EE 2015
(Partial) Differential Equations and Complex Variables

Ray-Kuang Lee*

Institute of Photonics Technologies,
Department of Electrical Engineering, and Department of Physics,
National Tsing-Hua University

*rklee@ee.nthu.edu.tw

Syllabus:

- ★ Course description and Introduction, 9/14
- 1. Ordinary Differential Equations: 4 weeks
  - First-order ODEs, Ch. 1: 9/16, 9/21
  - Second-order ODEs, Ch. 2: 9/23, ~~9/28~~, 9/30, 1
  - Higher-order ODEs, Ch. 3: 10/12
  - Systems of ODES, Ch. 4: 10/14
  - 1st EXAM, 10/15 (Friday night)
- 2. Transform Methods: 3+1 weeks
  - Laplace Transforms, Ch. 6: 10/19 - 11/11
  - 2nd EXAM, 11/12 (Friday night)
- 3. Series and Complex Variables: 3+3+3 weeks
  - Power Series, Ch. 5: 11/16, 11/18, 11/23
  - Fourier Series, Ch. 11: 11/25, 11/30, 12/2
  - 3rd EXAM, 12/3 (Friday night)
  - PDE by Fourier Series, Ch. 12, 12/7 - 12/22
  - 4th EXAM, 12/23 (in Class)
  - Taylor and Laurent Series, Ch. 13-16: 12/28, 12/30
  - Complex and Residue Integrations, Ch. 16: 1/4 - 1/13
  - 5th EXAM, 1/14 (Friday night)

Complex Variables:

PDEs, Ch. 12

- ODEs
  - 1st-order, Ch. 1
  - 2nd-order, Ch. 2
  - Higher-order, Ch. 3
  - Systems of ODEs, Ch. 4
  - Integral Transform
    - ☒ Laplace Trans., Ch. 6
    - ☐ Fourier Trans., Ch. 11
    - ☐ Z – Trans.
  - Series Solutions
    - ☒ Power Series, Ch. 5
    - ☒ Fourier Series, Ch. 11
    - ☐ Taylor Series, Ch. 15
    - ☐ Laurent Series, Ch. 16

Complex Variables, Ch. 13-16, 17,18

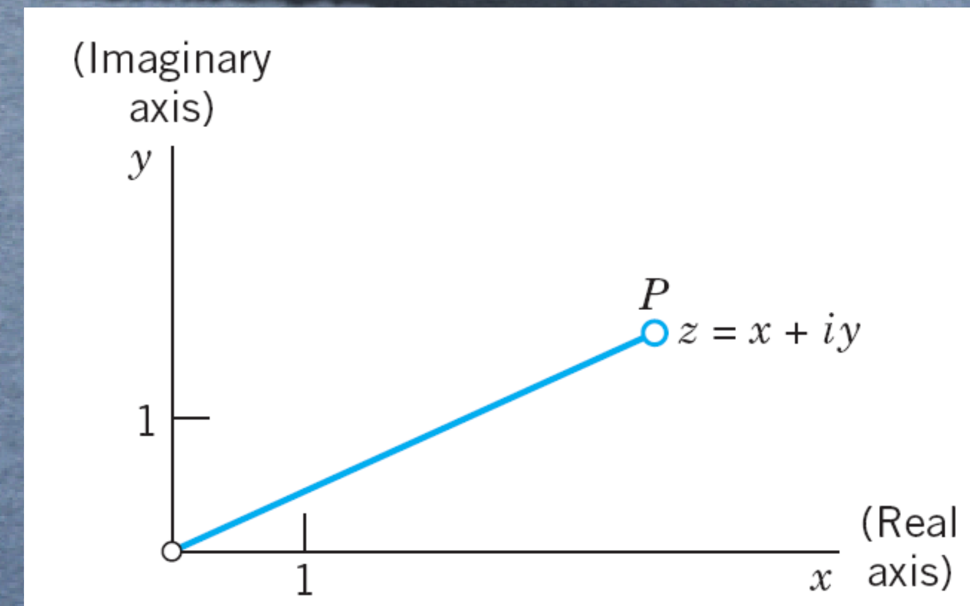
Complex Variables: Scope

Complex Variables

- Complex Numbers, Ch. 13
- Complex Function, Ch. 13
- Complex Integraton, Ch. 14
- Power Series, Ch. 15
- Taylor Series, Ch. 15

☒ Laurent Series, Ch. 16
☒ Residue Integration, Ch. 16
☐ Conformal Mapping, Ch. 17
☐ Complex Analysis, Ch. 18
☐ Potential Theory, Ch. 18

Complex Variables: complex plane, Ch. 13



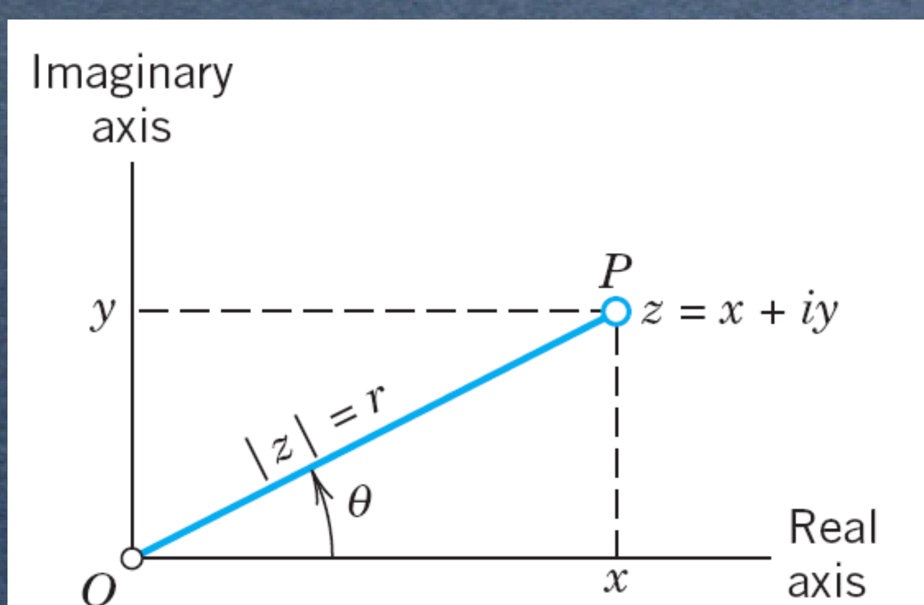
- Cartesian form: $\boxed{z = x + i\,y}$,

  where $x = \text{Re}\{z\}, y = \text{Im}\{z\}, i = \sqrt{-1}$.

- Polar form: $\boxed{z = r\,e^{i\theta}}$,

  where $r = |z| = \sqrt{x^2 + y^2}$ (absolute value or modulus),
  and $\theta = \arg(z) = \tan^{-1}(\frac{y}{x})$ (argument).

- Principle value $\text{Arg}(z)$:



$$\boxed{-\pi < \text{Arg}(z) \leq \pi.}$$

Complex Variables: Properties

- Addition: for two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$

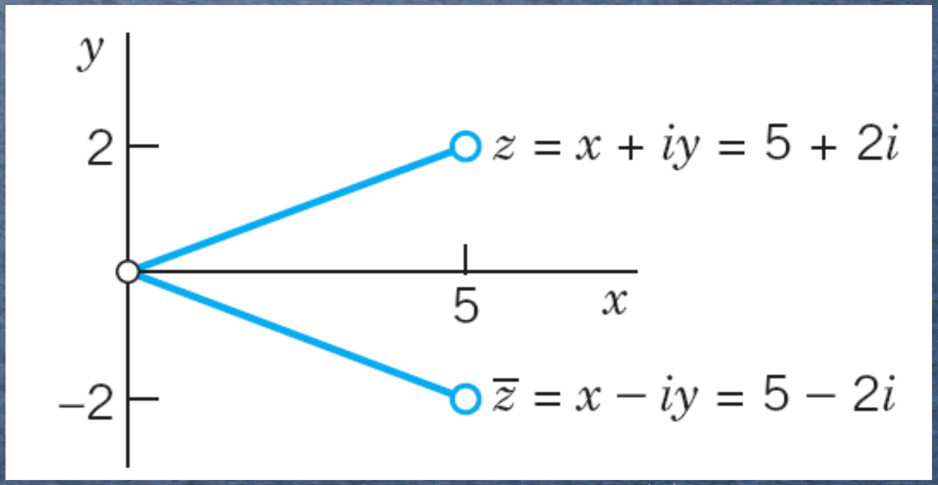
$$z_1 + z_2 = (x_1 + x_2, y_1 + y_2),$$

- Multiplication:

$$z_1\, z_2 = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1),$$

- Quotient:

$$z = \frac{z_1}{z_2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\,\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}$$



- Complex conjugate:

$$\bar{z} = z^* = x - i\, y,$$

- Real part and Imaginary part:

$$\begin{aligned} \mathrm{Re}(z) &= x = \frac{1}{2}(z + \bar{z}), \\ \mathrm{Im}(z) &= y = \frac{1}{2\, i}(z - \bar{z}), \end{aligned}$$

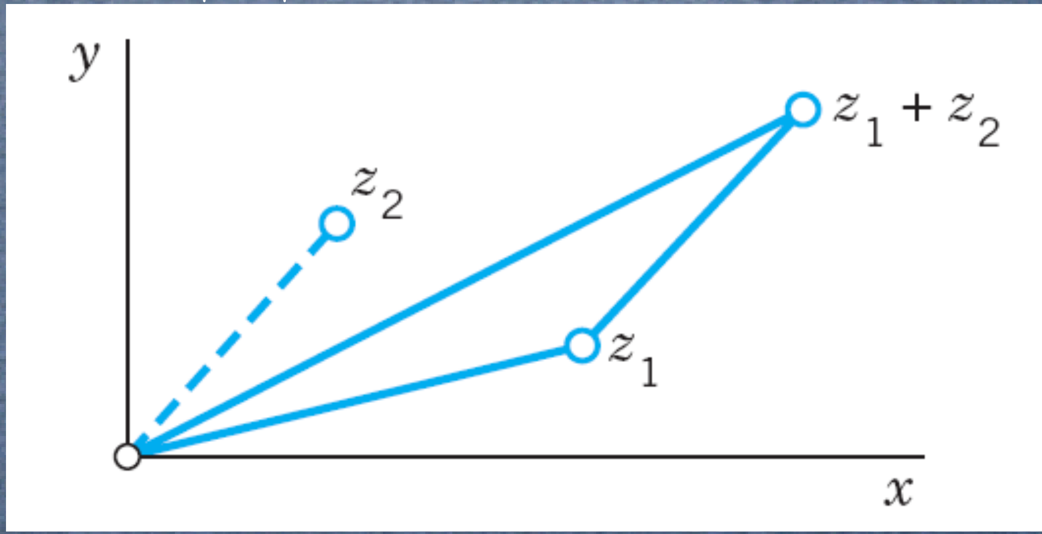
Complex Variables: Triangle Inequality

- Triangle inequality:

$$|z_1 + z_2| \le |z_1| + |z_2|,$$

By induction, we can obtain the generalized triangle inequality:

$$|z_1 + z_2 + \cdots + z_n| \le |z_1| + |z_2| + \cdots + |z_n|,$$



- Cauchy-Schwarz inequality:

$$|z_1\, z_2|^2 \le |z_1| \cdot |z_2|,$$

- Multiplication in polar form:

$$z_1\, z_2 = r_1\, r_2\, [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)],$$

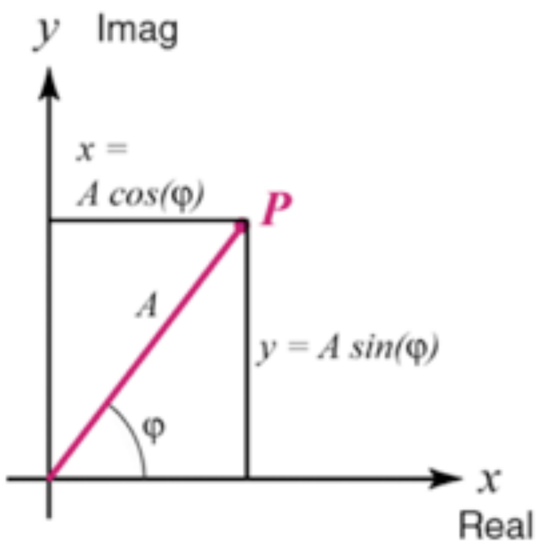
and

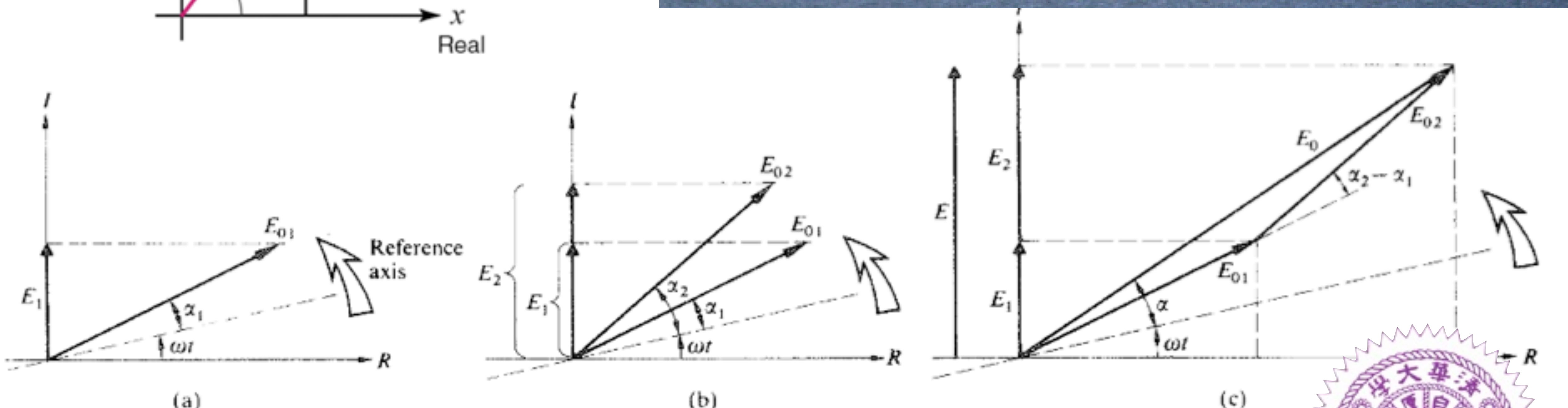
$$|z_1\, z_2| = |z_1||z_2|, \qquad |\frac{z_1}{z_2}| = \frac{|z_1|}{|z_2|},$$

$$\arg(z_1\, z_2) = \arg(z_1) + \arg(z_2), \qquad \arg(\frac{z_1}{z_2}) = \arg(z_1) - \arg(z_2),$$

Complex Variables: Phasor

$$E(x,t) = A_0 \cos[\omega t - k x + \theta] = \mathbf{Re}\{A_0 e^{i(\omega t - k x + \theta)}\} = \mathbf{Re}\{A(r)\, e^{i\omega t}\},$$





Complex Variables: Power

- Integer powers, De Moivre's formula:

$$z^n = r^n\,[\cos(n\theta) + i\sin(n\theta)],$$

- Roots, if $z = w^n$,

$$w = \sqrt[n]{z} = \sqrt[n]{r}[\cos(\frac{\theta + 2k\pi}{n}) + i\sin(\frac{\theta + 2k\pi}{n})],$$

  where $k = 0, 1, \ldots, n-1$.

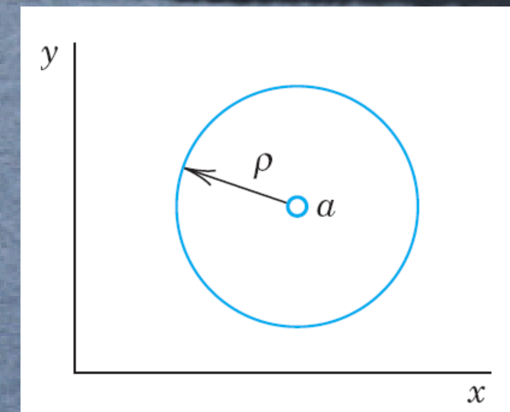
- The principal value of $w = \sqrt[n]{z}$ is obtained for the principal value of $\arg(z)$ and $k = 0$.

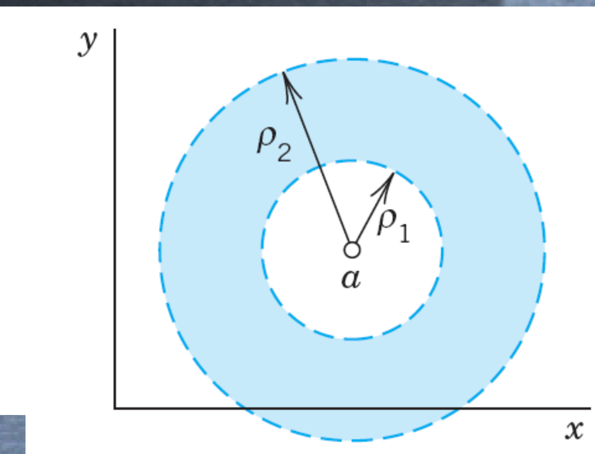
- Example: $w = \sqrt[3]{1}$,

$$w = 1, \omega, \omega^2,$$

  where $\omega = e^{i\frac{2\pi}{3}}$.

Complex Variables: Circle in the complex plane





- Circle:

$$|z - a| = \rho$$

- Open (closed) circular disk:

$$|z - a| < \rho \quad \text{(open)}, \qquad |z - a| \le \rho \quad \text{(closed)},$$

- An open circular disk $|z - a| < \rho$ is also called a *neighborhood* of a, or a ρ-neighborhood of a.

