin{aligned}
&\text{PDE:} && w_t = \alpha^2 w_{xx}, && 0 < x < L, \quad 0 < t < \infty \\
&\text{BCs:} && w(0,t) = 0 \quad \text{and} \quad w(L,t) = 0, && 0 < t < \infty \\
&\text{IC:} && w(x,0) = \phi(x), && 0 \le x \le L
\end{aligned}$$

with the solutions already known, i.e.

$$w(x,t) = \sum_{n=1} \left[ \frac{2}{L} \int_{-L}^{L} \phi(s) \mathrm{Sin}(\frac{n\pi}{L} s) \mathrm{d}\, s \right] e^{-(\frac{n\pi\alpha}{L})^2 t} \mathrm{Sin}(\frac{n\pi}{L} x).$$

# PDEs: Heat Equation, Non-homogeneous Boundary Conditions

**Example:**

PDE: $u_t = \alpha^2 u_{xx}, \qquad 0 < x < L, \quad 0 < t < \infty$

BCs: $\begin{cases} u(0,t) = k_1 \\ u(L,t) = k_2 \end{cases}, \quad 0 < t < \infty$

IC: $u(x,0) = \phi(x), \quad 0 \le x \le L$

**Hits:**

Transforming non-homogeneous BCs to homogeneous ones:

$$\begin{aligned} u(x,t) &= \text{steady state} \quad + \quad \text{transient} \\ &= [k_1 + \frac{x}{L}(k_2 - k_1) + U(x,t)], \end{aligned}$$

where

PDE $\quad U_t = \alpha^2 U_{xx}, \qquad 0 < x < L, \quad 0 < t < \infty$

BCs $\quad \begin{cases} U(0,t) = 0 \\ U(L,t) = 0 \end{cases}, \quad 0 < t < \infty$

IC $\quad U(x,0) = \phi(x) - [k_1 + \frac{x}{L}(k_2 - k_1)], \quad 0 \le x \le L$

# PDEs: Heat Equation, More

- Lateral heat loss proportional to the temperature difference:

$$u_t = \alpha^2 u_{xx} - \beta(u - u_0), \qquad \beta > 0.$$

Heat loss ($u > u_0$) or gain ($u < u_0$) is proportional to the difference between the temperature $u(x,t)$ of the rod and the surrounding medium $u_0$ (with $\beta$ the proportionality constant).

- Internal heat source:

$$u_t = \alpha^2 u_{xx} + f(x,t), \qquad \text{the nonhomogeneous equation.}$$

The rod is supplied with an internal heat source (everywhere along the rod and for all time $t$).

- Diffusion-convection equation:

$$u_t = \alpha^2 u_{xx} - \nu u_x.$$

E.g. a pollutant is carried along in a stream moving with velocity $\nu$.

- Nonhomogeneous material: $u_t = \alpha^2(x) u_{xx} +$ f(x,y).

# PDEs: Heat Equation, Non-homogeneous

**Example:**

PDE: $u_t = \alpha^2 u_{xx} + f(x,t), \quad 0 < x < 1, \quad 0 < t < \infty$

BCs: $u(0,t) = 0$ and $u(1,t) = 0, \quad 0 < t < \infty$

IC: $u(x,0) = \phi(x), \quad 0 \le x \le 1$

**Hits:**

- For $f(x,t) = 0$, we have solutions for the homogeneous problem, i.e.

$$u(x,t) = \sum_{n=1} a_n e^{-(\lambda_n \alpha)^2 t} X_n(x),$$

where $\lambda_n$ and $X_n(x)$ are the eigenvalues and the eigenfunctions of the Sturm-Liouville problem, i.e.

$$X'' + \lambda^2 X = 0, \quad \text{and} \quad X(0) = 0, X(1) = 0,$$

- For non-homogeneous problem, $f(x,t) \neq 0$, we try the slightly more general form,

$$u(x,t) = \sum_{n=1} T_n(t) X_n(x),$$



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# PDEs: Heat Equation, Non-homogeneous, Example

**Example:**

PDE: $u_t = \alpha^2 u_{xx} + \text{Sin}(3\pi x), \quad 0 < x < 1, \quad 0 < t < \infty$

BCs: $u(0,t) = 0 \quad \text{and} \quad u(1,t) = 0, \quad 0 < t < \infty$

IC: $u(x,0) = \text{Sin}(\pi x), \quad 0 \le x \le 1$

**Hits:**

- Since the BCs support $\text{Sin}(n\pi x)$ eigenfunctions,

$$u(x,t) = \sum_{n=1} T_n(t) X_n(x) = \sum_{n=1} T_n(t)\text{Sin}(n\pi x),$$

- Substitute this expansion into the problem,

PDE: $$\sum_{n=1} [T_n' + (n\pi\alpha)^2 T_n]\text{Sin}(n\pi x) = \text{Sin}(3\pi x),$$

IC: $$\sum_{n=1} T_n(0)\text{Sin}(n\pi x) = \text{Sin}(\pi x),$$

# PDEs: Heat Equation, Non-homogeneous, Example, cont.

- With the orthogonality for $\text{Sin}(n\pi x)$, we have,

PDE: $$T_n' + (n\pi\alpha)^2 T_n = 2\int_0^1 \text{Sin}(n\pi x)\text{Sin}(3\pi x)\,\mathrm{d}\,x = \begin{cases} 1 & ; \quad \text{for} \quad n = 3 \\ 0 & ; \quad \text{for} \quad n \neq 3 \end{cases}$$

IC: $$T_n(0) = 2\int_0^1 \text{Sin}(n\pi x)\text{Sin}(\pi x)\,\mathrm{d}\,x = \begin{cases} 1 & ; \quad \text{for} \quad n = 1 \\ 0 & ; \quad \text{for} \quad n \neq 1 \end{cases}$$

- Writing out these equations for $n = 1, 2, \ldots,$, wee see

$$(n=1) \qquad \left.\begin{array}{l} T_1' + (\pi\alpha)^2 T_1 = 0 \\ T_1(0) = 1 \end{array}\right\} \Rightarrow T_1(t) = e^{-(\pi\alpha)^2 t},$$

$$(n=2) \qquad \left.\begin{array}{l} T_2' + (2\pi\alpha)^2 T_2 = 0 \\ T_2(0) = 0 \end{array}\right\} \Rightarrow T_2(t) = 0,$$

$$(n=3) \qquad \left.\begin{array}{l} T_3' + (3\pi\alpha)^2 T_3 = 1 \\ T_3(0) = 0 \end{array}\right\} \Rightarrow T_3(t) = \frac{1}{(3\pi\alpha)^2}[1 - e^{-(3\pi\alpha)^2 t}],$$

$$(n \geq 4) \qquad \left.\begin{array}{l} T_n' + (n\pi\alpha)^2 T_n = 0 \\ T_n(0) = 0 \end{array}\right\} \Rightarrow T_n(t) = 0,$$

