EE 2020

Partial Differential Equations and Complex Variables

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EE 2020

- **Time**: M7M8R6 (3:20PM-5:10PM, Monday; 2:10AM-3:00PM, Thursday)
- **Teaching Method**: in-class lectures with examples.
  I would try to write in the black board, not the slides.
- **Evaluation**:
  1. Four Homeworks, 40%;
  2. Midterm 30%;
     Tentatively scheduled on 4/27,
     covering Ch.12 of the textbook.
  3. Final exam 30%:
     Tentatively scheduled on 6/15,
     covering Ch.13 - Ch.18 of the textbook.
  4. Bonus: just rise your hand in the classroom, 20%.

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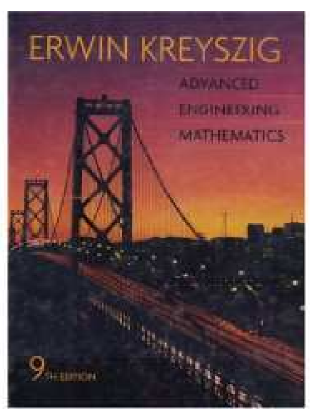
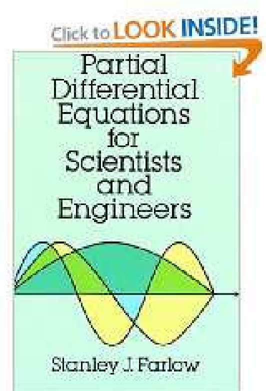
Textbook and Reference Books

- [Note]: Class handouts;
- Prof. S.D. Yang's note: http://www.ee.nthu.edu.tw/∼sdyang/Courses/PDE.htm
- [Textbook]: E. Kreyszig, "Advanced Engineering Mathematics", 9th Ed., John Wiley & Sons, Inc., (2006).
- [Ref.]: Stanley J. Farlow, "Partial Differential Equations for Scientists and Engineers", Dover Publications, (1993); (for PDE, but *optional*).
- [Ref.num]: Matthew P. Coleman, "An Introduction to Partial Differential Equations with MATLAB", Chapman & Hall/Crc Applied Mathematics & Nonlinear Science (2004); (*optional*).





Syllabus: for PDE

1. **Introduction to PDE and Complex variables**, (2/23, ~~2/26~~).
2. Diffusion-type problems: [Textbook] Ch.12, [Ref.] Ch.2.
   - Derivation of the Heat equation, (3/2).
   - Boundary conditions for Diffusion-type problems, (3/5).
   - Separation of variables, (3/9).
   - Solving nonhomogeneous PDEs, (3/12).
   - Integral transforms, (3/16, 3/19).
   - The Fourier transform, (3/23).
   - The Laplace Transform, (3/26).
3. Hyperbolic-type problems: [Textbook] Ch.12, [Ref.] Ch.3.
   - 1-D Wave equation, (~~4/2~~, 4/6).
   - D'Alembert solution of the Wave equation, (4/9).
   - Sturm-Liouville problems, (4/13).
   - 2-D Wave equation in Cartesian and polar coordinates, (4/16, 4/20).
   - Laplace's equation in Cartesian, polar, and spherical coordinates, (4/23).

Syllabus: for Complex variables

1. **Midterm**, (4/27).
2. Introduction to Numerical PDE (4/30): [Ref.num].
3. Complex variables: [Textbook]Ch.13-Ch.18.
   - Complex numbers and functions, (5/4).
   - Cauchy-Riemann equations, (5/7, 5/11).
   - Complex integration, (5/14, 5/18).
   - Complex power & Taylor series, (5/21, 5/25).
   - Laurent series & residue, (~~5/28~~, 6/1, 6/4).
   - Conformal mapping, (6/8, 6/11).
   - Applications: real integrals by residual integration, potential theory, (6/15, 6/18).
4. **Final exam**, (6/15).

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Related courses

1. Applied Mathematics (Phys.),
2. Complex Analysis (Math.),
3. Numerical Mehtods for Parital Differential Equations (Math.),
4. Numerical Analysis (EE),
5. Computational Methods for Optoelectronics (IPT),
6. ...