- Open (closed) annulus (circular ring):

$$\rho_1 < |z - a| < \rho_2 \quad \text{(open)}, \qquad \rho_1 \le |z - a| \le \rho_2 \quad \text{(closed)},$$

- Half-planes:

$$y > 0\,(y < 0), \qquad \text{upper half-plane (lower half-plane)},$$
$$x > 0\,(x < 0), \qquad \text{right half-plane (left half-plane)}.$$

Complex Variables: Sets in the complex plane

- Neighborhood of a:
  An open circular disk $|z - a| < \rho$ is also called a *neighborhood* of a, or a ρ-neighborhood of a.

- Open set S:
  Every point of S has a neighborhood only consisting of points belonging to S.
  E.g. $|z| < 1$ is open, $|z| \leq 1$ is not open.

- Connected set S:
  Any two of its points can be joined by a broken line (linear segments) within S.
  E.g. $\{|z| < 1 \text{ and } |z - 3| < 1\}$ is NOT connected.

- Domain: An open connected set.

- Complement of a set S:
  The set of all points of the complex plane that *do not belong* to S.

- Closed set S:
  A set S is called closed if its complement is open.

- Boundary point:
  A boundary point of a set S is a point every neighborhood of which contains both points that belong to S and points that do not belong to S.

Complex Variables: Complex function

- For a set of complex numbers, S,
- a function f defined on S is that

$$w = f(z),$$

- z varies in S, and is called a complex variable.
- The set S is called the domain of definition of f.
  In most cases S will be open and connected.
- The set of all values of a function f is called the *range* of f.
- Example:

$$w = f(z) = z^2 + 3\,z,$$

  is a complex function defined for all z.

Complex Variables: Complex function and Limit

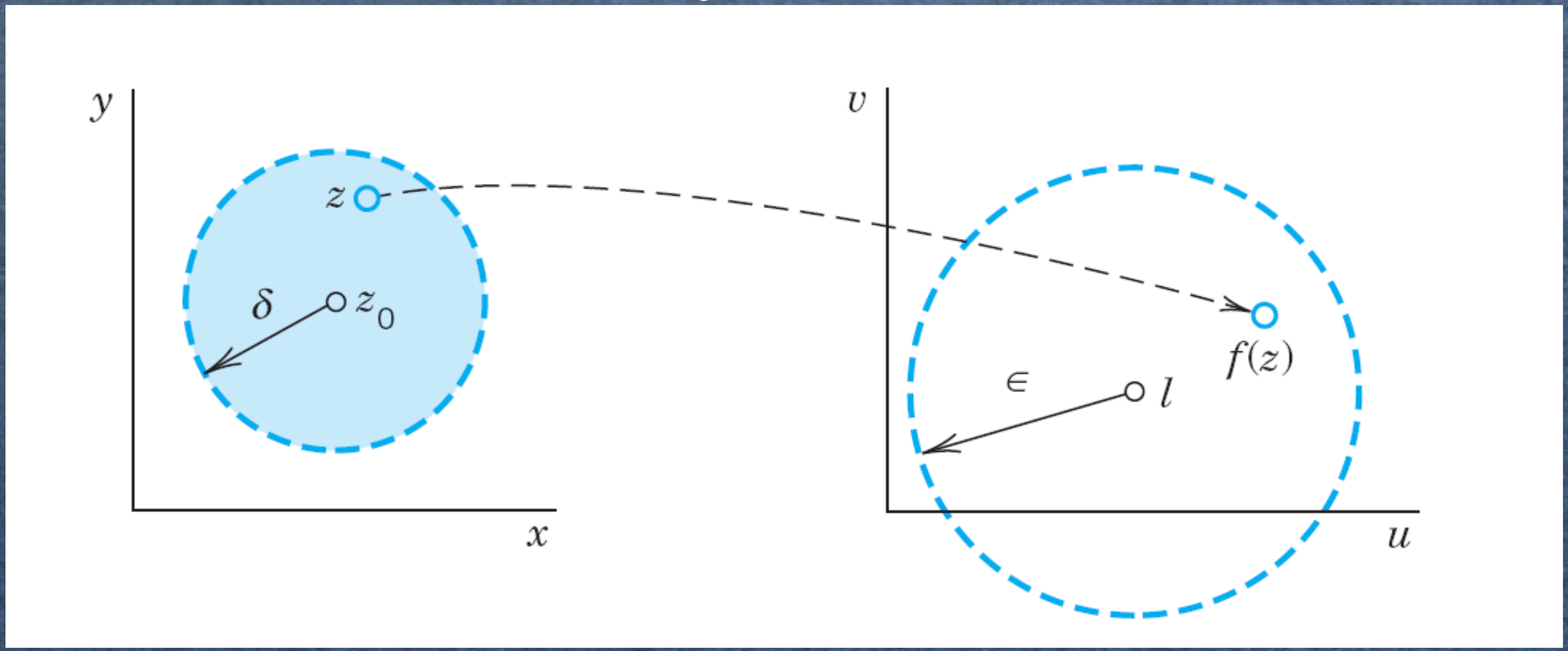
- A function $f(z)$ is said to have the limit l as z approaches a point z_0,

$$\lim_{z \to z_0} f(z) = l,$$

- Unlike the calculus, z may approach z_0 from any direction in the complex plane.

- Continuous: A function $f(z)$ is said to be continuous at $z = z_0$ if,

$$\lim_{z \to z_0} f(z) = f(z_0).$$



Complex Variables: Complex function, Derivative

- The derivative of a complex function f at a point z_0 is,

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

- Example 1: $f(z) = z^2$

$$f'(z) = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} = 2\,z$$

- Example 2: $f(z) = \overline{z}$

$$f'(z) = \lim_{\Delta z \to 0} \frac{\overline{(z + \Delta z)} - \overline{z}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z} = \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

not differentiable.

- Analyticity: A function $f(z)$ is said to be analytic in a domain D if $f(z)$ is defined and differentiable at all points of D.

Complex Variables: Cauchy-Riemann Equations

- For a complex function:

$$w = f(z) = u(x, y) + i\, v(x, y),$$

the criterion (test) for the analyticity is the Cauchy-Riemann equations,

$$u_x = v_y, \qquad \text{and} \qquad u_y = -v_x.$$

- f is analytic in a domain D if and only if the first partial derivatives of u and v satisfy the two Cauchy-Riemann equations.

- Example 1: $f(z) = z^2 = x^2 - y^2 + 2\, i\, x\, y,$

$$u_x = 2x = v_y, \qquad u_y = -2y = -v_x.$$

- Example 2: $f(z) = \bar{z} = x - i\, y,$

$$u_x = 1 \neq v_y = -1, \qquad u_y = -v_x = 0.$$

Complex Variables: Cauchy-Riemann Eq., Proof

- By assumption, the derivative $f'(z)$ at z exists, $f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z}$.
- Write $\Delta z = \Delta x + i\Delta y$,

$$f'(z) = \lim_{\Delta z \to 0} \frac{[u(x+\Delta x, y+\Delta y) + i\,v(x+\Delta x, y+\Delta y)] - [u(x,y) + i\,v(x,y)]}{\Delta x + i\Delta y},$$

- For the first path I: let $\Delta y \to 0$ first and then $\Delta x \to 0$,

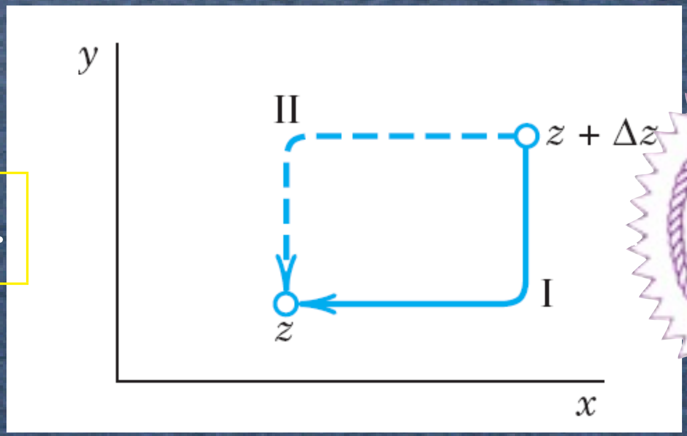
$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x} = u_x + i\,v_x,$$

- For the second path II: let $\Delta x \to 0$ first and then $\Delta y \to 0$,

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\,\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{i\,\Delta y} = -i\,u_y + v_y,$$

- the Cauchy-Riemann equations,

$$u_x = v_y, \qquad \text{and} \qquad u_y = -v_x.$$



Complex Variables: Exponential function

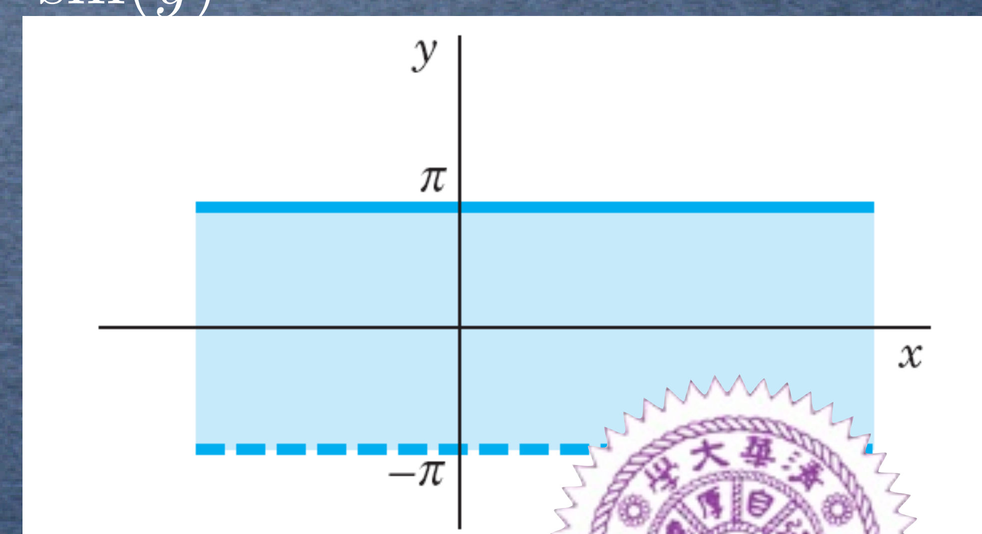
- The complex exponential function: e^z, or written as $\exp(z)$.
- e^z is analytical for all z, i.e., an entire function.
- Proof: by using the Cauchy-Riemann equations, i.e.,

$$e^z = e^x[\cos(y) + i\sin(y)],$$

where

$$u = e^x\cos(y), \qquad v = e^x\sin(y)$$

Many to 1



- The derivative of e^z is also e^z, i.e., $(e^z)' = e^z$.
- The expansion of $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots$
- $e^{z_1}e^{z_2} = e^{z_1+z_2}$,
- Periodicity of e^z with period $2\pi i$, i.e.,

$$e^{z+2\pi i} = e^z, \qquad \text{for all} z$$

Complex Variables: Trigonometric function

- Euler formulas:

$$
\begin{aligned}
e^{iz} &= \cos(z) + i\,\sin(z),\\
e^{-iz} &= \cos(z) - i\,\sin(z),
\end{aligned}
$$

- the complex trigonometric functions:

$$
\cos z = \frac{e^{iz} + e^{-iz}}{2},
$$
$$
\sin z = \frac{e^{iz} - e^{-iz}}{2i},
$$

- $\cos z$, $\sin z$ are analytic for all z, but $\tan z$ is not wherever $\cos z = 0$.

- General formulas of real trigonometric functions remain valid for complex counterparts, i.e.,

$$
\begin{aligned}
\cos(z_1 \pm z_2) &= \cos z_1\, \cos z_2 \mp \sin z_1\, \sin z_2,\\
\sin(z_1 \pm z_2) &= \sin z_1\, \cos z_2 \pm \cos z_1\, \sin z_2.
\end{aligned}
$$

Complex Variables: Hyperbolic function

- The complex hyperbolic cosine and sine:

$$\cosh z = \frac{e^z + e^{-z}}{2},$$
$$\sinh z = \frac{e^z - e^{-z}}{2},$$
$$\tanh z = \frac{\sinh z}{\cosh z},$$

- These functions are *entire*, with derivatives

$$(\cosh z)' = \sinh z,$$
$$(\sinh z)' = \cosh z,$$

- Relations between complex trigonometric and hyperbolic functions,

$$\cosh(i\,z) = \cos z, \qquad \sinh(i\,z) = i \sin z,$$
$$\cos(i\,z) = \cosh z, \qquad \sin(i\,z) = i \sinh z,$$

Complex Variables: Logarithm

- The natural logarithm of $z = x + i\,y$ is denoted by $\ln z$ or $\log z$,
- Define:

$$e^{w} = e^{u+i\,v} = r\,e^{i\theta} = z,$$

then

$$\ln z = \ln r + i\theta, \qquad (r = |z| > 0, \theta = \arg z),$$

- Since the argument of z is determined only determined only up to integer multiples of 2π,
- the complex natural logarithm $\ln z (z \neq 0)$ is infinitely many-valued.
- Principal value of $\ln z$,

$$\boxed{\mathrm{Ln} z = \ln|z| + i\,\mathrm{Arg}(z)}$$

- Other values of $\ln z$ are

$$\ln z = \mathrm{Ln} z \pm 2n\pi i, \qquad n = 1, 2, \ldots$$

Complex Variables: Logarithm, Example

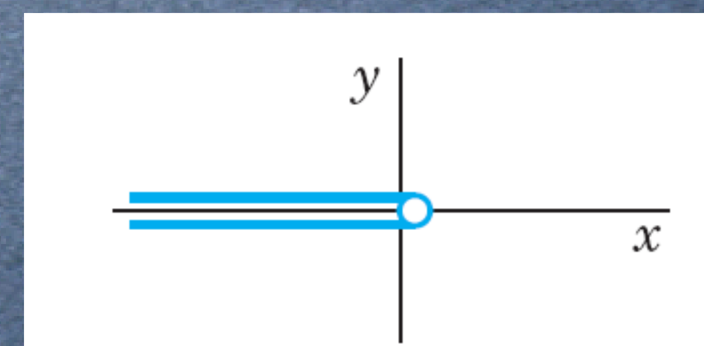
- Examples:

$$\mathrm{Ln}(1) = 0, \qquad \ln 1 = 0, \pm 2\pi i, \pm 4\pi i, \ldots$$
$$\mathrm{Ln}(-1) = \pi i, \qquad \ln 1 = \pm \pi i, \pm 3\pi i, \pm 5\pi i, \ldots$$
$$\mathrm{Ln}(i) = \pi i/2, \qquad \ln i = \pi i/2, -3\pi i/2, 5\pi i/2, \ldots$$

- Relations:

1 to Many

$$\ln(z_1\, z_2) = \ln z_1 + \ln z_2,$$
$$\ln\left(\frac{z_1}{z_2}\right) = \ln z_1 - \ln z_2,$$



- Analyticity of the Logarithm: for every $n = 0, \pm 1, \pm 2, \ldots,$

$$\ln z = \mathrm{Ln} z \pm 2n\pi i, \qquad n = 1, 2, \ldots$$

is analytic, except at 0 and on the negative real axis.

- Each of the infinitely n is call a brach of the logarithm.

Complex Variables: Logarithm, Derivative

- If z is negatively real (where real logarithm is undefined), $\text{Ln}z = \ln|z| + i\pi$.
- The derivative:

$$(\ln z)' = \frac{1}{z}, \qquad (z \quad \text{not } 0 \text{ or negative real}).$$

- Proof:

$$\ln z = \ln r + i(\theta + c) = \frac{1}{2}\ln(x^2 + y^2) + i[\tan^{-1}\frac{y}{x} + c],$$

where the constant c is a multiple of 2π. By the Cauchy-Riemann equations, i.e., $u_x = v_y, \qquad u_y = -v_x$, and

$$(\ln z)' = u_x + i\, v_x = \frac{x - i\, y}{x^2 + y^2} = \frac{1}{z}.$$

Complex Variables: General Powers

- The general powers of a complex number,

$$z^c = e^{c \ln z}, \qquad c \quad \text{is complex and} \quad z \neq 0.$$

- Principle value:

$$z^c = e^{c\,\mathrm{Ln} z},$$

- Example 1:

$$i^i = e^{i \ln i} = \exp[i(\frac{\pi}{2} i \pm 2\, n\pi i)] = e^{-\pi/2 \mp 2n\pi},$$

the principal value (n = 0) is $e^{-\pi/2}$.