# PDEs: Heat Equation, Non-homogeneous, Example

## Solution:

- The total solution for our problem is

$$\begin{aligned} u(x,t) &= e^{-(\pi\alpha)^2 t}\mathrm{Sin}(\pi x) + \frac{1}{(3\pi\alpha)^2}[1 - e^{-(3\pi\alpha)^2 t}]\mathrm{Sin}(3\pi x) \\ &= \text{transient} \quad + \quad \text{steady state} \end{aligned}$$

- The first term represents transient behavior, due to the initial conditions.
- The second term represents steady state behavior, due to the right-hand side of the PDE (non-homogeneous term).

# Homework #16:

1. [**Vibrating of a stretched string with fixed ends**:
   Find the solutions for one-dimensional wave equation,

$$\text{PDE:} \quad u_{tt} = \alpha^2 u_{xx}, \qquad 0 < x < 1, \quad 0 < t < \infty$$

$$\text{BCs:} \quad \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases}, \quad 0 < t < \infty$$

$$\text{ICs:} \quad \begin{cases} u(x,0) = \begin{cases} \frac{3}{10}x & ; \text{ for } 0 \le x \le \frac{1}{3} \\ \frac{3(1-x)}{20} & ; \text{ for } \frac{1}{3} \le x \le 1 \end{cases} \\ u_t(x,0) = 0 \end{cases}, \quad 0 < t < \infty$$

   Hint: use the separation of variables first.

2. [**Homogeneous Heat Equations in 2D Rectangular coordinates**:
   Solve the temperature distribution $u(x,y,t)$ described by the heat equation,

$$\text{PDE:} \quad u_t = \alpha^2 (u_{xx} + u_{yy}), \qquad 0 < x < a, \quad 0 < y < b, \quad 0 < t < \infty$$

$$\text{BCs:} \quad u(0,y,t) = u(a,y,t) = 0 \quad \text{and} \quad u(x,0,t) = u(x,b,t) = 0, \quad 0 < t$$

$$\text{IC:} \quad u(x,y,0) = c_0, \qquad 0 < x < a, \quad 0 < y < b.$$

   where $\alpha$ and $c_0$ are both constants.
   Hint: try the separation of variables $u(x,y,t) = X(x)Y(y)T(t)$ first.

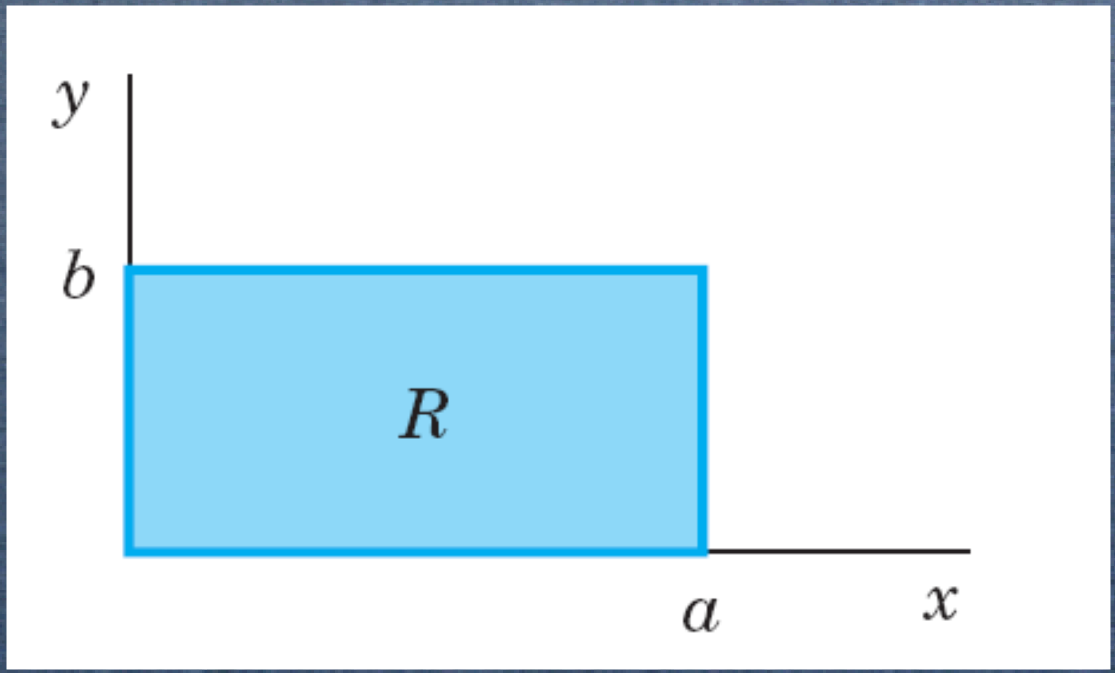
# PDEs: 2D Heat Equation,

PDE: $u_t = \alpha^2 (u_{xx} + u_{yy}) \equiv \alpha^2 \nabla_\perp^2, \quad 0 < x < a, \quad 0 < y < b, \quad 0 < t < \infty$

BCs: $u(0, y, t) = u(a, y, t) = 0$ and $u(x, 0, t) = u(x, b, t) = 0, \quad 0 < t < \infty$

IC: $u(x, y, 0) = c_0, \quad 0 < x < a, \quad 0 < y < b.$

where $\alpha$ and $c_0$ are both constants.



# PDEs: 2D Heat Equation, solution

- Find elementary solutions to the PDE:

$$u(x,y,t) = X(x)\,Y(y)\,T(t),$$

- Fundamental solutions to match the BCs:

$$u(x,y,t) = \sum_m \sum_n A_{mn} e^{-(m^2+n^2)\alpha^2 t} \sin(\frac{m\pi}{a}x)\sin(\frac{n\pi}{b}y),$$

- For the IC:

$$u(x,y,0) = c_0 = \sum_m \sum_n A_{mn} \sin(\frac{m\pi}{a}x)\sin(\frac{n\pi}{b}y),$$

where the *double* Sine series coefficient $A_{mn}$ is

$$A_{mn} = \frac{4}{ab}\int_0^b \int_0^a c_0 \sin(\frac{m\pi}{a}x)\sin(\frac{n\pi}{b}y)\,dxdy.$$

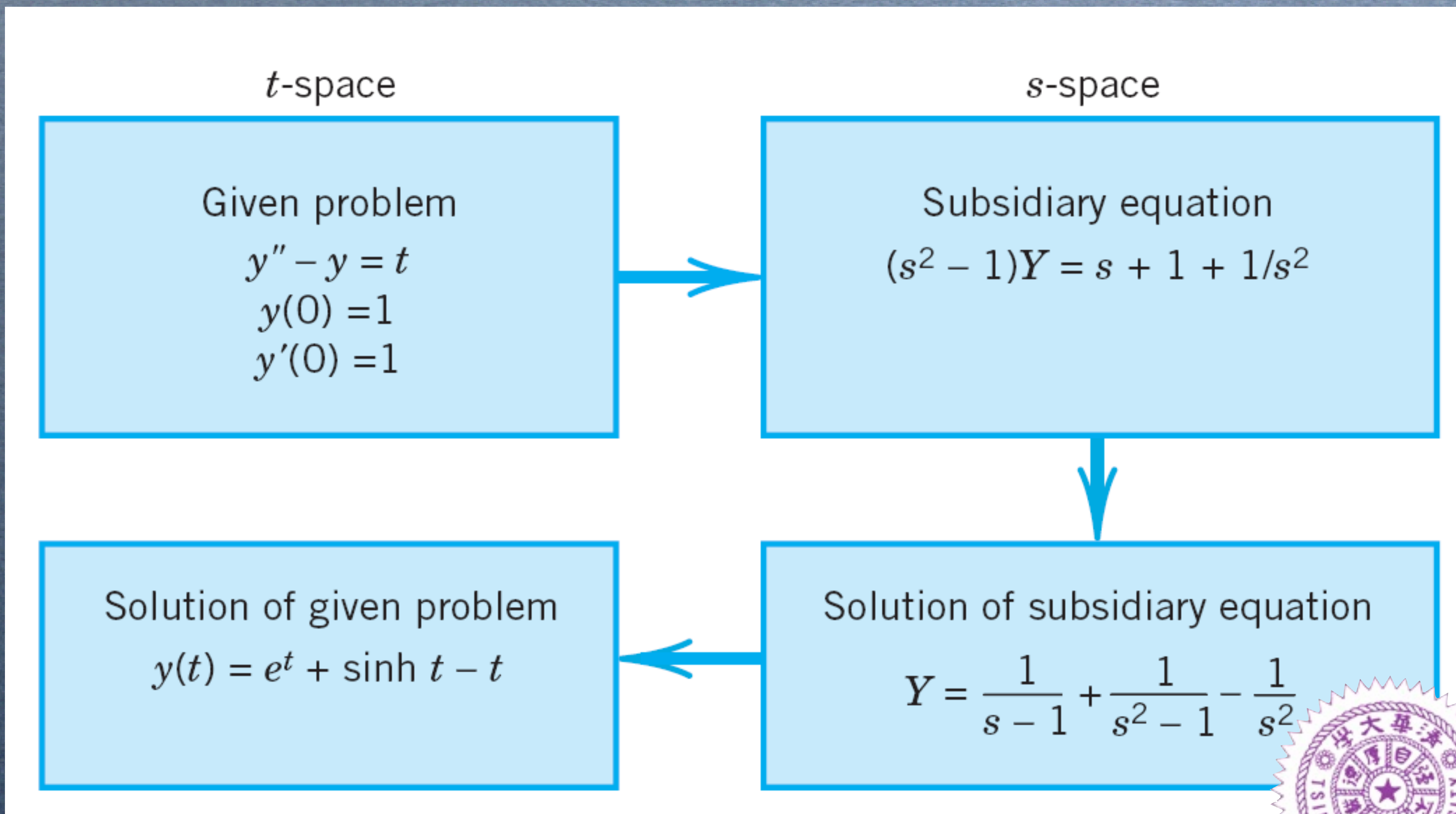
# PDEs: Wave Equation, Semi-infinite media, Ch. 12.11

PDE: $w_{tt} = c^2 w_{xx}, \qquad 0 < x < \infty, \quad \text{and} \quad 0 < t < \infty$

BCs: $w(0,t) = f(t) = \begin{cases} \sin t & ; \quad \text{for} \quad 0 \le t \le 2\pi \\ 0 & ; \quad \text{otherwise} \end{cases}$

AC: $\lim_{x \to \infty} w(x,t) = 0, \quad t \ge 0 \quad \text{Asymptotic Condition.}$

# Laplace Transform: ODE with IVP, Example



# Laplace Transform: Definition

- If $f(t)$ is a function defined for all $t \geq 0$, its Laplace transform is defined as

$$F(s) = \mathcal{L}(f) \equiv \int_0^\infty e^{-st} f(t)\, \mathrm{d}t.$$

- Here we must assume that $f(t)$ is such that the integral exists (that is, has some finite value).
- This assumption is usually satisfied in applications.

# Laplace Transform: Derivatives

- The transform of first derivative of $f$ satisfies

$$\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0).$$

- The transform of second derivative of $f$ satisfies

$$\mathcal{L}[f''(t)] = s^2\mathcal{L}[f(t)] - sf(0) - f'(0).$$

- The transform of $n$th-derivative of $f$ satisfies

$$\mathcal{L}[f^{(n)}(t)] = s^n\mathcal{L}[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0).$$