- Example 2:

$$\begin{aligned}(1+i)^{2-i} &= \exp[(2-i)(\ln\sqrt{2} + \frac{1}{4}\pi i \pm 2\, n\pi i)] \\ &= 2e^{\pi/4 \pm 2\, n\pi}[\sin(\frac{1}{2}\ln 2) + i\, \cos(\frac{1}{2}\ln 2)]\end{aligned}$$

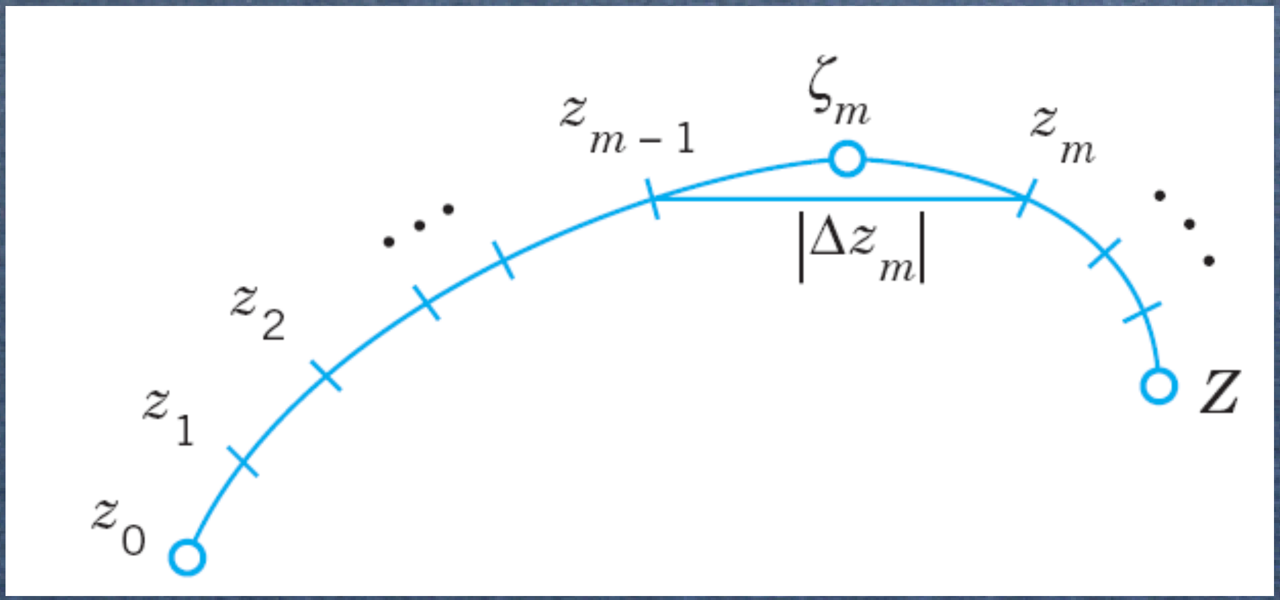
Complex Variables: Complex Integral, Ch. 14

- Complex definite integral are called line integrals,

$$\int_C f(z) \mathrm{d}\, z,$$

- The line integral over curve $C : \{z(t) = x(t) + i\, y(t)\}$ in the complex plane is defined as the limit of partial sum,

$$\int_C f(z) \mathrm{d}\, z \approx \sum_{m=1}^{n} f(\xi_m)\, \Delta z_m$$



$$\int_{z_0}^{z_1} f(z)\mathrm{d}\, z = F(z_1) - F(z_2), \qquad F'(z) = f(z)$$

- Examples:

$$\int_0^{1+i} z^2 \mathrm{d}\, z = \frac{-2}{3} + \frac{2}{3} i,$$

$$\int_{-\pi i}^{\pi i} \cos z \mathrm{d}\, z = 2 \sin \pi i,$$

$$\int_{-i}^{i} \frac{1}{z} \mathrm{d}\, z = \mathrm{Ln} z|_{-i}^{i} = i\pi,$$

Complex Variables: Complex Integral, Path

- Let C be a piecewise smooth path, represented by $z = z(t)$, where $a \leq t \leq b$,

$$\int_C f(z)\mathrm{d}\,z = \int_a^b f[z(t)]\dot{z}(t)\mathrm{d}\,t, \qquad \dot{z} = \frac{\mathrm{d}\,z}{\mathrm{d}\,t}$$

- Example 1: around the Unit Circle

$$\oint_C \frac{\mathrm{d}\,z}{z},$$

- Use $z(t) = \cos t + i \sin t = e^{it}$, $0 \leq t \leq 2\pi$, i.e.,

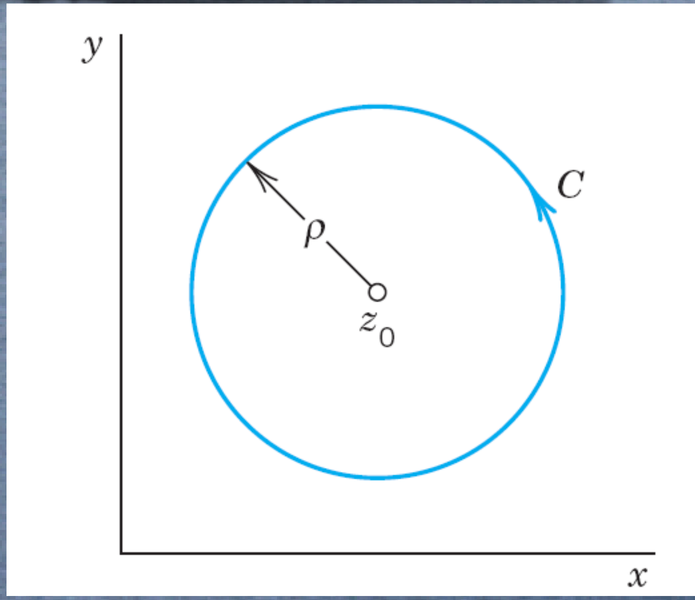
$$\oint_C \frac{\mathrm{d}\,z}{z} = \int_0^{2\pi} e^{-it} i\, e^{it}\mathrm{d}\,t = i\int_0^{2\pi} \mathrm{d}\,t = 2\pi i,$$

- Compare with $\mathrm{Ln} z|_{z_1}^{z_1} = 0$.
- Now $1/z$ is not analytic at $z = 0$. But any simply connected domain containing the unit circle must contain $z = 0$.

Complex Variables: Complex Integral, around a circle

- Example 2:

$$\oint_C (z - z_0)^m \mathrm{d}\, z$$



- Represent C in the form,

$$z(t) = z_0 + \rho(\cos t + i\,\sin t) = z_0 + \rho\, e^{i\,t},$$

then

$$\begin{aligned}\oint_C (z - z_0)^m \mathrm{d}\, z &= \int_0^{2\pi} \rho^m e^{i\,mt} i\rho e^{i\,t} \mathrm{d}\, t, \\ &= i\rho^{m+1} \int_0^{2\pi} e^{i(m+1)t} \mathrm{d}\, t, \\ &= \begin{cases} 2\pi i & ; \quad m = -1 \\ 0 & ; \quad m \neq -1 \text{ and integer} \end{cases}\end{aligned}$$

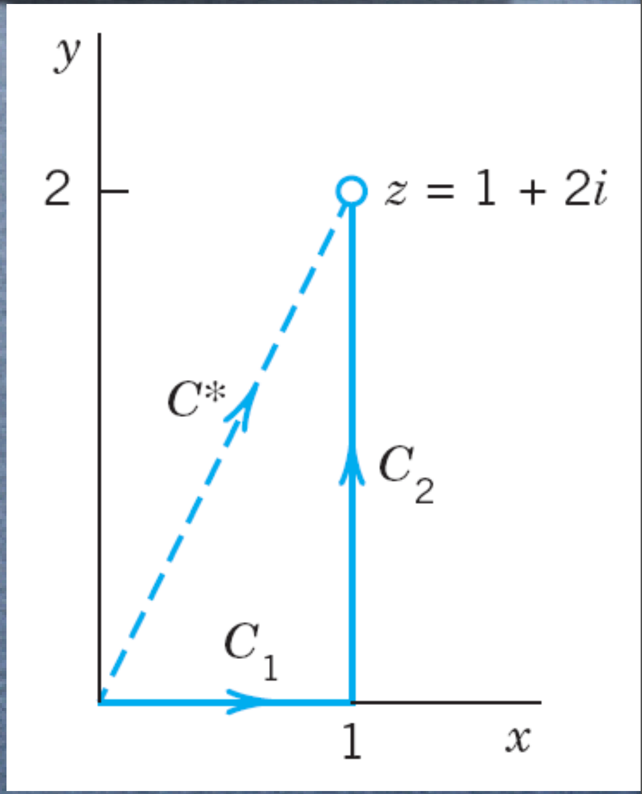
Complex Variables: Complex Integral, Non-analytic

- Example 3:

$$\int_C \mathrm{Re}(z)\mathrm{d}\,z$$

- Path 1: C^*, where $z(t) = t + 2i\,t$, $(0 \le t \le 1)$,

$$\int_C \mathrm{Re}z\mathrm{d}\,z = \int_0^1 t(1+2i)\mathrm{d}\,t = \frac{1}{2} + i,$$



- Path 2: C_1 and C_2, where $z(t) = t$ and $z(t) = 1 + i\,t$,

$$\int_C \mathrm{Re}z\mathrm{d}\,z = \int_{C_1} \mathrm{Re}z\mathrm{d}\,z + \int_{C_2} \mathrm{Re}z\mathrm{d}\,z,$$

$$= \int_0^1 t\mathrm{d}\,t + \int_0^2 i\mathrm{d}\,t = \frac{1}{2} + 2i.$$

Homework #17:

1. **[Complex Numbers and Functions]**:
   Find the principal value

   (a) $(1-i)^{1+i}$

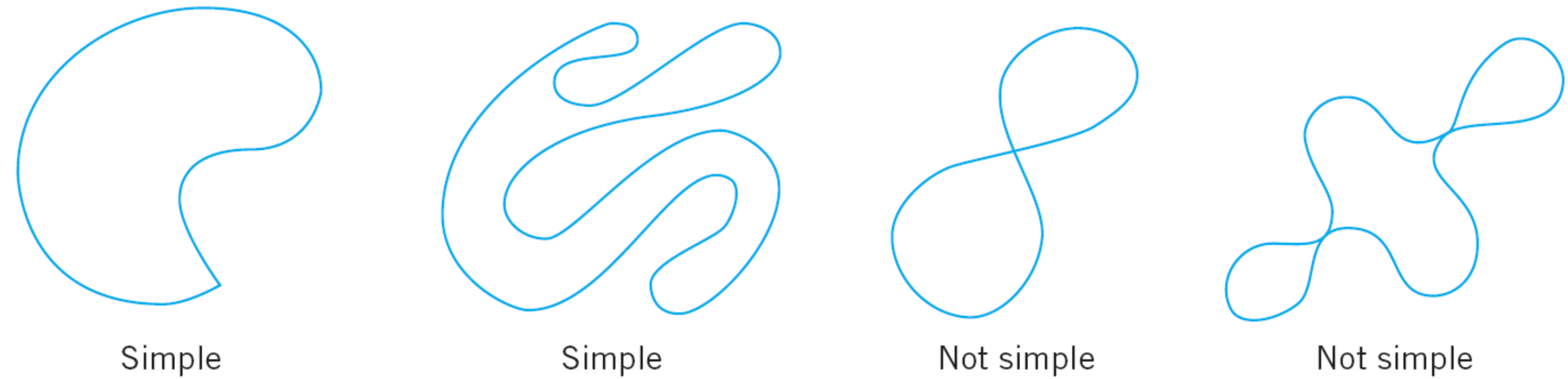
2. **[Line Integrations]**:

   (a) $\int_C (z+z^{-1})\,\mathrm{d}\,z$, C the unit circle (counterclockwise),

   (b) $\int_C \cosh(4z)\,\mathrm{d}\,z$, C any path from $-\pi i/8$ to $\pi i/8$,

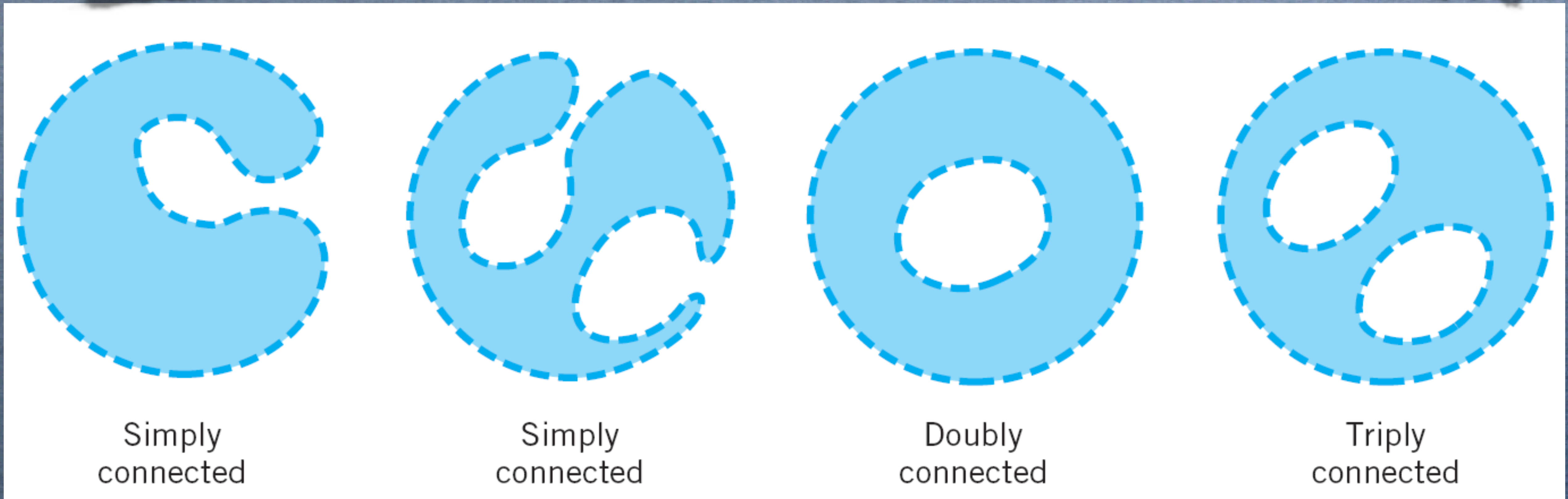
   (c) $\int_C \mathrm{Im}(z^2)\,\mathrm{d}\,z$, C around the triangle with vertices $z = 0, 1, i$ (counterclockwise),

Complex Variables: Closed paths



Simple closed path: a closed path that does not intersect or touch itself.

Complex Variables: Domain

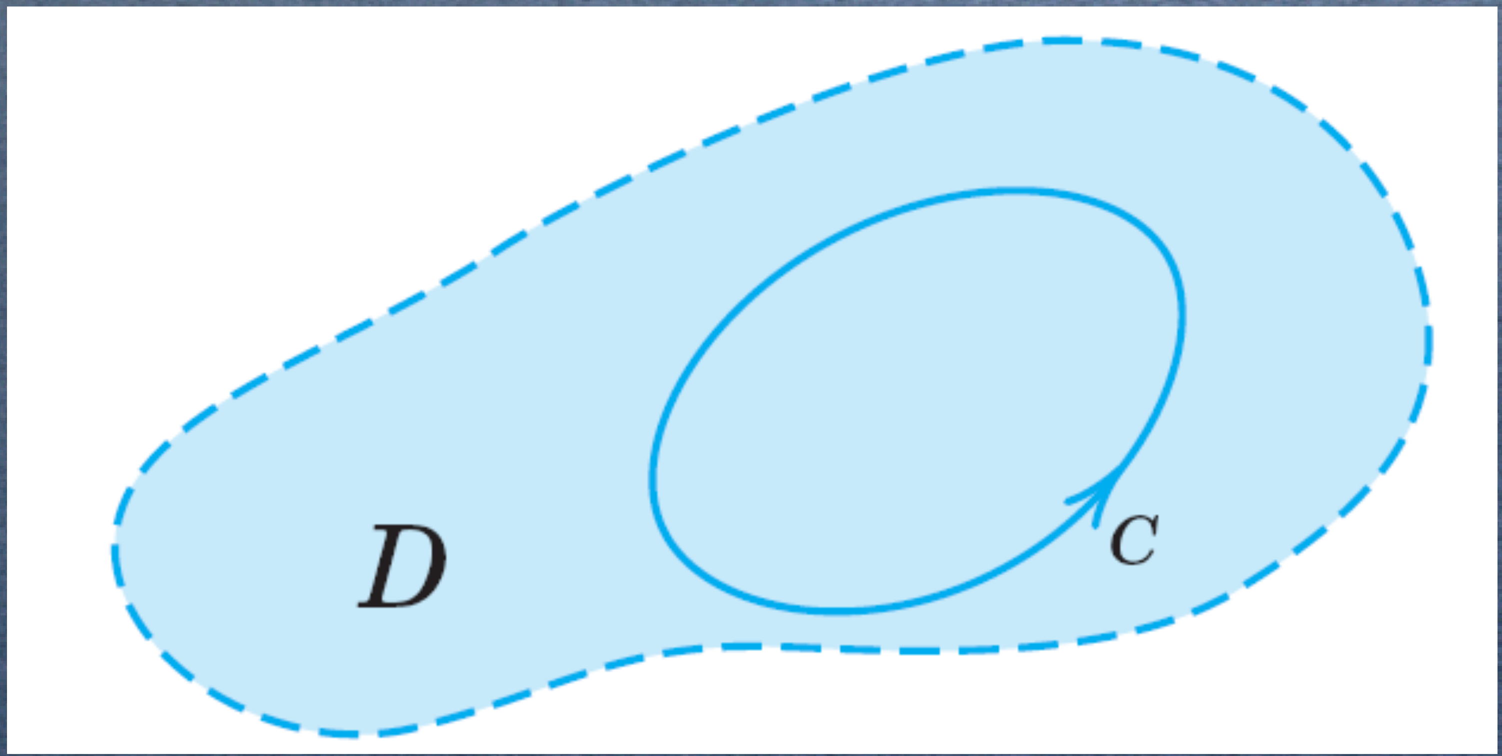


Simply connected domain D: a domain such that every simple closed path in D enclosed only points of D.

Complex Variables: Cauchy integral theorem

- If $f(z)$ is analytic in a simply connected domain D, then for every simple closed path C in D,

$$\oint_C f(z)\,\mathrm{d}\,z = 0.$$



Complex Variables: Cauchy integral theorem, Proof

- Proof:

$$\oint_C f(z)\,\mathrm{d}\,z = \oint_C [u\,\mathrm{d}\,x - v\,\mathrm{d}\,y] + i\oint_C [u\,\mathrm{d}\,y + v\,\mathrm{d}\,x].$$

Since $f(z)$ is analytic in D, its derivative $f'(z)$ exists in D.

- Green's theorem:

$$\oint_C \vec{F}\cdot \mathrm{d}\vec{r} = \iint_R \nabla\times\vec{F}\cdot\vec{k}\,\mathrm{d}\,x\mathrm{d}\,y$$

$$\oint_C [F_1\,\mathrm{d}\,x + F_2\,\mathrm{d}\,y] = \iint_R [\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}]\mathrm{d}\,x\mathrm{d}\,y$$

- Replace F_1 and F_2 in the Green's theorem by u and $-v$,

$$\oint_C [u\,\mathrm{d}\,x - v\,\mathrm{d}\,y] = \iint_R [-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}]\mathrm{d}\,x\mathrm{d}\,y = 0.$$

by using the Cauchy-Riemann equations.

- Replace F_1 and F_2 in the Green's theorem by v and u,

$$\oint_C [u\,\mathrm{d}\,y + v\,\mathrm{d}\,x] = \iint_R [\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}]\mathrm{d}\,x\mathrm{d}\,y = 0.$$

by using the Cauchy-Riemann equations again.

Complex Variables: Cauchy integral theorem, Example

- Example 1: No singularities (entire functions),

$$\oint_C e^z \,\mathrm{d}\, z = 0, \qquad \oint_C \cos z \,\mathrm{d}\, z = 0, \qquad \oint_C z^n \,\mathrm{d}\, z = 0, \qquad (n = 0, 1, \dots).$$

- Example 2: Singularities outside the contour,

$$\oint_C \sec(z) \,\mathrm{d}\, z = 0, \qquad C \text{ is the unit circle.}$$

$$\oint_C \frac{1}{z^2+4}, \mathrm{d}\, z = 0.$$

- Example 3: nonanalytic function, C is the unit circle,

$$\oint_C \mathrm{Re}(z) \,\mathrm{d}\, z = \int_0^{2\pi} e^{-it}\, i\, e^{it} \,\mathrm{d}\, t = 2\pi i.$$

- Example 4: Simple connectedness essential,

$$\oint_C \frac{1}{z} \,\mathrm{d}\, z = 2\pi i.$$

Complex Variables: Independence of Path

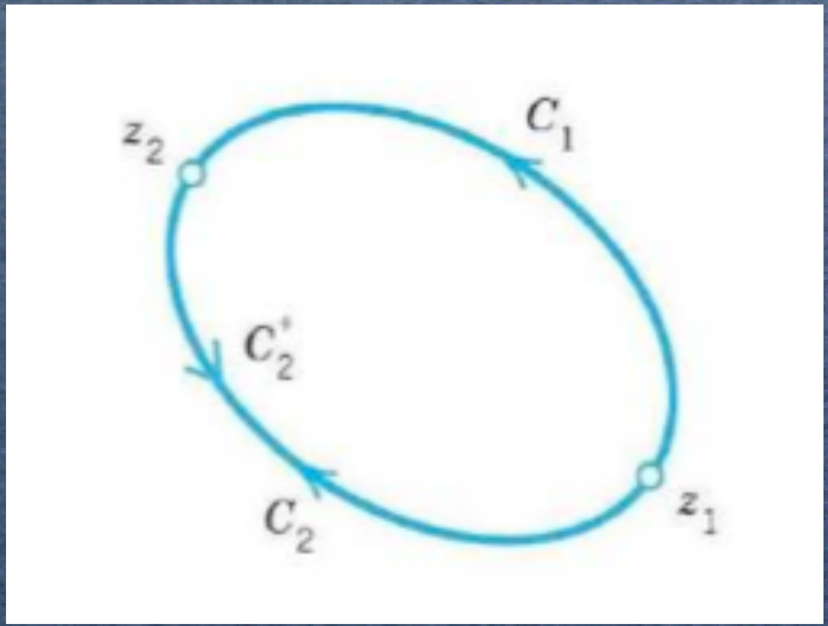
- Independence of Path:
  If $f(z)$ is analytic in a simply connected domain D, then the integral of $f(z)$ is independent of path in D.

- Proof:

$$\int_{C_1} f\,\mathrm{d}\,z + \int_{C_2^*} f\,\mathrm{d}\,z = 0$$

thus,

$$\int_{C_1} f\,\mathrm{d}\,z = -\int_{C_2^*} f\,\mathrm{d}\,z = \int_{C_2} f\,\mathrm{d}\,z$$

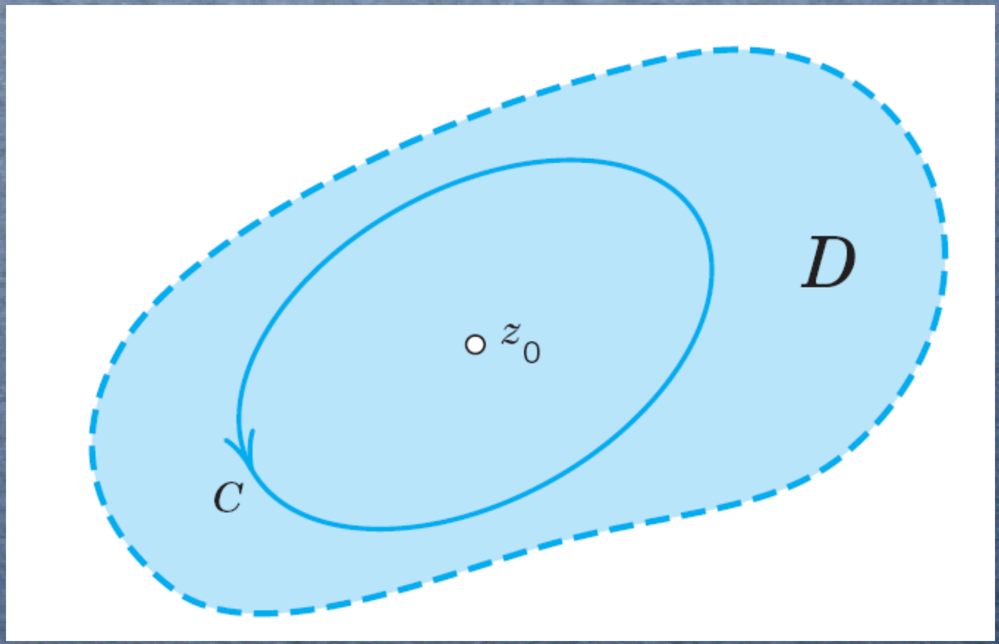


Complex Variables: Cauchy's integral formula

- Let $f(z)$ be analytic in a simply connected domain D. Then for any points z_0 in D and any simple closed path C in D that enclose z_0,

$$\oint_C \frac{f(z)}{z - z_0} \mathrm{d}\, z = 2\pi\, i\, f(z_0).$$

where the integration being taken *counterclockwise*.



- Alternatively for $f(z_0)$,

$$f(z_0) = \frac{1}{2\pi\, i} \oint_C \frac{f(z)}{z - z_0} \mathrm{d}\, z,$$

- Proof: by replacing $f(z) = f(z_0) + [f(z) - f(z_0)]$,

$$\begin{aligned} \oint_C \frac{f(z)}{z - z_0} \mathrm{d}\, z &= f(z_0) \oint_C \frac{1}{z - z_0} \mathrm{d}\, z + \oint_C \frac{f(z) - f(z_0)}{z - z_0} \mathrm{d}\, z \\ &= 2\pi\, i\, f(z_0) + \frac{\epsilon}{\rho} 2\pi \rho, \\ &= 2\pi\, i\, f(z_0) \end{aligned}$$

Complex Variables: Complex Integral, ML - Inequality

- ML-inequality:

$$\left|\int_C f(z)\,\mathrm{d}\,z\right| \le M\,L$$

where L is the length of C and M is a constant such that $|f(z)| \le M$ everywhere on C.

- Example: estimation of an integral (find an upper bound),

$$\int_C z^2\,\mathrm{d}\,z, \qquad C \quad \text{the straight-line segment from 0 to} \quad 1+i$$

- Solution: $L = \sqrt{2}$ and $|f(z)| = |z^2| \le 2$ on C,

$$\left|\int_C z^2\,\mathrm{d}\,z\right| \le 2\sqrt{2}.$$

Complex Variables: Cauchy integral formula, Example

- Example 1:

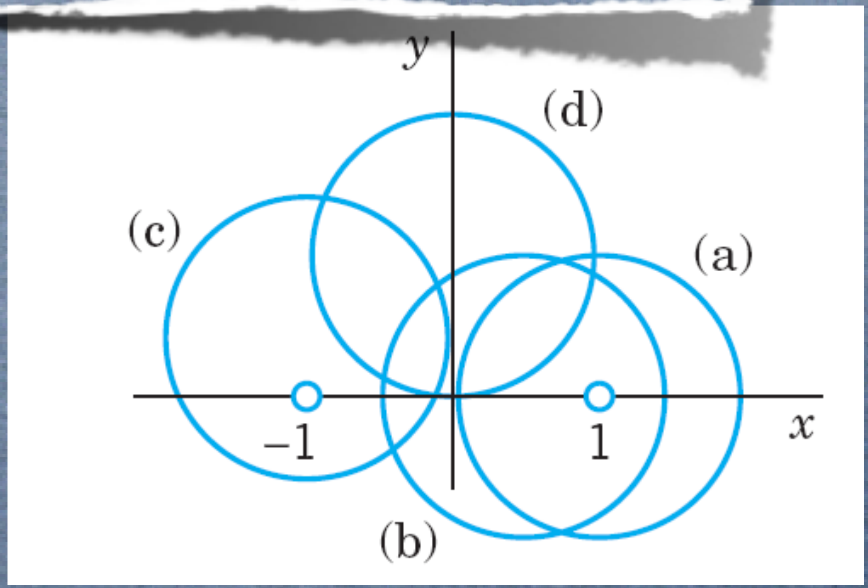
$$\oint_C \frac{e^z}{z-2}\mathrm{d}\,z = 2\pi\,i\,e^z|_{z=2} = 2\pi\,i\,e^2.$$

- Example 2:

$$\oint_C \frac{z^3-6}{2z-i}\mathrm{d}\,z = 2\pi\,i\,[\frac{1}{2}z^3-3]|_{z=i/2} = \frac{\pi}{8} - 6\pi\,i.$$

Complex Variables: Cauchy's integral formula, cont.

$$\oint_C \frac{z^2+1}{z^2-1}\mathrm{d}\,z$$



- (a): The circle $|z-1|=1$, enclosing the point $z_0=+1$

$$\oint_C \frac{z^2+1}{z^2-1}\mathrm{d}\,z = \oint_C [\frac{z^2+1}{z+1}][\frac{1}{z-1}]\mathrm{d}\,z = 2\pi\,i\,[\frac{z^2+1}{z+1}]|_{z=1} = 2\pi\,i.$$

- (b): Enclosing the point $z_0=+1$, gives the same as (a) by the principle of deformation of path.

- (c): The path encloses the point $z_0=-1$,

$$\oint_C \frac{z^2+1}{z^2-1}\mathrm{d}\,z = \oint_C [\frac{z^2+1}{z-1}][\frac{1}{z+1}]\mathrm{d}\,z = 2\pi\,i\,[\frac{z^2+1}{z-1}]|_{z=-1} = -2\pi\,i.$$

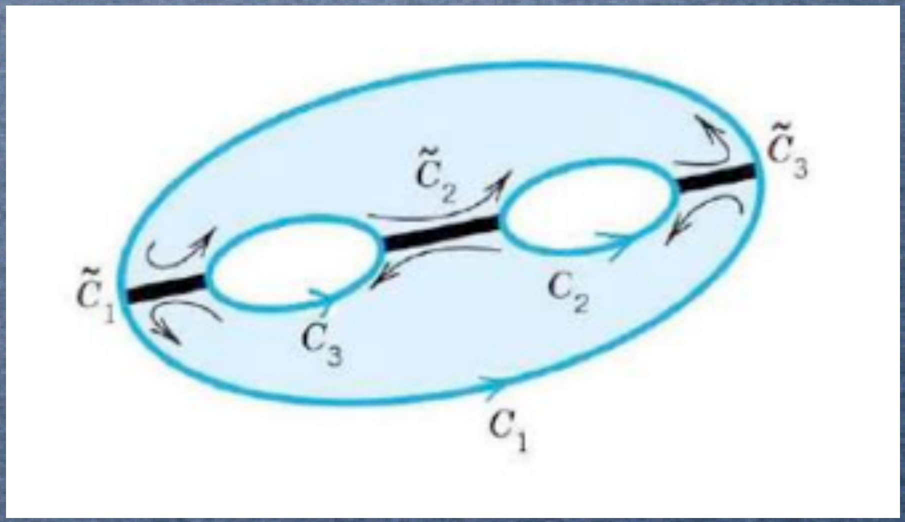
- (d): Don't enclose any singular point,

$$\oint_C \frac{z^2+1}{z^2-1}\mathrm{d}\,z = 0.$$

Complex Variables: Multiple-connected domain

- Cauchy's theorem applies to multiply connected domains.
- If $f(z)$ is analytic in a multiply connected domain D defined by an outer contour C_1 and multiple inner contours $C_i, i = 2, 3, \ldots, n$ (all are in counterclockwise sense),

$$\oint_{C_1} f \,\mathrm{d}\, z = \sum_{i=2}^{n} \oint_{C_i} f \,\mathrm{d}\, z$$

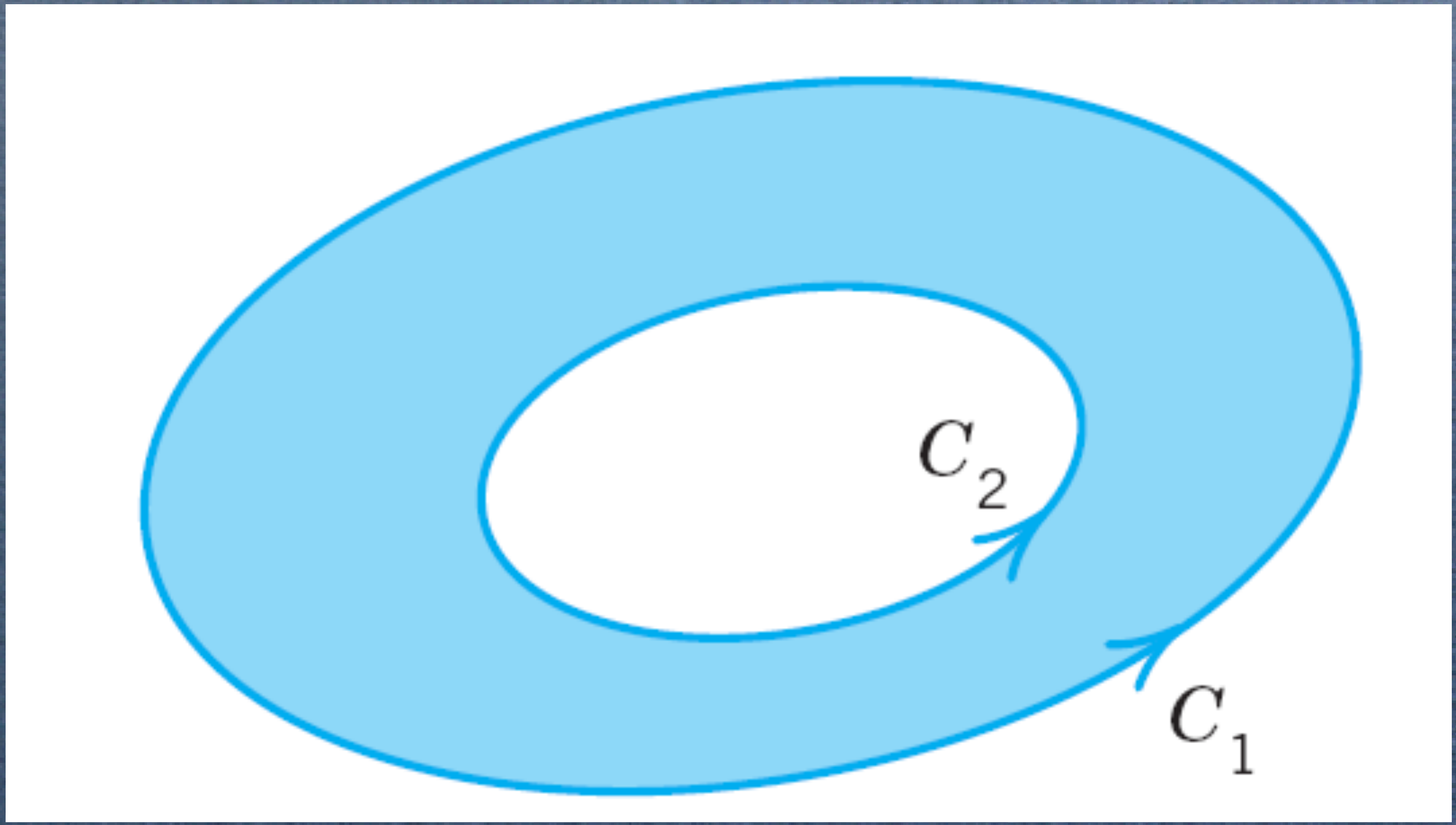


- Proof:
  Introducing three inner cuts $\tilde{C}_1, \tilde{C}_2, \tilde{C}_3$ to divide the domain D into two simply connected domains. Apply Theorem 1 to them, integral over cuts will be canceled.

Complex Variables: Multiple-connected domain

- If $f(z)$ is analytic in a doubly connected domain D bounded by two *counterclockwise* contours C_1, C_2,

$$f(z_0) = \frac{1}{2\pi i}[\oint_{C_1} \frac{f(z)}{z - z_0} \mathrm{d}\, z - \oint_{C_2} \frac{f(z)}{z - z_0} \mathrm{d}\, z].$$



Complex Variables: Derivatives of Analytic function

- If $f(z)$ is analytic in a domain D, then it has derivatives of all orders in D, which are then also analytic functions in D.
- The values of these derivatives at a point z_0 in D are given by

$$f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}\,\mathrm{d}\,z,$$
$$f''(z_0) = \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^3}\,\mathrm{d}\,z,$$

- and in general

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}\,z,$$

where C is any simple closed path in D that encloses z_0 and whose full interior belongs to D;

- We integrate *counterclockwise* around C.