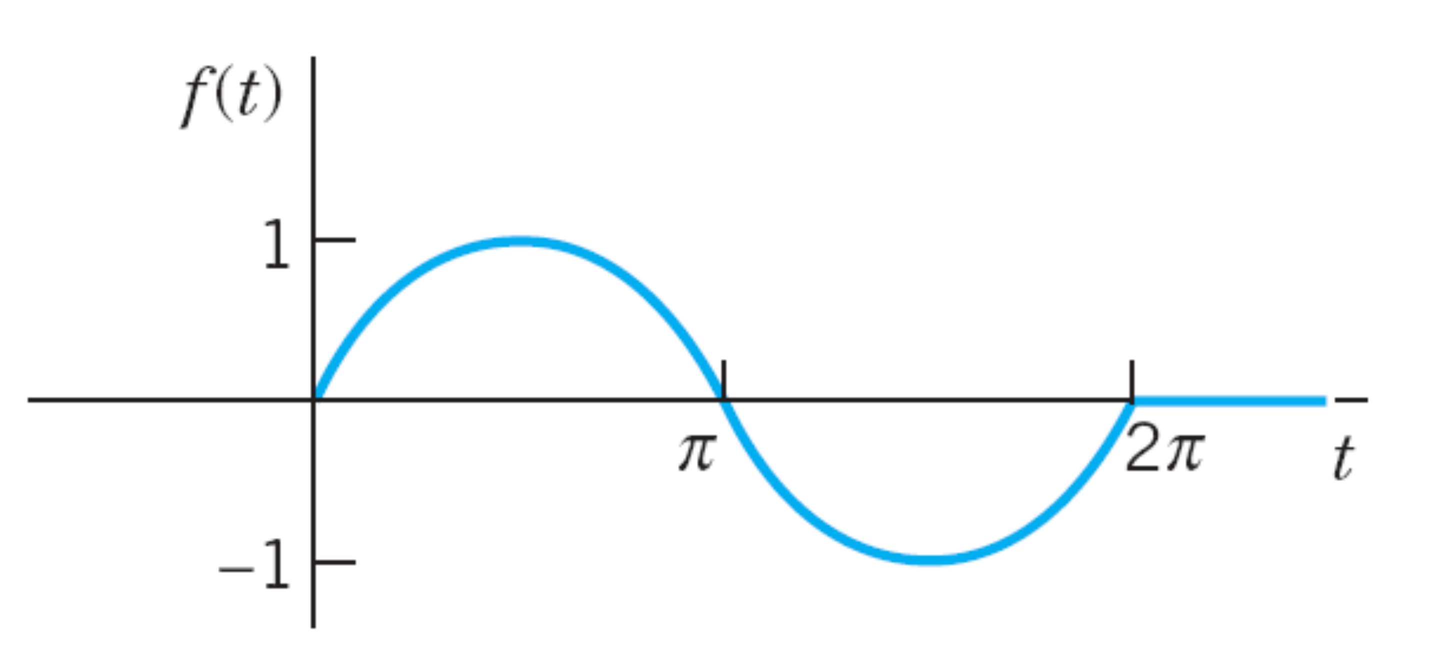
# PDEs: Wave Equation, Semi-infinite media, Ch. 12.11

PDE: $w_{tt} = c^2 w_{xx}, \quad 0 < x < \infty, \quad \text{and} \quad 0 < t < \infty$

BC: $w(0,t) = f(t) = \begin{cases} \sin t & ; \quad \text{for} \quad 0 \le t \le 2\pi \\ 0 & ; \quad \text{otherwise} \end{cases}$

ICs: $w(x,0) = w_t(x,0) = 0, \quad 0 \le x \le \infty$

AC: $\lim_{x\to\infty} w(x,t) = 0, \quad t \ge 0$ Asymptotic Condition.



# PDEs: Wave Equation, Semi-infinite media, cont.

- Transform $t$-variable via the Laplace transform, i.e. $\mathcal{L}[w(x,t)] = W(x)$,

$$\mathcal{L}[w_{tt}] = s^2 W(x,s) - sw(x,0) - w_t(x,0),$$

- For the ODE,

$$s^2 W(x) = c^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} W(x),$$

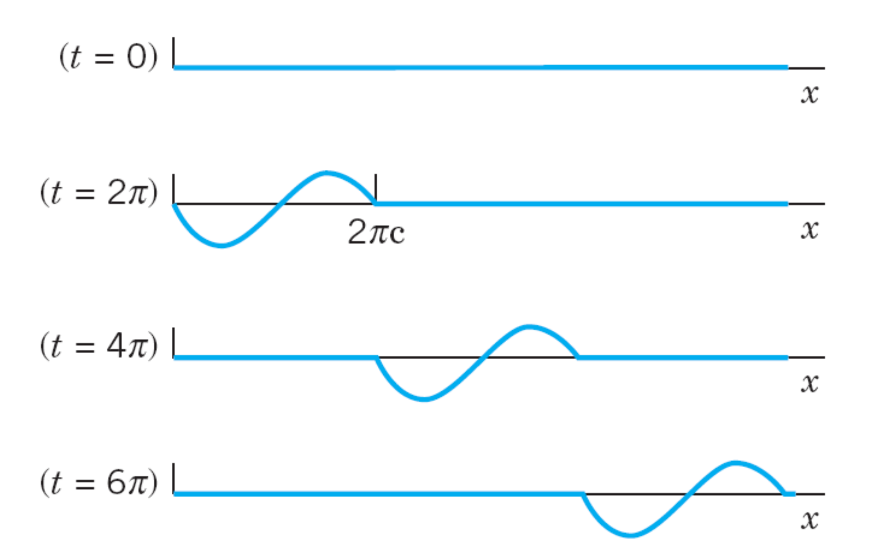
we have the general solution (homogeneous),

$$W(x,s) = c_1 e^{sx/c} + c_2 e^{-sx/c},$$

# PDEs: Wave Equation, Semi-infinite media, cont.

**Solution:**

- From the AC, we have $c_1 = 0$.
- From the BS:




$$W(0, s) = c_2 = \mathcal{L}[f(t)] \equiv F(s)$$

we have the coefficients,

$$W(x, s) = F(s)e^{-sx/c}.$$

- By the inverse Laplace transform, i.e., $\boxed{\mathcal{L}^{-1}[e^{-as}F(s)] = f(t-a)u(t-a).}$

$$\begin{aligned} u(x,t) &= f(t - \frac{x}{c})u(t - \frac{x}{c}) \\ &= \sin(t - \frac{x}{c}). \end{aligned}$$

# PDEs: Wave Equation, Semi-infinite media, cont.

- For the another general solution,

$$W(x,s) = G(s)\, e^{sx/c},$$

- By the inverse Laplace transform, i.e., $\mathcal{L}^{-1}[e^{-as}F(s)] = f(t-a)u(t-a).$

$$u(x,t) \quad = \quad g(t+\frac{x}{c})\, u(t+\frac{x}{c}).$$

**Backward wave !**

# PDEs: Wave Equation, Infinite media

**Fourier Integral !**

# PDEs: Scope

ODEs
- 1st, 2nd, higher-orders, and Systems, Ch. 1-4
- Integral Transform
  - ☒ Laplace Trans., Ch. 6
  - □ Fourier Trans., Ch. 11
- Series Solutions
  - ☒ Power Series, Ch. 5
  - ☒ Fourier Series, Ch. 11