Complex Variables: Derivatives, Proof

- The definition of the Derivative,

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z},$$

- By Cauchy's integral formula,

$$\begin{aligned} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} &= \frac{1}{2\pi i\, \Delta z}\left[\oint_C \frac{f(z)}{z - (z_0 + \Delta z)} \mathrm{d}\, z - \oint_C \frac{f(z)}{z - z_0} \mathrm{d}\, z\right], \\ &= \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0 - \Delta z)(z - z_0)} \mathrm{d}\, z \end{aligned}$$

- as $\Delta z \to 0$,

$$f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2} \mathrm{d}\, z,$$

Complex Variables: Contour integral

- Example 1: for any contour enclosing the point πi (counterclockwise)

$$\oint_C \frac{\cos z}{(z-\pi i)^2} \mathrm{d}\, z = 2\pi\, i\, (\cos z)'|_{z=\pi i} = 2\pi \sinh \pi .$$

- Example 2: for any contour enclosing the point $-i$ (counterclockwise)

$$\oint_C \frac{z^4 - 3z^2 + 6}{(z+i)^3} \mathrm{d}\, z = \pi i (z^4 - 3z^2 + 6)''|_{z=-i} = -18\pi i.$$

- Example 3: for any contour for which 1 lies inside and $\pm 2i$ lie outside (counterclockwise)

$$\oint_C \frac{e^z}{(z-1)^2(z^2+4)} \mathrm{d}\, z = 2\pi\, i\, (\frac{e^z}{z^2+4})'|_{z=1} = \frac{6\pi e}{25} i.$$

Complex Variables: Cauchy's inequality

- For $|f(z)| \leq M$ on C,

$$|f^{(n)}(z_0)| = \frac{n!}{2\pi\, i} | \oint_C \frac{f(z)}{(z-z_0)^{n+1}} \mathrm{d}\, z| \leq \frac{n!}{2\pi\, i} M \frac{1}{r^{n+1}} 2\pi r$$

- Cauchy's inequality

$$|f^{(n)}(z_0)| \leq \frac{n! M}{r^2}.$$

Complex Variables: Liouville's & Morera's theorems

- Liouville's theorem:
  If an entire function is bounded in absolute value in the whole complex plane, then this function must be a constant.

- Proof: By assumption, $|f(z)|$ is bounded, say, $|f(z)| < K$ for all z. Using Cauchy's inequality, wee see that $|f'(z_0) < K/r$.

- Since $f(z)$ is entire, this holds for every r, so that we can take r as large as we please and conclude that $f'(z_0) = 0$. Since z_0 is arbitrary, $f'(z) = 0$ for all z, then $f(z)$ is a constant.

- Morera's theorem:
  If $f(z)$ is continuous in a simply connected domain D and if

$$\oint_C f(z)\,\mathrm{d}\,z = 0,$$

for every closed path in D, then $f(z)$ is analytic in D.

Power Series: Ch. 15

- A power series, in powers of $x - x_0$, is an infinite series of the form:

$$\sum_{m=0}^{\infty} a_m (x - x_0)^m = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots$$

- x is a variable.
- $a_0, a_1, a_2, \cdots$ are constants, called the *coefficients* of the series.
- x_0 is a constant, called the *center* of the series.
- We shall assume that all variables and constants are *real*, c.p. Laurent Series.

Power Series: Theory

- A power series, in powers of $x - x_0$, is an infinite series of the form:

$$\sum_{m=0}^{\infty} a_m (x - x_0)^m = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots$$

- The nth partial sum is

$$S_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n.$$

- The remainder is

$$\begin{aligned} R_n(x) &= \sum_{m=0}^{\infty} a_m (x - x_0)^m - S_n(x) \\ &= a_{n+1}(x - x_0)^{n+1} + a_{n+2}(x - x_0)^{n+2} + \cdots \end{aligned}$$

Power Series: Convergence, Chap. 15.1

- A sequence:

$$z_1, z_2, \cdots$$

- A Series:

$$S_n(x) = \sum_{m=0}^{n} z_m.$$

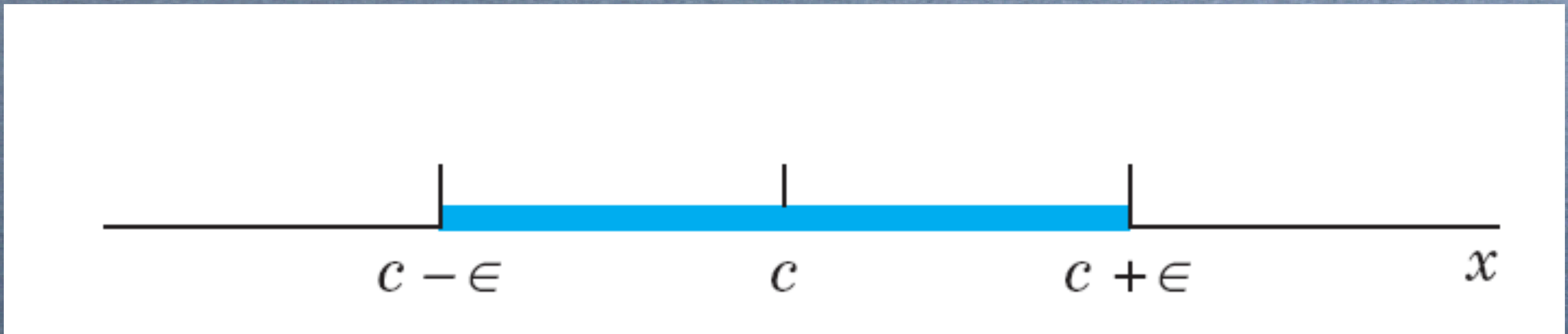
- A convergent sequence is one that has a limit,

$$\lim_{n\to\infty} z_n = c.$$

- By definition of *limit*, this means that for every $\epsilon > 0$, we can find an N such that,

$$|z_n - c| < \epsilon, \qquad \text{for all} \quad n > N;$$

Power Series: Convergent Sequence



- The sequence: $\{\frac{i^n}{n}\}$

$$i, -1/2, -i/3, 1/4, \cdots$$

is convergent with limit 0.

- The sequence: $\{i^n\}$

$$i, -1, -i, 1, \cdots$$

is divergent.

Power Series: Convergent Series

- A convergent series is one whose sequence of partial sums converges, say

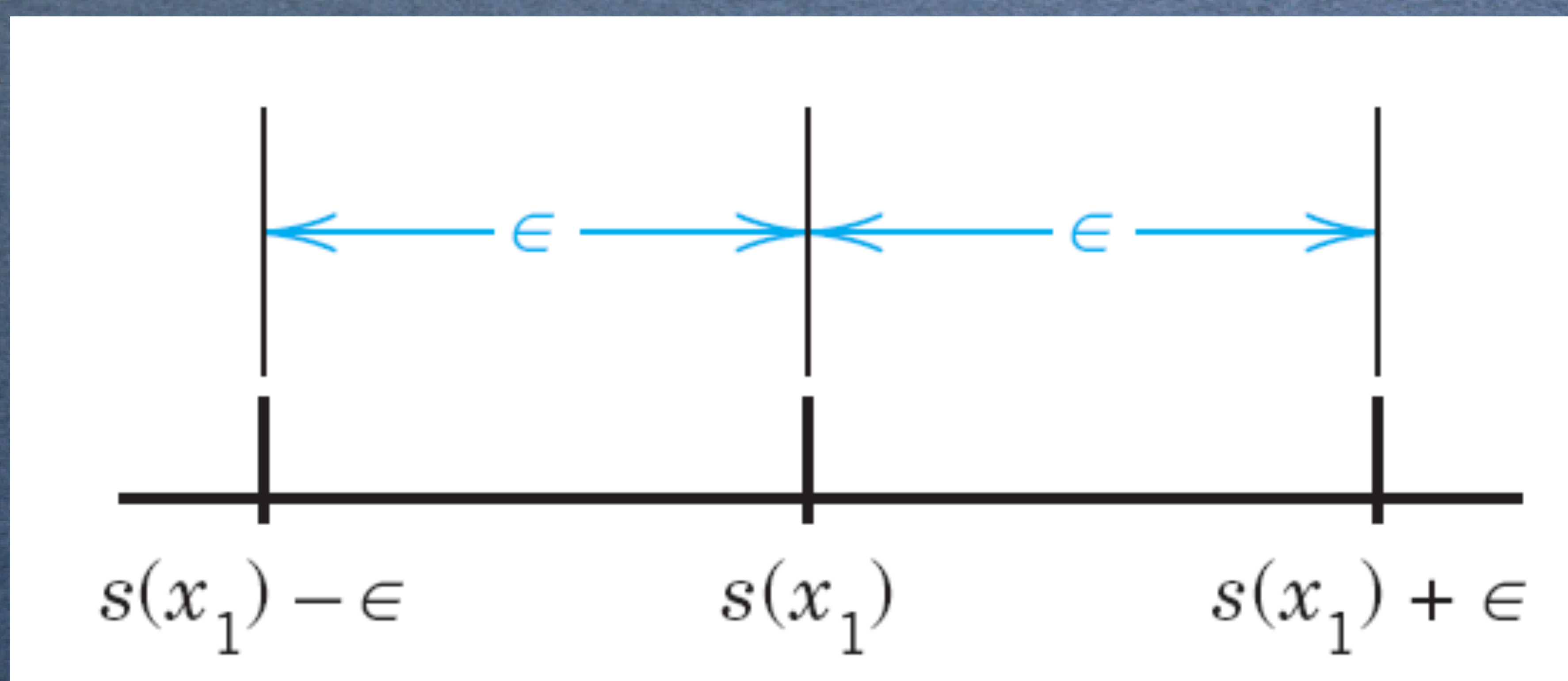
$$\lim_{n\to\infty} S_n = s = \sum_{m=1}^{\infty} z_m$$

- s is called the sum or *value* of the series.
- A series that is not convergent is called *divergent series.*

Power Series: Convergent Interval

- Cauchy's convergence principle for series:
  A series $z_1 + z_2 + \ldots$ is *convergent* if and only if for every given $\epsilon < 0$ (no matter how small) we can find an N (which depends on ϵ, in general) such that

$$|z_{n+1} + z_{n+2} + \cdots + z_{n+p}| < \epsilon, \qquad \text{for every} \quad n > N \quad \text{and} \quad p = 1, 2, \ldots$$



Power Series: Convergence Test

1. Comparison Test: i.e., *Harmonic Series*

$$\sum_{m=1}^{\infty} \frac{1}{m} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots, \qquad \text{divergent}$$

2. Integral Test: i.e., $\int_a^b \frac{1}{x^2}\,\mathrm{d}x$ converges,

$$\sum_{m=1}^{\infty} \frac{1}{m^2} = 1 + \frac{1}{4} + \frac{1}{9} + \ldots, \qquad \text{convergent}$$

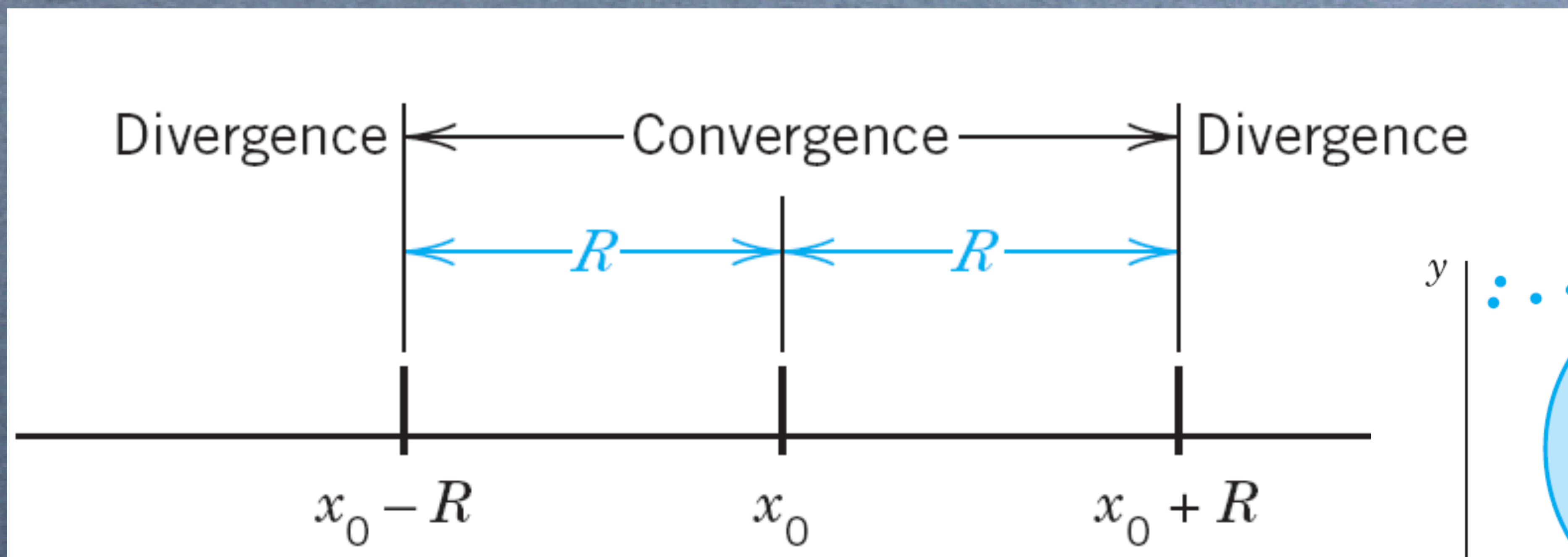
3. Ratio Test:

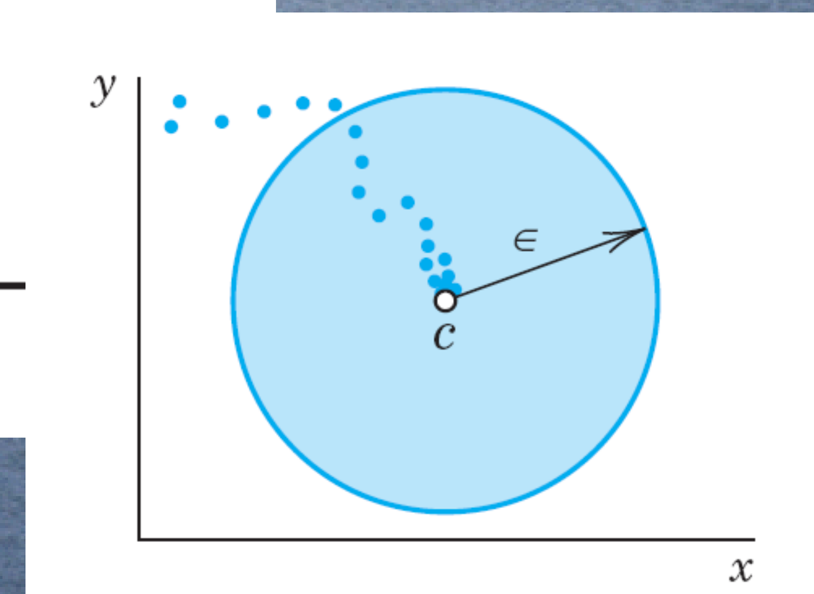
$$\sum_{m=1}^{\infty} \frac{n!n!}{(2m)!}, \qquad \text{convergent}$$

4. nth-Root Test:

$$\sum_{m=1}^{\infty} \frac{m^2}{2^m}, \qquad \text{convergent}$$

Power Series: Radius of Convergence, ROC





- For a series converges for all x such that $|x - x_0| < R$,
- and diverges for all x such that $|x - x_0| > R$, the interval defined by R is called the *convergence interval.*
- No general statement about convergence or divergence can be made for $x - x_0 = R$ or $-R$.
- The number R is called the radius of convergence.
- R is called "radius" because for a *complex* power series it is the radius of a disk of convergence.

Power Series: Radius of Convergence, cont.

- The radius of convergence R is defined as

$$R \equiv \frac{1}{\lim_{m\to\infty} \sqrt[m]{|a_m|}}, \qquad \text{or} \qquad R \equiv \frac{1}{\lim_{m\to\infty} \left|\frac{a_{m+1}}{a_m}\right|}.$$

Example:

$$\sum_{m=0}^{\infty} m! x^m$$

Solution:

- by Ratio test:

$$\frac{a_{m+1}}{a_m} = m + 1 \to \infty, \qquad \text{as} \quad m \to \infty$$

- This series converges only at the center $x = 0$, i.e., a useless series.

Power Series: Radius of Convergence, Example

Example 1:

Geometric series:

$$\sum_{m=0}^{\infty} x^m = \frac{1}{1-x}$$

Solution:

- by Ratio test:

$$\frac{a_{m+1}}{a_m} = 1,$$

- This series converges with a ROC, $R = 1$.

Example 2:

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{8^m} x^{3m}$$

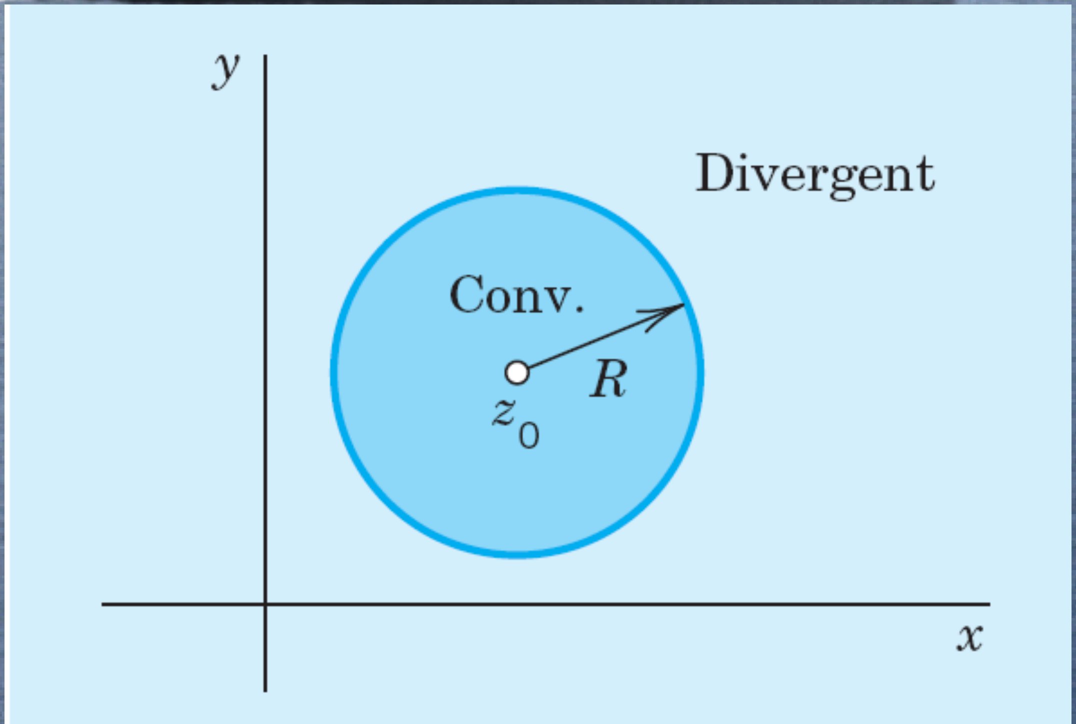
Solution:

- by Ratio test:

$$\frac{a_{m+1}}{a_m} = \frac{1}{8},$$

- This series converges with a ROC, $|x^3| < R = 8$, or $|x| < 2$.