PDEs
- ☒ Basic concepts, Ch. 12.1
- □ Modeling, Ch. 12.2, Ch. 12.7
- ☒ Separating Variables, Ch. 12.3
- ☒ Solved by Fourier Series, Ch. 12.5
- ☒ Heat Eq., Ch. 12.5
- □ D'Alember's solutions for Wave Eq., Ch. 12.4
- □ Solved by Fourier Integrals, Ch. 12.6
- ☒ 2D Cartesian, Ch. 12.8
- □ Laplace Eq., Ch. 12.9
- □ Cylindrical Coordinates, Ch. 12.10
- □ Spherical Coordinates, Ch. 12.10
- ☒ Solved by Laplace Transforms, Ch. 12.11

# 4th EXAM:

1. Topics cover "Partial Differential Equations":
   - Heat equations, Wave equations, and Laplacian equations:
   - 12.1, 12.3, 12.5, 12.8, 12.11
   - skip:
     Modeling (12.2, 12.7), D'Alembert's solution (12.4),
     Solution by Fourier Integrals (12.6)
     Polar Coordinates (12.9), Spherical Coordinate (12.10)

2. You can bring an ONE-PAGE NOTE.

**Dec. 23 (this Thursday), in class**
**1-3PM, at Room106!!**

# PDEs: Wave Equation, d'Alembert Solution, Ch. 12.4

- For the diffusion problems (the parabolic case), we solve the bounded case $(0 \le x \le L)$ by separation of variables while solve the unbounded case $(-\infty < x < \infty)$ by the Fourier transform.
- For the wave problems (the hyperbolic case), we will do the **opposite**.

PDE: $$u_{tt} = \alpha^2 u_{xx}, \qquad -\infty < x < \infty, \quad 0 < t < \infty$$

ICs: $$\begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}, \qquad -\infty < x < \infty$$

- Replace $(x, t)$ by new canonical coordinates $(\xi, \eta)$, i.e. the moving-coordinate,

$$\xi = x + \alpha\, t \qquad \eta = x - \alpha\, t$$

- the PDE becomes

$$\boxed{u_{\xi\eta} = 0,}$$

with the solution of arbitrary functions of $\xi$ or $\eta$, i.e.

$$\boxed{u(\xi, \eta) = \phi(\eta) + \psi(\xi).}$$

# PDEs: Wave Equation, d'Alembert Solution, cont.

- In the original coordinates $x$ and $t$, we have

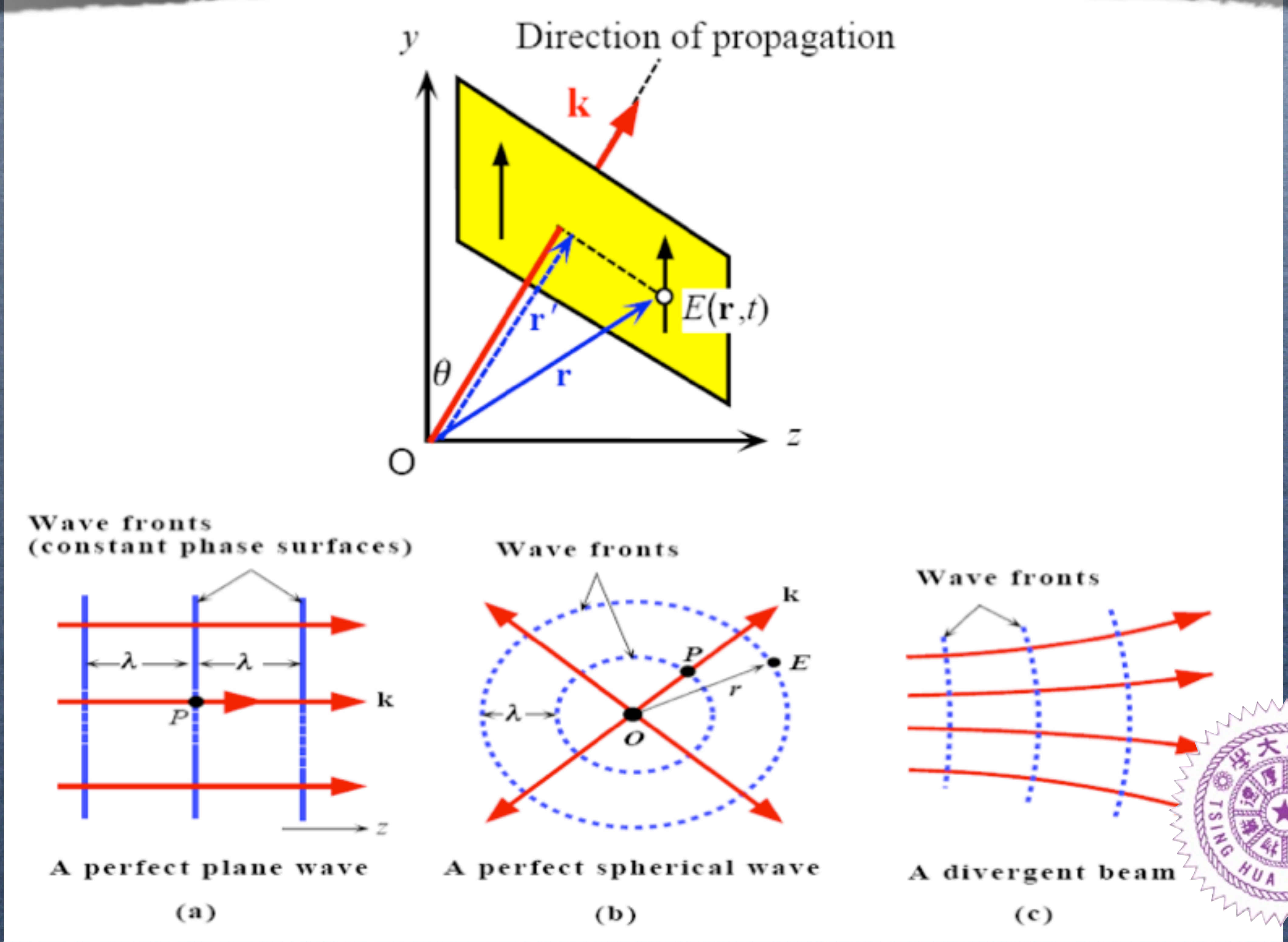
$$u(x,t) = \Phi(x - \alpha\, t) + \Psi(x + \alpha\, t),$$

this is the general solution of the wave equation.