Taylor Series: Circle of Convergence



- The Taylor series of a function $f(z)$ is,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \qquad \text{where} \qquad a_n = \frac{1}{n!} f^{(n)}(z_0),$$

- A Maclaurin series is a Taylor series with center $z_0 = 0$,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad \text{where} \qquad a_n = \frac{1}{n!} f^{(n)}(0),$$

Taylor Series: Proof

- The fundamental theorem of calculus states that

$$\int_a^x f'(t)\,\mathrm{d}t = f(x) - f(a),$$

which can be rearranged to,

$$f(x) = f(a) + \int_a^x f'(t)\,\mathrm{d}t.$$

- By integration by parts,

$$\begin{aligned} f(x) &= f(a) + (x-a)f'(a) + \int_a^x (x-t)f''(t)\,\mathrm{d}t, \\ &= f(a) + (x-a)f'(a) + \frac{1}{2}(x-a)^2 f''(a) + \frac{1}{2}\int_a^x (x-t)^2 f'''(t)\,\mathrm{d}t, \end{aligned}$$

- The Taylor series of a function $f(z)$ is,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \qquad \text{where} \qquad a_n = \frac{1}{n!} f^{(n)}(z_0),$$

Power Series: Common-used Maclaurin Series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{m=0}^{\infty} x^m,$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots = \sum_{m=0}^{\infty} (-1)^m x^m$$

Complex numbers:

x -> z

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots = \sum_{m=0}^{\infty} \frac{x^m}{m!}, \qquad |x| < \infty.$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!}, \qquad |x| < \infty.$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!}, \qquad |x| < \infty.$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = \sum_{m=1}^{\infty} \frac{(-1)^{m-1} x^m}{m}, \qquad -1 < x \le 1.$$

Taylor Series: with Cauchy's integral formula

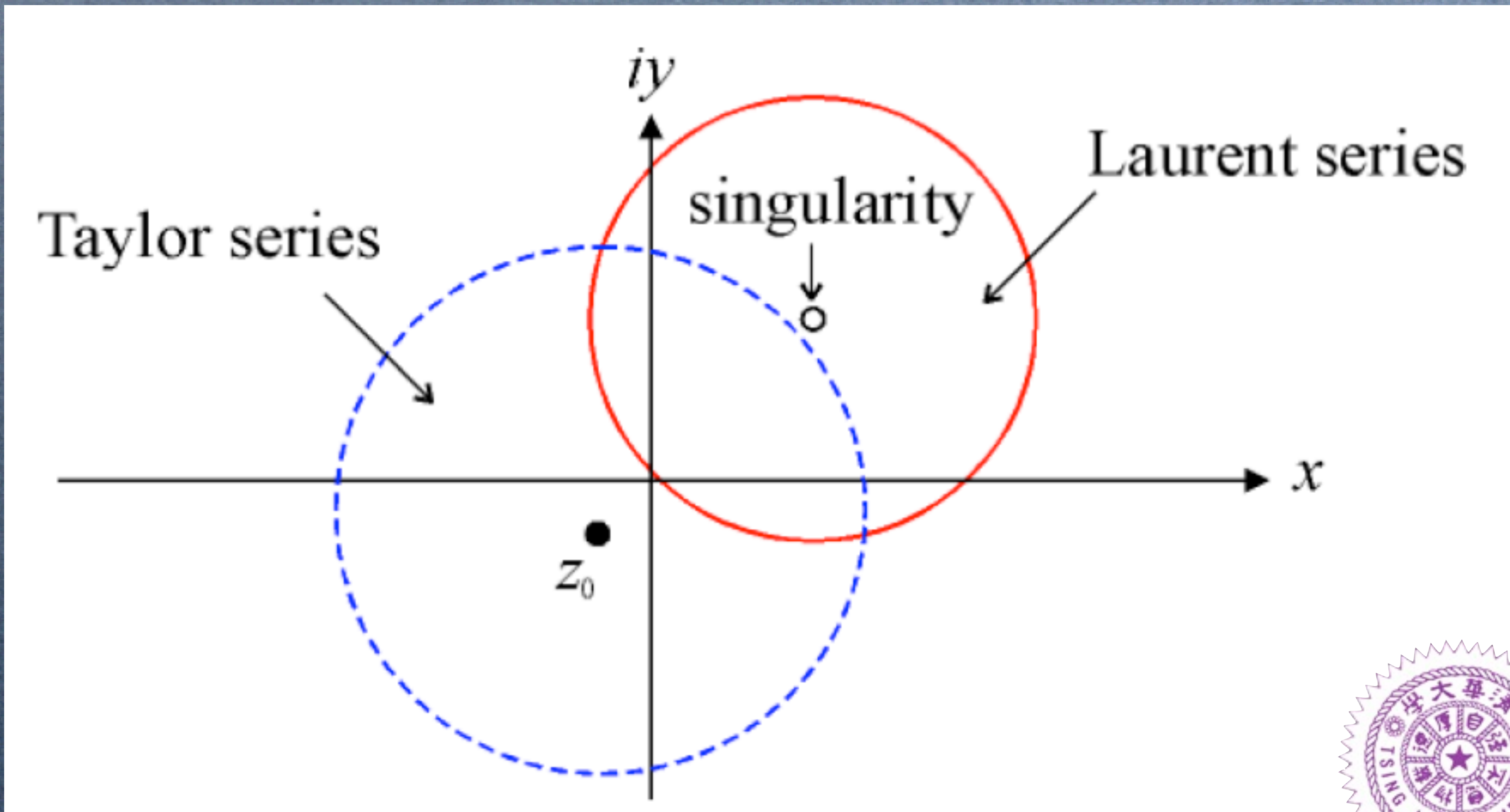
- The Taylor series of a function $f(z)$ is,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \qquad \text{where} \qquad a_n = \frac{1}{n!} f^{(n)}(z_0),$$

- By Cauchy's integral formula,

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} \mathrm{d}\, z.$$

Laurent Series: Expansion around the singularity

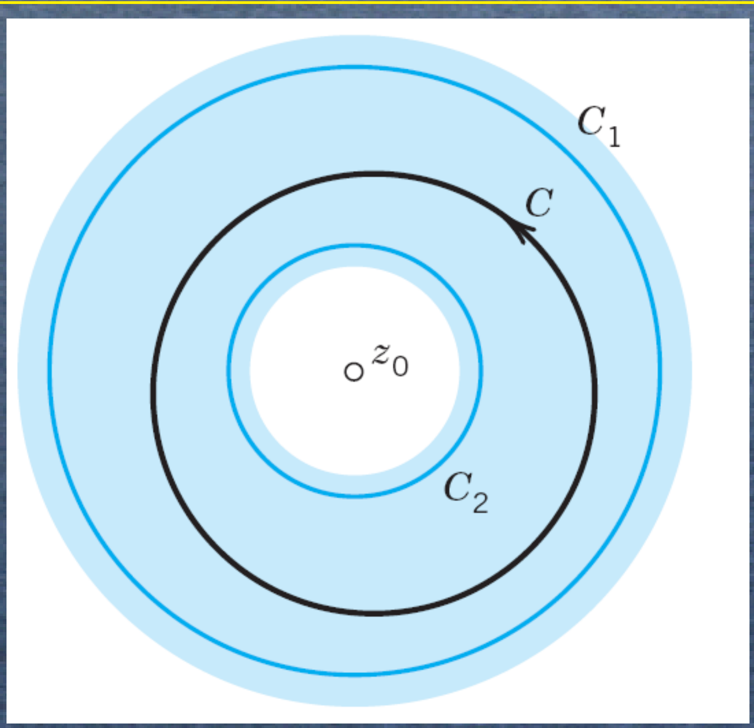


Laurent Series: Ch. 16

- Let $f(z)$ be analytic in a domain containing two concentric circles C_1 and C_2 with center z_0 and the annulus between them. Then $f(z)$ can be represented by the Laurent series,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n},$$

$$= a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots$$



Laurent Series: Laurent theorem

- The coefficients of this Laurent series are given by the integrals,

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}} \mathrm{d}\, z^*,$$
$$b_n = \frac{1}{2\pi i} \oint_C (z^* - z_0)^{n-1} f(z^*) \mathrm{d}\, z^*,$$

taken counterclockwise around any simple closed path C that lies in the annulus and encircles the inner circle.

- This series converges.

- Denote b_n by a_{-n}, one have

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$
$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z)^{n+1}} \mathrm{d}\, z^*, \qquad (n = 0, \pm 1, \pm 2, \cdots)$$

Laurent Series: Laurent theorem, proof

- By the Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z^*)}{z^* - z}\mathrm{d}\, z^* - \frac{1}{2\pi i}\oint_{C_2} \frac{f(z^*)}{z^* - z}\mathrm{d}\, z^* = g(z) + h(z),$$

- The non-negative powers, the Taylor series of $g(z)$ is,

$$g(z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z^*)}{(z^* - z)}\mathrm{d}\, z^* = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

$$a_n = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z^*)}{(z^* - z_0)^{n+1}}\mathrm{d}\, z^*,$$

- The negative powers, now instead of $|\frac{z - z_0}{z^* - z_0}| < 1$, we have $|\frac{z^* - z_0}{z - z_0}| < 1$, i.e.

$$\frac{1}{z^* - z} = \frac{-1}{z - z_0}\frac{1}{(1 - \frac{z^* - z_0}{z - z_0})}$$

$$= \frac{-1}{z - z_0}\{1 + \frac{z^* - z_0}{z - z_0} + (\frac{z^* - z_0}{z - z_0})^2 + \cdots + (\frac{z^* - z_0}{z - z_0})^n\} - \frac{1}{z - z^*}(\frac{z^* - z_0}{z - z_0})^{n+1}$$

Laurent Series: Laurent theorem, proof, cont.

- Multiplication by $-f(z^*)/2\pi\, i$ and integration over C_2,

$$\begin{aligned} h(z) &= -\frac{1}{2\pi\, i}\oint_{C_2}\frac{f(z^*)}{z^*-z}\mathrm{d}\, z^* \\ &= \frac{1}{2\pi\, i}\{\frac{1}{z-z_0}\oint_{C_2} f(z^*)\mathrm{d}\, z^* + \frac{1}{(z-z_0)^2}\oint_{C_2}(z^*-z_0)f(z^*)\mathrm{d}\, z^* + \cdots \\ &+ \frac{1}{(z-z_0)^n}\oint_{C_2}(z^*-z_0)^{n-1}f(z^*)\mathrm{d}\, z^* + \frac{1}{(z-z_0)^{n+1}}\oint_{C_2}(z^*-z_0)^n f(z^*)\mathrm{d}\, z^*\} \\ &+ R_n^*(z) \end{aligned}$$

- where the last term,

$$\lim_{n\to\infty} R_n^*(z) = \lim_{n\to\infty}\{\frac{1}{2\pi\, i}\frac{1}{(z-z_0)^{n+1}}\oint_{C_2}\frac{(z^*-z_0)^{n+1}}{z-z^*}f(z^*)\mathrm{d}\, z^*\} = 0$$

- Convergence
- Uniqueness

Laurent Series: Laurent theorem, Example

- Example 1: $z^{-5}\sin z$

$$z^{-5}\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n-4},$$

with the annulus of convergence $|z| > 0$, the whole complex plane without the origin and the principal part of the series at 0 is $z^{-4} - \frac{1}{6}z^{-2}$.

- Example 2: $z^2 e^{1/z}$,

$$z^2 e^{1/z} = z^2\left(1 + \frac{1}{1!z} + \frac{1}{2!z^2} + \cdots\right) = z^2 + z + \frac{1}{2} + \frac{1}{3!z} + \frac{1}{4!z^2} + \cdots$$

with the annulus of convergence $|z| > 0$.

- Example 3: $1/(1-z)$,

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \qquad \text{valid if} \qquad |z| < 1,$$

$$\frac{1}{1-z} = \frac{-1}{z(1-z^{-1})} = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}}, \qquad \text{valid if} \qquad |z| > 1,$$

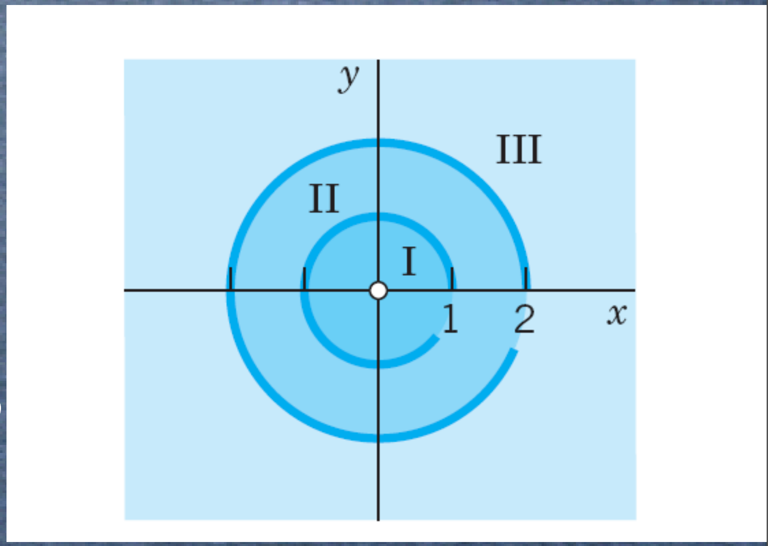
Laurent Series: Laurent theorem, Example

- Example 4: $1/(z^3 - z^4)$,

$$\frac{1}{z^3 - z^4} = \sum_{n=0}^{\infty} z^{n-3}, \qquad \text{valid if} \qquad 0 < |z| < 1,$$

$$\frac{1}{z^3 - z^4} = \frac{-1}{z^4(1 - z^{-1})} = -\sum_{n=0}^{\infty} \frac{1}{z^{n+4}}, \qquad \text{valid if} \qquad |z| > 1,$$

- Example 5: $f(z) = \frac{-2z+3}{z^2-3z+2} = \frac{-1}{z-1} + \frac{-1}{z-2}$,



$$f(z) = \sum_{n=1}^{\infty} (1 + \frac{1}{2^{n+1}}) z^n, \qquad \text{valid if} \qquad |z| < 1,$$

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} z^n - \sum_{n=1}^{\infty} \frac{1}{z^{n+1}}, \qquad \text{valid if} \qquad 1 < |z| < 2,$$

$$f(z) = -\sum_{n=1}^{\infty} (2^n + 1) \frac{1}{z^{n+1}}, \qquad \text{valid if} \qquad |z| > 2,$$

Complex Variables: Scope

Complex Variables

- Complex Numbers, Ch. 13
- Complex Function, Ch. 13
- Complex Integraton, Ch. 14
- Power Series, Ch. 15
- Taylor Series, Ch. 15

☐ Uniform Convergence, Ch. 15.5

☒ Laurent Series, Ch. 16

☒ Residue Integration, Ch. 16

☐ Conformal Mapping, Ch. 17

☐ Complex Analysis, Ch. 18

☐ Potential Theory, Ch. 18

Homework #18:

1. **[Contour Integrations]**:

   (a) $\oint_C \frac{1}{2z-i}\,\mathrm{d}\,z$, C the unit circle $|z|=3$ (counterclockwise),

   (b) $\oint_C \frac{\mathrm{Ln}(z+3)+\cos z}{(z+1)^2}\,\mathrm{d}\,z$, C the unit circle $|z|=2$ (counterclockwise),

2. **[Power Series]**:
   Find the radius of convergence for the power series,

$$\sum_{n=1}^{\infty} \frac{n}{2^n}(z+i)^{2n}, \tag{1}$$

3. **[Taylor and Laurent Series]**:
   Find all Taylor and Laurent series with center $z=z_0$ and determine the precise regions of convergence.

$$\frac{1}{1-z^3}, \qquad z_0 = 0, \tag{2}$$

$$\frac{1}{1-z^2}, \qquad z_0 = 1, \tag{3}$$

Laurent Series: Singularity, Zero and Pole

- A singularity point of an analytic function $f(z)$ is a z_0 at which $f(z)$ ceases to be analytic.
- A zero is a z at which $f(z) = 0$.
- Laurent series can be used for classifying singularities and Taylor series for discussing zeros.
- An isolated singularity of $f(z)$ is a z_0, which has a neighborhood without further singularities of $f(z)$.
- The singularity of $f(z)$ at $z = z_0$ is called a pole, and m is called its *order*,

$$f(z) = \frac{b_1}{z - z_0} + \cdots + \frac{b_m}{(z - z_0)^m}$$

- Poles of the first order are also known as *simple poles*.