- Physically it represents the sum of any two moving waves, each moving in opposite direction with the velocity $\alpha$. Eg.

$$\begin{aligned} u(x,t) &= Sin(x - \alpha\, t), && \text{(one right-moving wave)} \\ u(x,t) &= (x + \alpha\, t)^2, && \text{(one left-moving wave)} \\ u(x,t) &= Sin(x - \alpha\, t) + (x + \alpha\, t)^2, && \text{(two oppositely moving waves)} \end{aligned}$$

# PDE: Wave equation



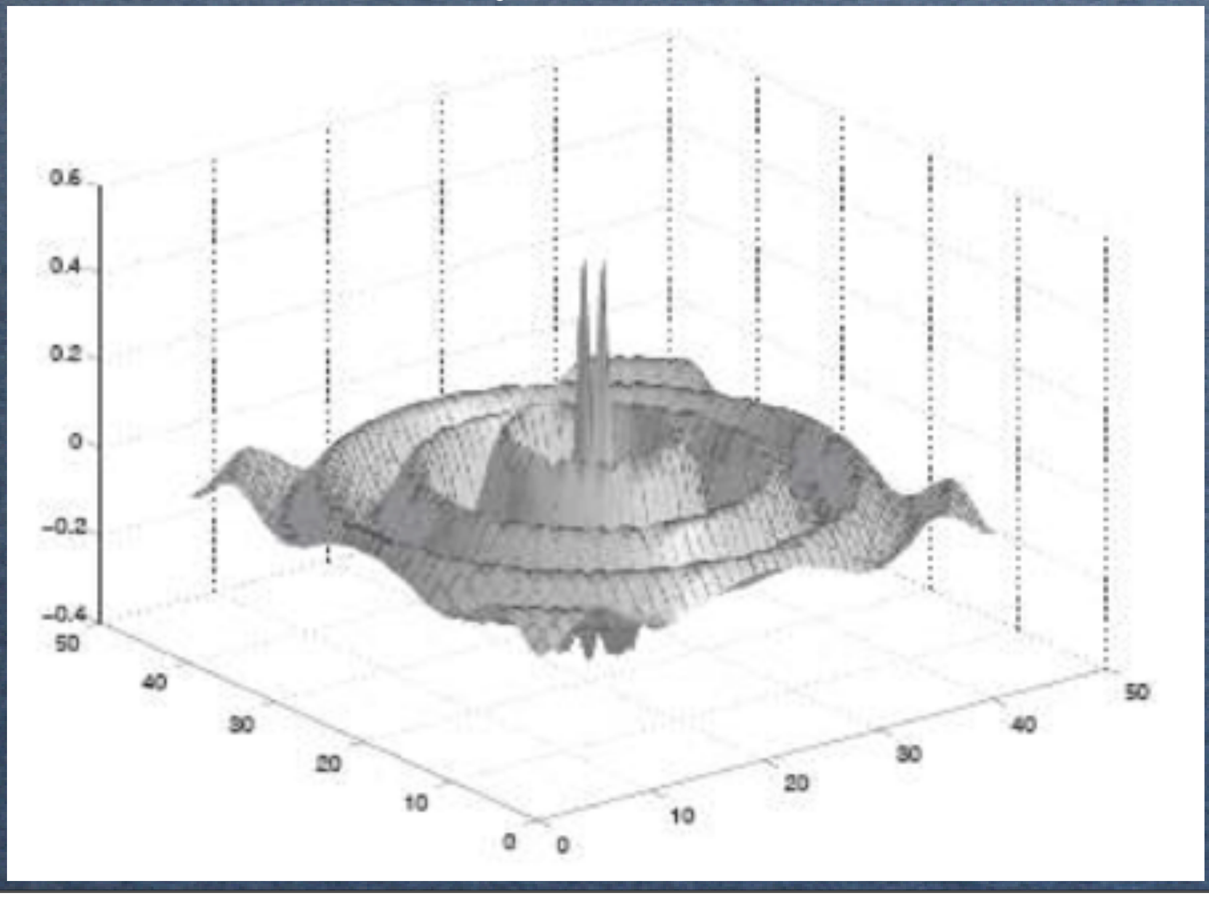
# PDE: Wave equation, Spherical wave

- spherical wave:

$$U(r) = \frac{A}{|r - r_0|} \exp(-ik|r - r_0|),$$

where $k|r - r_0|$ = constant, wavefronts resemble sphere surfaces,

- intensity:




$$I(r) = \frac{|A|^2}{r^2},$$

# PDEs: Wave Equation, d'Alembert Solution, IC

- Substitute the general solution into the two ICs,

$$\text{ICs:} \quad \begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases},$$

- for arbitrary functions $\phi$ and $\psi$, we have

$$\begin{aligned} \phi(x) + \psi(x) &= f(x), \\ -\alpha\,\phi'(x) + \alpha\,\psi'(x) &= g(x), \end{aligned}$$

- then by integrating from $x_0$ to $x$,

$$-\alpha\,\phi(x) + \alpha\,\psi(x) = \int_{x_0}^{x} g(\xi)\mathrm{d}\,\xi + K$$

where $K$ is an integration constant.

# PDEs: Wave Equation, d'Alembert Solution

- The solutions for $\phi$ and $\psi$ are

$$\begin{aligned} \phi(x) &= \frac{1}{2} f(x) - \frac{1}{2\alpha} \int_{x_0}^{x} g(\xi) \mathrm{d}\,\xi, \\ \psi(x) &= \frac{1}{2} f(x) + \frac{1}{2\alpha} \int_{x_0}^{x} g(\xi) \mathrm{d}\,\xi, \end{aligned}$$

- The D'Alembert solution,

$$\boxed{u(x,t) = \frac{1}{2}[f(x-\alpha, t) + f(x+\alpha\, t)] + \frac{1}{2\alpha}\int_{x-\alpha\, t}^{x+\alpha\, t} g(\xi)\mathrm{d}\,\xi.}$$

# PDEs: Wave Equation, d'Alembert Solution, Example

## Example 1:

Motion of an initial Sine wave,

PDE: $u_{tt} = \alpha^2 u_{xx}, \quad -\infty < x < \infty, \quad 0 < t < \infty$

ICs: $\begin{cases} u(x,0) = Sin(x) \\ u_t(x,0) = 0 \end{cases}, \quad -\infty < x < \infty$

## Solution:

- D'Alembert's solution:

$$u(x,t) = \frac{1}{2}[Sin(x - \alpha\, t) + Sin(x + \alpha\, t)].$$

# PDEs: Wave Equation, d'Alembert Solution, Example

## Example 2:

Motion of an initial Sine wave,

$$\text{PDE:} \qquad u_{tt} = \alpha^2 u_{xx}, \qquad -\infty < x < \infty, \quad 0 < t < \infty$$

$$\text{ICs:} \qquad \begin{cases} u(x,0) = 0 \\ u_t(x,0) = Sin(x) \end{cases}, \qquad -\infty < x < \infty$$

Initial velocity is given.