Laurent Series: Pichard's theorem

- If the principal part of $f(z)$ has infinitely many terms, we say that $f(z)$ has at $z = z_0$ an isolated essential singularity.
- If $f(z)$ is analytic and has a pole at $z = z_0$, then $|f(z)| \to \infty$ as $z \to z_0$ in any manner.
- Picard's Theorem:
  If $f(z)$ is analytic and has an isolated essential singularity at a point z_0, it takes on every value, with at most one exceptional value, in an arbitrarily small ϵ-neighborhood of z_0.

Laurent Series: Removable Singularity

- Removable singularities: a function $f(z)$ has a removable singularity at $z = z_0$ if $f(z)$ is not analytic at $z = z_0$, but can be mad analytic there by assigning a suitable value $f(z_0)$.
- Example: $f(z) = (\sin z)/z$ becomes analytic at $z = 0$ if we define $f(0) = 1$.

Laurent Series: Zero

- A zero of an analytic function $f(z)$ in a domain D is a $z = z_0$ in D such that $f(z_0) = 0$.
- A zero has order n if not only f but also the derivatives $f', f'', \cdots, f^{(n-1)}$ are all 0 at $z = z_0$ but $f^{(n)} \neq 0$.
- A first-order zero is also called a simple zero.
- Taylor Series at a n-th order Zero: At an n-th order zero $z = z_0$ of $f(z)$, the derivatives $f'(z_0), \cdots f^{(n-1)}(z_0)$ are zero, by definition.

$$f(z) = a_n(z - z_0)^n + a_{n+1}(z - z_0)^{n+1} + \cdots$$

- The zeros of an analytic function $f(z)$ are isolated, that is, each of them has a neighborhood that contains no further zeros of $f(z)$.
- Poles and Zeros:
  Let $f(z)$ be analytic at $z = z_0$ and have a zero of n-th order at $z = z_0$. Then $1/f(z)$ has a pole of n-th order at $z = z_0$.

Laurent Series: Residue integration method

$$\oint_C f(z) \mathrm{d}\, z$$

- If $f(z)$ is analytic everywhere on C and inside C, $\oint_C f(z) \mathrm{d}\, z = 0$.
- If $f(z)$ has a singularity at a point $z = z_0$ inside C, but is otherwise analytic on C and inside C, then $f(z)$ has a Laurent series,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n},$$

that converges for all points near $z = z_0$, except at $z = z_0$ itself.

- With the first negative power, $b_1 = \frac{1}{2\pi\, i} \oint_C f(z) \mathrm{d}\, z$, we can evaluate the integral,

$$\boxed{\oint_C f(z) \mathrm{d}\, z = 2\pi\, i\, b_1,}$$

- The coefficient b_1 is called the residue of $f(Z)$ at $z = z_0$,

$$\boxed{b_1 = \mathrm{Res}_{z=z_0} f(z).}$$

Laurent Series: Residue integral, Example

- Example 1: Integrate the function $f(z) = z^{-4}\sin z$ counterclockwise around the unit circle C,

$$\text{Laurent series}: \qquad f(z) = \frac{\sin z}{z^4} = \frac{1}{z^3} - \frac{1}{3!z} + \frac{z}{5!} - \frac{z^3}{7!} + \cdots$$

which converges for $|z| > 0$, with the residue $b_1 = -1/3!$, then

$$\oint_C \frac{\sin z}{z^4} \mathrm{d}\, z = 2\pi\, i b_1 = -\frac{\pi\, i}{3}.$$

- Example 2: Integrate the function $f(z) = 1/(z^3 - z^4)$ counterclockwise around the circle $C : |z| = 1/2$,

$$\text{Laurent series}: \qquad f(z) = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + \cdots$$

which converges $0 < |z| < 1$, with the residue $b_1 = 1$, then

$$\oint_C \frac{1}{z^3 - z^4} \mathrm{d}\, z = 2\pi\, i.$$

Laurent Series: Residue integral, Simple pole

- For the residue of $f(z)$ at a simple pole at z_0,

$$\text{Res}_{z=z_0} f(z) = b_1 = \lim_{z \to z_0} (z - z_0) f(z),$$

- Proof: For a simple pole at $z = z_0$ the Laurent series is

$$f(z) = \frac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots, \qquad 0 < |z - z_0| < R$$

Multiplying both sides by $z - z_0$ and then letting $z \to z_0$, we obtain

$$\lim_{z \to z_0} (z - z_0) f(z) = b_1 + \lim_{z \to z_0} (z - z_0)[a_0 + a_1(z - z_0) + \cdots] = b_1.$$

Residue Integral: Simple pole

- If $f(z) = p(z)/q(z)$, $p(z_0) \neq 0$ and $q(z)$ has a simple zero at z_0 (so that $f(z)$ has at z_0 a simple pole),

$$\text{Res}_{z=z_0} f(z) = \text{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}.$$

- Proof: The Taylor series of $q(z)$ at a simple zero z_0 is

$$q(z) = (z - z_0)q'(z_0) + \frac{(z - z_0)^2}{2!} q"(z_0) + \cdots$$

substituting this into $f = p/q$ and then

$$\begin{aligned}\text{Res}_{z=z_0} f(z) &= \lim_{z=z_0} (z - z_0)\frac{p(z)}{q(z)} = \lim_{z=z_0} \frac{(z - z_0)p(z)}{(z - z_0)[q'(z_0) + (z - z_0)q"(z_0)/2 + \cdots]} \\ &= \frac{p(z_0)}{q'(z_0)}.\end{aligned}$$

Residue Integral: Simple poles, Example

- Example: $f(z) = \frac{(9z+i)}{(z^3+z)}$ has a simple pole at i because $z^2+1 = (z+i)(z-i)$,
- Using $\text{Res}_{z=z_0} f(z) = b_1 = \lim_{z\to z_0}(z-z_0)f(z)$,

$$\text{Res}_{z=i}\frac{(9z+i)}{(z^3+z)} = \lim_{z\to i}(z-i)\frac{(9z+i)}{z(z+i)(z-i)} = [\frac{9z+i}{z(z+i}]_{z=i} = -5i,$$

- Using $\text{Res}_{z=z_0} f(z) = \text{Res}_{z=z_0}\frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$ with $p(z) = 9z+i$ and $q'(z) = 3z^2+1$,

$$\text{Res}_{z=i}\frac{(9z+i)}{(z^3+z)} = [\frac{9z+i}{3z^2+1}]_{z=i} = -5i.$$

Residue Integral: Poles of Any Order

- The residue of $f(z)$ at an m-th order pole at z_0 is

$$\mathrm{Res}_{z=z_0} = \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{\mathrm{d}^{m-1}}{\mathrm{d}\, z^{m-1}}[(z-z_0)^m f(z)],$$

- In particular, for a 2nd-order pole, $\mathrm{Res}_{z=z_0} = \lim_{z\to z_0} \frac{\mathrm{d}}{\mathrm{d}\, z}[(z-z_0)^2 f(z)]$.

- Proof: The Laurent series converging near z_0 is

$$f(z) = \frac{b_m}{(z-z_0)^m} + \frac{b_{m-1}}{(z-z_0)^{m-1}} + \cdots \frac{b_1}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots,$$

Multiplying both sides by $(z-z_0)^m$ gives

$$(z-z_0)^m f(z) = b_m + b_{m-1}(z-z_0) + \cdots b_1(z-z_0)^{m-1} + a_0(z-z_0)^m \cdots,$$

Now b_1 now the coefficient of the power $(z-z_0)^{m-1}$ of the power series of $g(z) = (z-z_0)^m f(z)$, then

$$b_1 = \frac{1}{(m-1)!} g^{(m-1)}(z_0) = \frac{1}{(m-1)!} \frac{\mathrm{d}^{m-1}}{\mathrm{d}\, z^{m-1}}[(z-z_0)^m f(z)].$$

Residue Integral: Poles of any order, Example

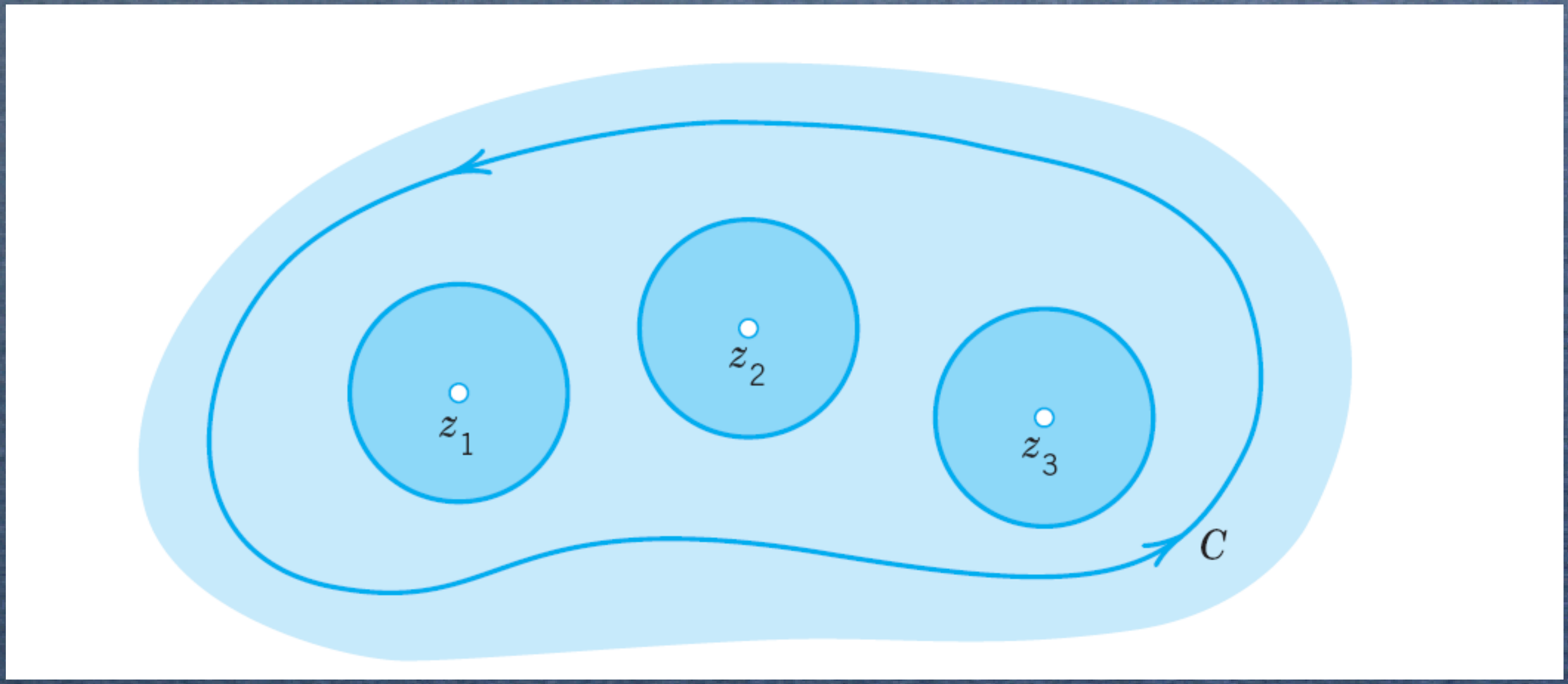
- $f(z) = \frac{50z}{(z^3+2z^2-7z+4)}$ has a pole of second order at $z = 1$ because the denominator equals $(z+4)(z-1)^2$.
- The Residue of $f(z)$ is

$$\begin{aligned} \mathrm{Res}_{z=1} f(z) &= \lim_{z\to 1} \frac{\mathrm{d}}{\mathrm{d}\,z}[(z-1)^2 f(z)] \\ &= \lim_{z\to 1} \frac{\mathrm{d}}{\mathrm{d}\,z}(\frac{50z}{z+4}) \\ &= 8. \end{aligned}$$

Residue Integral: Several singularities

- Let $f(z)$ be analytic inside a simple closed path C and on C, except for finitely many singular points $z_1, z_2, \cdots, z_k$ inside C. Then the integral of $f(z)$ taken counterclockwise around C equals $2\pi i$ times the sum of the residues of $f(z)$ at $z_1, z_2, \cdots, z_k$:

$$\oint_C f(z)\,\mathrm{d}\,z = 2\pi\,i \sum_{j=1}^{k} \operatorname{Res}_{z=z_j} f(z).$$



Residue Integral: Several singularities, Example

- Example 1: Evaluate the integral counterclockwise

$$\oint_C \frac{4-3z}{z^2-z}\,\mathrm{d}\,z$$

1. 0 and 1 are inside C; Ans. $2\pi\, i(-4+1) = -6\pi\, i$.
2. 0 is inside, and 1 outside C; Ans. $2\pi\, i(-4) = -8\pi\, i$.
3. 1 is inside, and 0 outside C; Ans. $2\pi\, i$.
4. 0 and 1 are outside C; Ans. 0.

$$\mathrm{Res}_{z=0}\frac{4-3z}{z(z-1)} = -4, \qquad \mathrm{Res}_{z=1}\frac{4-3z}{z(z-1)} = 1.$$

- Example 2: Integrate $\frac{\tan z}{z^2-1}$ counterclockwise around the circle $C : |z| = 3/2$,

$$\oint_C \frac{\tan z}{z^2-1}\,\mathrm{d}\,z = 2\pi\, i[\mathrm{Res}_{z=1}\frac{\tan z}{z^2-1} + \mathrm{Res}_{z=-1}\frac{\tan z}{z^2-1}] = 2\pi\, i\tan 1.$$

Homework #19:

1. **[Contour Integrations]**:

   (a) $\oint_C \frac{1}{2z-i}\,\mathrm{d}\,z$, C the unit circle $|z|=3$ (counterclockwise),

   (b) $\oint_C \frac{\mathrm{Ln}(z+3)+\cos z}{(z+1)^2}\,\mathrm{d}\,z$, C the unit circle $|z|=2$ (counterclockwise),

2. **[Power Series]**:
   Find the radius of convergence for the power series,

$$\sum_{n=1}^{\infty} \frac{n}{2^n}(z+i)^{2n}, \tag{1}$$

3. **[Taylor and Laurent Series]**:
   Find all Taylor and Laurent series with center $z = z_0$ and determine the precise regions of convergence.

$$\frac{1}{1-z^3}, \qquad z_0 = 0, \tag{2}$$

$$\frac{1}{1-z^2}, \qquad z_0 = 1, \tag{3}$$

Residue Integral: Rational sine and cosine functions

- Consider the integrals of the type

$$J = \int_0^{2\pi} F(\cos\theta, \sin\theta) \mathrm{d}\,\theta$$

where $F(\cos\theta, \sin\theta)$ is a *real rational function* of $\cos\theta$ and $\sin\theta$ and is finite on the interval of integration.

- Setting $e^{i\theta} = z$, we obtain

$$\begin{aligned} \cos\theta &= \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}(z + \frac{1}{z}), \\ \sin\theta &= \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}(z - \frac{1}{z}), \end{aligned}$$

- Since $\mathrm{d}\,z/\mathrm{d}\,\theta = ie^{i\theta}$, the given integral takes the form,

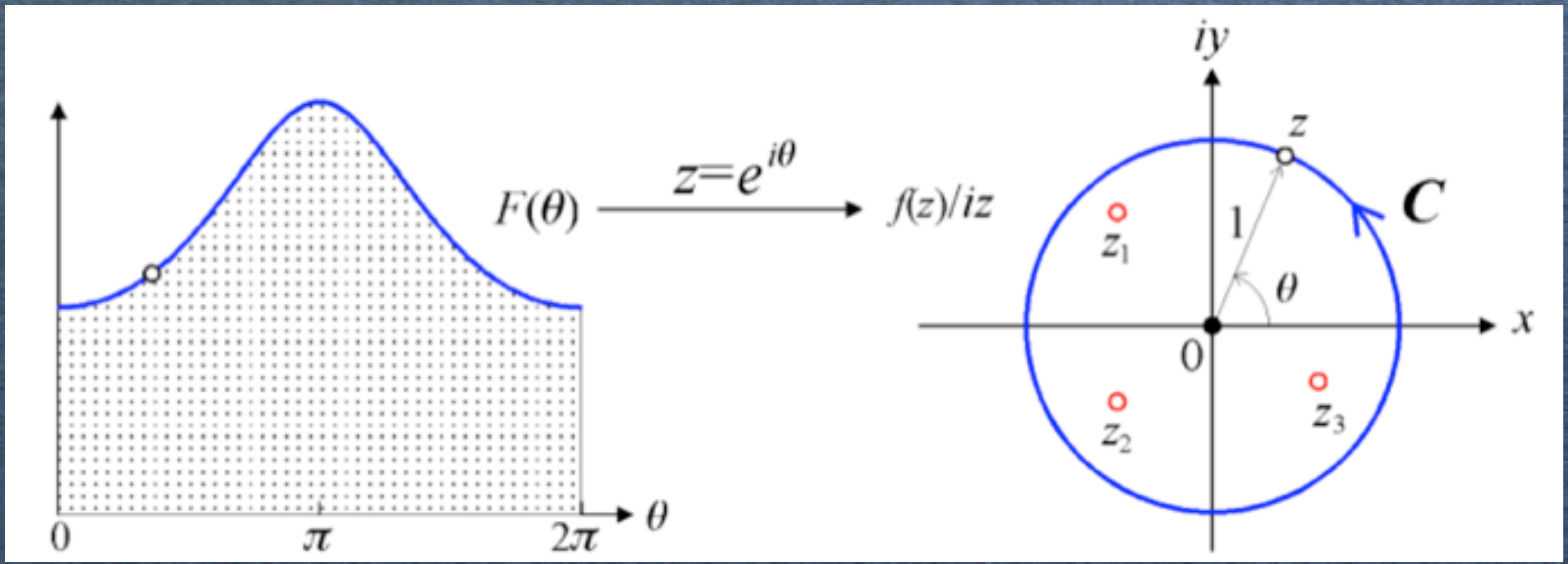
$$J = \oint_C f(z) \frac{\mathrm{d}\,z}{i\,z},$$

counterclockwise around the unit circle $|z| = 1$.