## Solution:

$$u(x,t) = \frac{1}{2\alpha}\int_{x-\alpha t}^{x+\alpha t} Sin(\xi)\,\mathrm{d}\,\xi$$

$$= \frac{1}{2\alpha}\left[Cos(x+\alpha t) - \mathrm{Cos}(x-\alpha t)\right].$$

# Power Series: Laplace's Equation, Chap. 12.10

Laplace's equation

$$\nabla^2 u \equiv u_{xx} + u_{yy} + u_{zz} = 0.$$

- Laplacian in Spherical Coordinates, i.e. $x = r\cos\theta\sin\phi$, $y = r\sin\theta\sin\phi$, and $z = r\cos\phi$,

$$\begin{aligned}\nabla^2 u &= \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\cot\phi}{r^2}\frac{\partial u}{\partial \phi} + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 u}{\partial \theta^2} \\ &= \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial u}{\partial \phi}\right) + \frac{1}{\sin^2\phi}\left(\frac{\partial^2 u}{\partial \theta^2}\right)\right].\end{aligned}$$

- Dirichlet problem with the boundary conditions:

$$u(R,\phi) = f(\phi), \qquad \text{and} \qquad \lim_{r\to\infty} u(r,\phi) = 0.$$

# Power Series: Laplace's Equation, Chap. 12.10

- Assume $u(r, \phi) = G(r)H(\phi)$, one has

$$r^2 \frac{\mathrm{d}^2 G}{\mathrm{d}r^2} + 2r \frac{\mathrm{d}G}{\mathrm{d}r} = n(n+1)G, \qquad \text{Euler-Cauchy equation,}$$

$$\frac{1}{\sin\phi} \frac{\mathrm{d}}{\mathrm{d}\phi}(\sin\phi \frac{\mathrm{d}}{\mathrm{d}\phi} H) + n(n+1)H = 0.$$

- By setting $\cos\phi \equiv w$, we have the Legendre's equation

$$(1 - w^2)\frac{\mathrm{d}^2 H}{\mathrm{d}w^2} - 2w \frac{\mathrm{d}H}{\mathrm{d}w} + n(n+1)H = 0.$$

# Power Series: Legendre's Function

**Hint:**

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0.$$

- The coefficients $\frac{-2x}{(1-x^2)}$ and $\frac{n(n+1)}{(1-x^2)}$ are analytic at $x=0$, then Legendre's equation has power series solutions of the form

$$y(x) = \sum_{m=0}^{\infty} a_m x^m.$$

- Recursive relation:

**Solution:** $$a_{s+2} = -\frac{(n-s)(n+s+1)}{(s+2)(s+1)} a_s, \qquad s = 0, 1, 2, \ldots$$

$$\begin{aligned} y_1(x) &= 1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - \ldots \\ y_2(x) &= x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - \ldots \end{aligned}$$

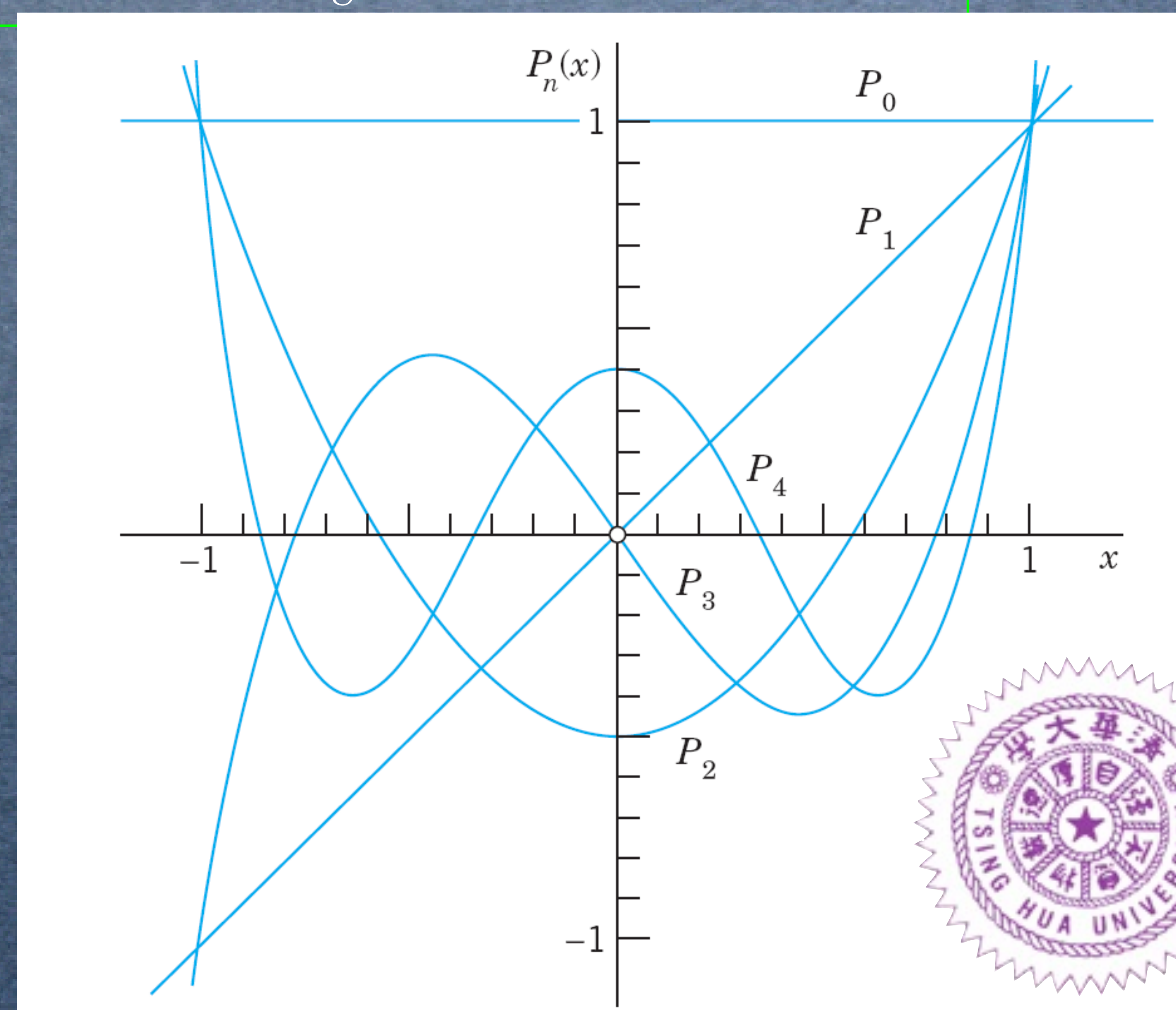
# Power Series: Legendre's Polynomials, cont.

Legendre's polynomial of degree $n$ :

$$P_0(x) = 1 \quad ; \quad P_1(x) = x$$
$$P_2(x) = \frac{1}{2}(3x^2 - 1) \quad ; \quad P_3(x) = \frac{1}{2}(5x^3 - 3x)$$
$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) \quad ; \quad P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

to meet the boundary condition

$$P_n(x) = 1.$$



# Laplace's Equation, Chap. 12.10

Laplace's equation in *spherical* coordinates:

$$\nabla^2 u \equiv u_{xx} + u_{yy} + u_{zz} = 0.$$

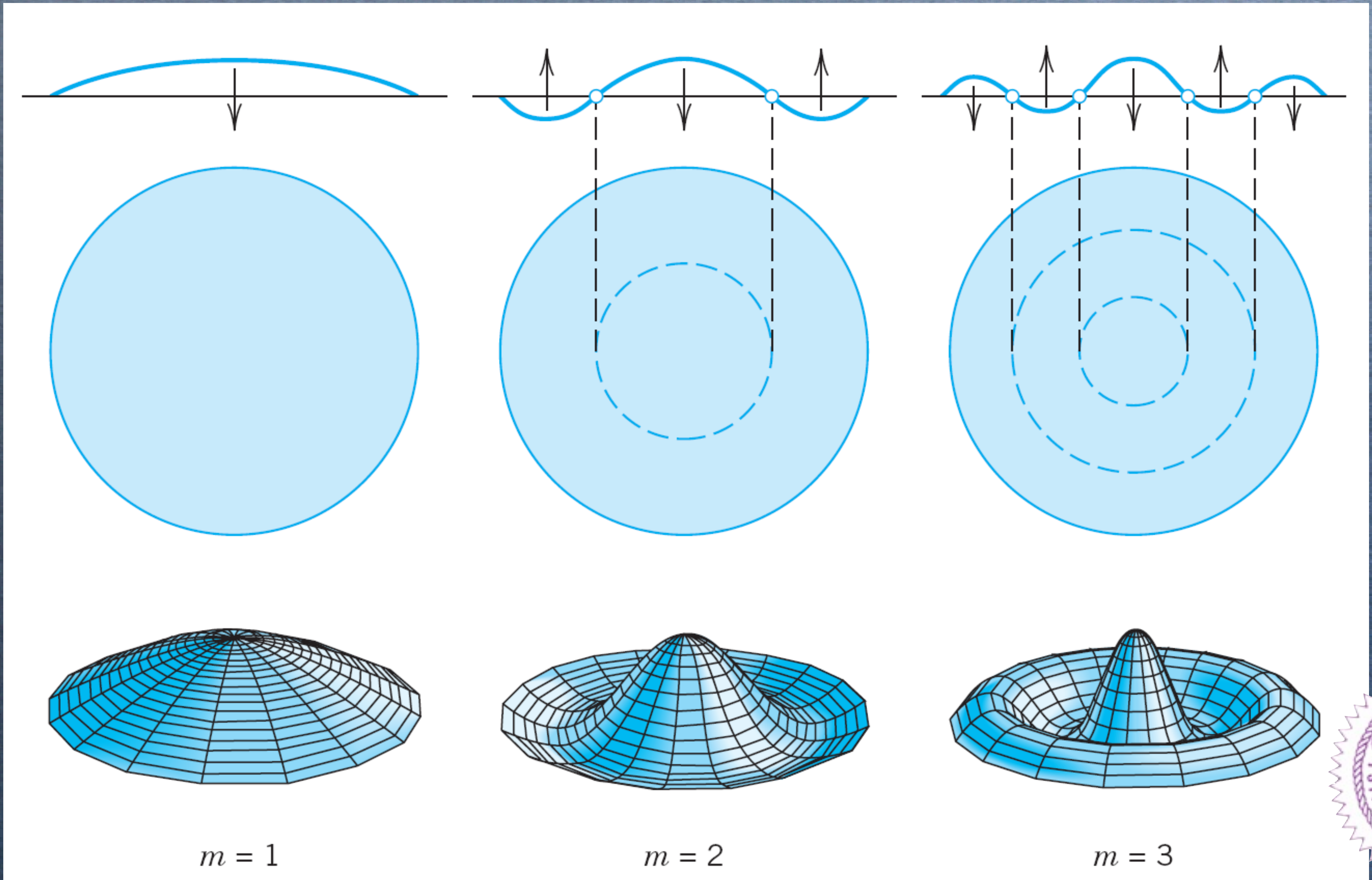
- Laplacian in Spherical Coordinates, i.e. $x = r\cos\theta\sin\phi$, $y = r\sin\theta\sin\phi$, and $z = r\cos\phi$,

$$\begin{aligned}\nabla^2 u &= \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\cot\phi}{r^2}\frac{\partial u}{\partial \phi} + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 u}{\partial \theta^2} \\ &= \frac{1}{r^2}[\frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r}) + \frac{1}{\sin\phi}\frac{\partial}{\partial \phi}(\sin\phi\frac{\partial u}{\partial \phi}) + \frac{1}{\sin^2\phi}(\frac{\partial^2 u}{\partial \theta^2})].\end{aligned}$$

- Fundamental solutions:

$$u(r,\phi) = \sum_n A_n r^n P_n(\cos\phi) + \frac{B_n}{r^{n+1}} P_n(\cos\phi).$$

# Laplace's Equation, Chap. 12.9



# Laplace's equation in a Disk