Residue Integral: Rational functions, Example

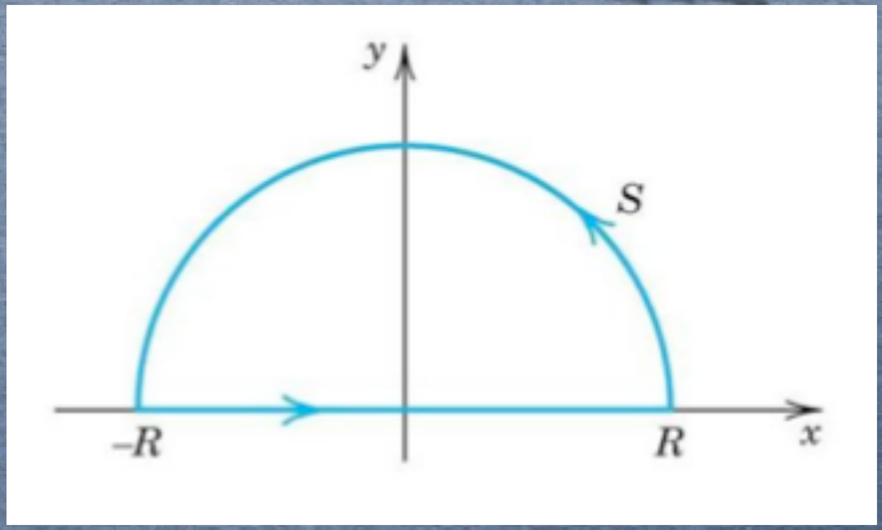
- Integrate

$$\begin{aligned}\int_0^{2\pi} \frac{d\,\theta}{\sqrt{2}-\cos\theta} &= \oint_C \frac{d\,z/iz}{\sqrt{2}-\frac{1}{2}(z+\frac{1}{z})} \\ &= \frac{i}{2}\oint_C \frac{1}{(z-\sqrt{2}-1)(z-\sqrt{2}+1)} d\,z \\ &= 2\pi\, i\mathrm{Res}_{z=\sqrt{2}-1}\left[\frac{i}{2}\frac{1}{(z-\sqrt{2}-1)(z-\sqrt{2}+1)}\right] \\ &= 2\pi.\end{aligned}$$



Residue Integral: Improper Integral of Rational functions

- Consider real integrals of the form,



$$\int_{-\infty}^{\infty} f(x)\mathrm{d}\,x = \lim_{R\to\infty} \int_{-R}^{R} f(x)\mathrm{d}\,x$$

- The limit is called the Cauchy principal value of the integral, i.e., P.V $\int_{-\infty}^{\infty} f(x)\mathrm{d}\,x$
- If the function $f(x)$ is a real rational function whose denominator is different from zero for all x and is of degree at least two units higher than the degree of the numerator, then the limit exists.
- We may consider the corresponding contour integral,

$$\oint_C f(z)\mathrm{d}\,z = \oint_S f(z)\mathrm{d}\,z + \int_{-R}^{R} f(x)\mathrm{d}\,x = 2\pi\,i \sum \mathrm{Res} f(z).$$

Residue Integral: Improper Integral, cont.

- The sum consists of all the residues of $f(z)$ at the points in the upper half-plane.

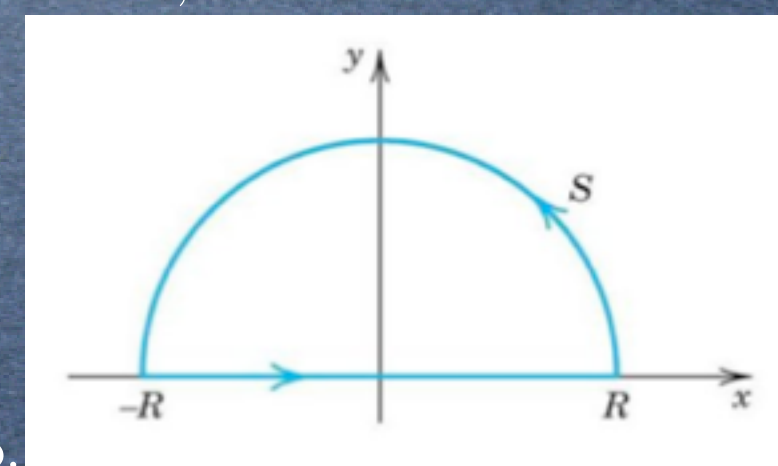
$$\int_{-R}^{R} f(x)\,\mathrm{d}\,x = 2\pi\, i \sum \mathrm{Res} f(z) - \oint_S f(z)\,\mathrm{d}\,z$$

- If $R \to \infty$, the value of the integral over the semicircle S approaches zero.

- Proof: If we set $z = Re^{i\theta}$, and by assumption, the degree of the denominator of $f(z)$ is at least two units higher than the degree of the numerator, we have

$$|f(z)| < \frac{k}{|z|^2}, \qquad |z| = R > R_0$$

for sufficiently large constant k and R_0.



- By the ML-inequality, $|\oint_S f(z)\,\mathrm{d}\,z| < \frac{k}{R^2}\pi R = \frac{k\pi}{R}, \qquad R > R_0$
Hence, as $R \to 0$, the value of the integral over S approaches zero.

- Then the integral

$$\boxed{\int_{-\infty}^{\infty} f(x)\,\mathrm{d}\,x = 2\pi\, i \sum \mathrm{Res} f(z).}$$

Residue Integral: Improper Integral, Example

- Integrate

$$\int_0^\infty \frac{1}{1+x^4}\mathrm{d}\,x$$

- first consider the improper integral from $-\infty$ to ∞

$$\int_{-\infty}^\infty \frac{1}{1+x^4}\mathrm{d}\,x = 2\pi\, i\{\mathrm{Res}_{z=e^{\pi i/4}}[\frac{1}{1+x^4}] + \mathrm{Res}_{z=e^{3\pi i/4}}[\frac{1}{1+x^4}]\} = \frac{\pi}{\sqrt{2}}$$

- since $1/(1+x^4)$ is an even function, we obtain,

$$\int_0^\infty \frac{1}{1+x^4}\mathrm{d}\,x = \frac{\pi}{2\sqrt{2}}.$$

Residue Integral: Fourier Integral

- If $f(z)$ is a rational function satisfying the assumption on the degrees, we may consider the corresponding integral

$$\int_{-\infty}^{\infty} f(x)e^{ikx}\mathrm{d}\,x = \oint_C f(z)e^{ikz}\mathrm{d}\,z = 2\pi\,i\sum \mathrm{Res}[f(z)e^{ikz}], \qquad k \text{ real and positive.}$$

- The real and imaginary parts are,

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\cos kx\mathrm{d}\,x &= -2\pi\sum \mathrm{Im}\{\mathrm{Res}[f(z)e^{ikz}]\},\\ \int_{-\infty}^{\infty} f(x)\sin kx\mathrm{d}\,x &= 2\pi\sum \mathrm{Re}\{\mathrm{Res}[f(z)e^{ikz}]\},\end{aligned}$$

Residue Integral: Fourier Integral, Example

- Integrate $\int_{-\infty}^{\infty} \frac{\cos kx}{a^2+x^2} \mathrm{d}\,x$ for $k > 0, a > 0$.
- For $\frac{e^{ikz}}{a^2+z^2}$ has only one pole in the upper half-plane, i.e., a simple pole at $z = ia$, then

$$\mathrm{Res}_{z=ia}[\frac{e^{ikz}}{a^2+z^2}] = \frac{e^{-ak}}{2ia}$$

- Thus

$$\int_{-\infty}^{\infty} \frac{e^{ikz}}{a^2+x^2} \mathrm{d}\,x = \frac{\pi}{a} e^{-ak}.$$

- Fourier cosine integral becomes

$$\int_{-\infty}^{\infty} \frac{\cos kx}{a^2+x^2} \mathrm{d}\,x = \frac{\pi}{a} e^{-ak}.$$

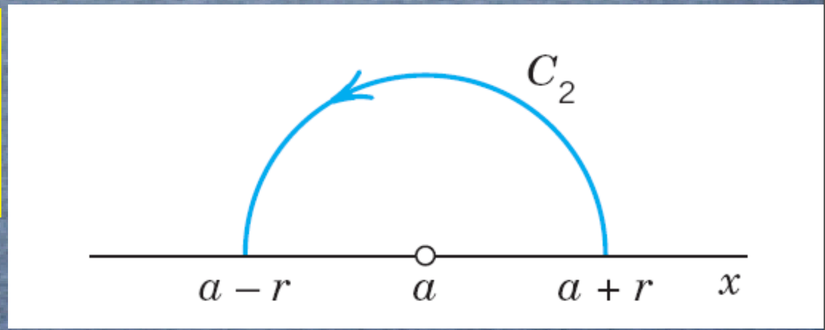
- Fourier sine integral becomes

$$\int_{-\infty}^{\infty} \frac{\sin kx}{a^2+x^2} \mathrm{d}\,x = 0.$$

Residue Integral: Simple poles on the Real axis

- If $f(z)$ has a simple pole at $z = a$ on the real axis, then

$$\lim_{r \to 0} \int_{C_2} f(z) \mathrm{d}\, z = \pi i \mathrm{Res}_{z=a} f(z).$$



- Proof: By the definition of a simple pole (poles of the first order), the integrand $f(z)$ has the Laurent series for $0 < |z - a| < R$,

$$f(z) = \frac{b_1}{z - a} + g(z), \qquad b_1 = \mathrm{Res}_{z=a} f(z).$$

- Here $g(z)$ is analytic on the semicircle of integration, $C_2 : z = a + re^{i\theta}$, $0 \le \theta \le \pi$, and for all z between C_2 and the x-axis.
- By integration $\int_{C_2} f(z) \mathrm{d}\, z = \int_0^{\pi} \frac{b_1}{re^{i\theta}} i\, re^{i\theta} \mathrm{d}\, \theta + \int_{C_2} g(z) \mathrm{d}\, z = \pi\, i b_1$.

Residue Integral: Simple poles on the Real axis, P.V.

- For sufficient large R, the integral over the entire contour has the value J given by

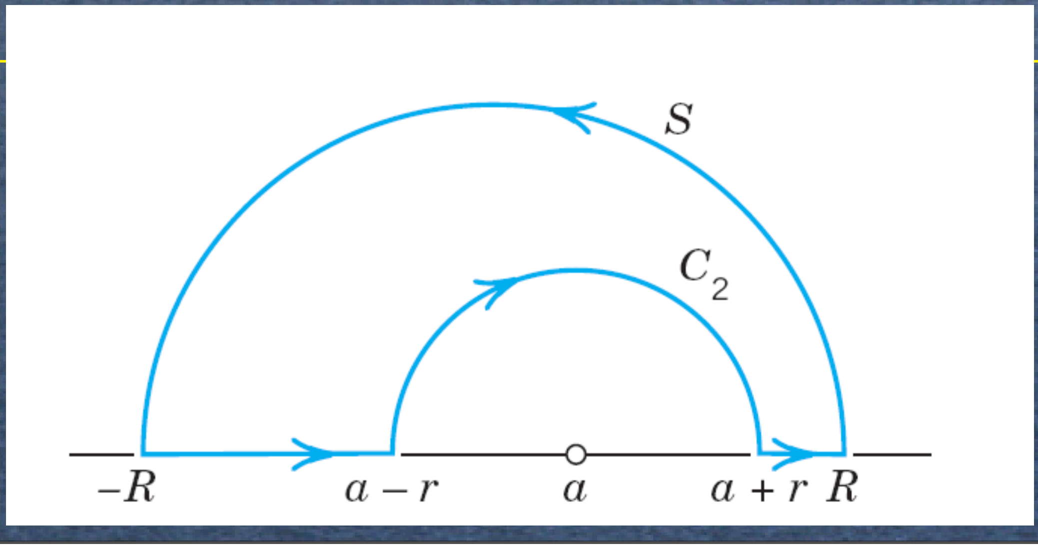
$$J = \int_{-\infty}^{\infty} f(x)\mathrm{d}\,x = 2\pi\,i \sum \mathrm{Res}[f(z)],$$

- For a simple pole on the real axis, the integral over C_2 (clockwise !) approaches the value,

$$K = -\pi\,i \sum_{z=a_i} \mathrm{Res}[f(z)],$$

- The principal value P of the integral is

$$\mathrm{P.V.} \int_{-\infty}^{\infty} f(x)\mathrm{d}\,x = J - K = 2\pi\,i \sum \mathrm{Res}[f(z)] + \pi\,i \sum_{z=a_i} \mathrm{Res}[f(z)].$$



Residue Integral: Simple poles on the Real axis, Example

- Integrate

$$\int_{-\infty}^{\infty} \frac{1}{(x^2-3x+2)(x^2+1)} \mathrm{d}\, x$$

- In the upper half-plane, the integrand $f(z)$ has simple pols at

$$\begin{aligned} z = 1, &\quad \mathrm{Res}_{z=1} f(z) = \frac{-1}{2}, \\ z = 2, &\quad \mathrm{Res}_{z=2} f(z) = \frac{1}{5}, \\ z = i, &\quad \mathrm{Res}_{z=1} f(z) = \frac{3-i}{20}, \end{aligned}$$

- the principal value of $f(z)$ is

$$\begin{aligned} \mathrm{P.V.} \int_{-\infty}^{\infty} \frac{1}{(x^2-3x+2)(x^2+1)} \mathrm{d}\, x &= 2\pi\, i(\frac{3-i}{20}) + \pi\, i(\frac{-1}{2} + \frac{1}{5}) \\ &= \frac{\pi}{10}. \end{aligned}$$

Residue Integral: Fractional function

- Evaluate

$$I = \int_{-\infty}^{\infty} \frac{e^{a\,x}}{1+e^{x}}\mathrm{d}\,x, \qquad 0 < a < 1$$

where the limits on a are necessary (and sufficient) to prevent the integral from diverging as $x \to \pm\infty$.

- The principle value

$$\begin{aligned}
\oint \frac{e^{a\,z}}{1+e^{z}}\mathrm{d}\,z &= \lim_{R\to\infty}\left(\int_{-R}^{R} \frac{e^{a\,x}}{1+e^{x}}\mathrm{d}\,x - e^{2\pi\,i\,a}\int_{-R}^{R} \frac{e^{a\,x}}{1+e^{x}}\mathrm{d}\,x\right) \\
&= (1-e^{2\pi\,i\,a})\int_{-\infty}^{\infty} \frac{e^{a\,x}}{1+e^{x}}\mathrm{d}\,x \\
&= 2\pi\,i\mathrm{Res}_{z=i\pi}\left[\frac{e^{a\,z}}{1+e^{z}}\right] = -2\pi i e^{i\pi\,a}
\end{aligned}$$

- Then the integral

$$I = \int_{-\infty}^{\infty} \frac{e^{a\,x}}{1+e^{x}}\mathrm{d}\,x = \frac{\pi}{\sin a\pi}.$$

Residue Integral: Inverse Laplace transform

- Laplace transform

$$F(s) = \int_0^{\infty} f(t)\, e^{-s\,t} \mathrm{d}t,$$
$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\, e^{st} \mathrm{d}s,$$

- For the inverse Laplace transform, $\mathcal{L}^{-1}$, i.e., $f(t) = \mathcal{L}^{-1}\{F(s)\}$.
- To avoid the exponential divergence, one write

$$f(t) = e^{c\,t}\, g(t).$$

If $f(t)$ diverges as $e^{\alpha\,t}$, we require c to be great than α so that $g(t)$ will be convergent.

- For $g(t) = 0$ for $t < 0$, and may be represented by a Fourier integral,

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} \mathrm{d}\omega \int_0^{\infty} g(\tau) e^{-i\omega\tau} \mathrm{d}\,\tau$$

Residue Integral: Inverse Laplace transform, cont.

- Then

$$f(t) = \frac{e^{c\,t}}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\mathrm{d}\omega \int_0^{\infty} f(\tau)e^{-c\,\tau}e^{-i\omega\tau}\mathrm{d}\,\tau$$

- With the change of variable, $s = c + i\,\omega$, the integral over τ is thrown into the form of a Laplace transform,

$$\int_0^{\infty} f(\tau)e^{-c\,\tau}e^{-i\omega\tau}\mathrm{d}\,\tau = \int_0^{\infty} f(\tau)e^{-s\tau}\mathrm{d}\,\tau \equiv F(s)$$

  s is now a complex variable and $\mathrm{Re}(s) \geq c$ to guarantee convergence.

- With c as a constant, $\mathrm{d}\,s = i\mathrm{d}\,\omega$, we obtain

$$\boxed{f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\,e^{st}\mathrm{d}s,}$$

Residue Integral: Inverse Laplace transform, Example

- Find the inverse Laplace transformation for

$$F(s) = \frac{a}{s^2 - a^2},$$

- For the integrand,

$$e^{s\,t} F(s) = \frac{a e^{s\,t}}{(s+a)(s-a)}$$

has the Residues at one simple pole $s = a$ and the other simple pole at $s = -a$ are,

$$\mathrm{Res}_{s=a} = \frac{1}{2} e^{at}, \qquad \mathrm{Res}_{s=-a} = \frac{-1}{2} e^{-at},$$

- Then the inverse Laplace transformation is

$$\begin{aligned} f(t) &= \sum_i \mathrm{Res}_{z=z_i} \frac{a e^{z\,t}}{(z+a)(z-a)} \\ &= \frac{1}{2}(e^{at} - e^{-at}) = \sinh at. \end{aligned}$$

5th EXAM:

1. Topics cover "Complex Analysis" Ch. 13-16:
   - Complex numbers, functions, Ch. 13
   - Complex integration, Ch. 14
   - Power, Taylor series, Ch. 15 (skip 15.5)
   - Laurent series, Ch. 16
   - Residue Integration, Ch. 16

   Skip: Conformal Mapping (Ch. 17), Potential Theory (Ch. 18).

2. You can bring an ONE-PAGE NOTE.

Jan. 14 (this Friday night), 7-10PM.