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Partial Differential Equations

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} = f(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}),$$

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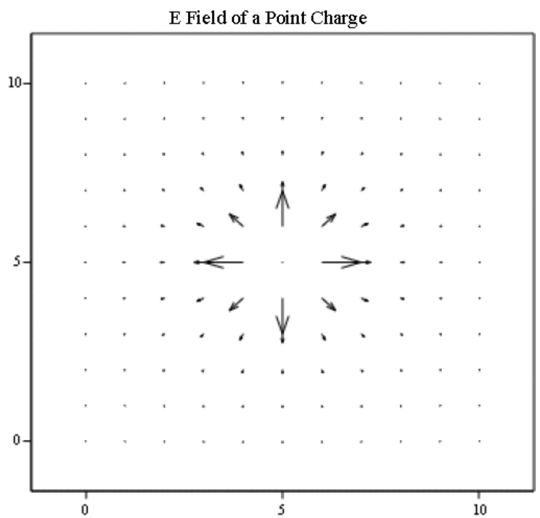
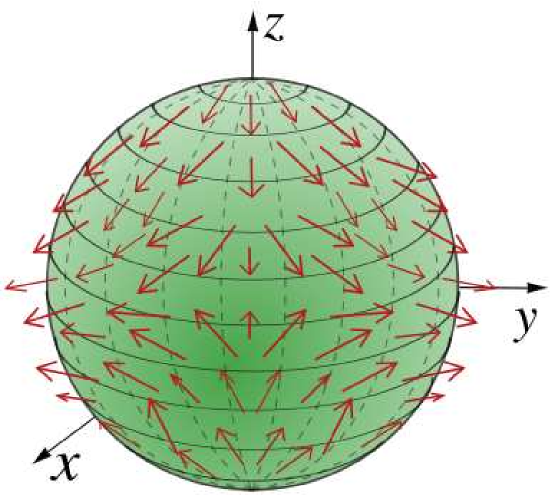
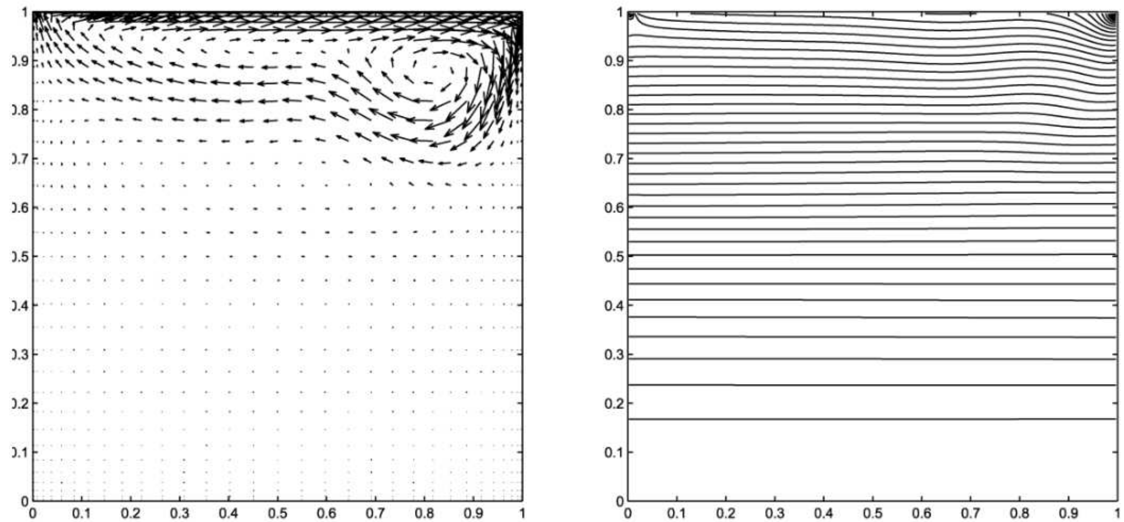
Vector calculus: scalar and vector fields







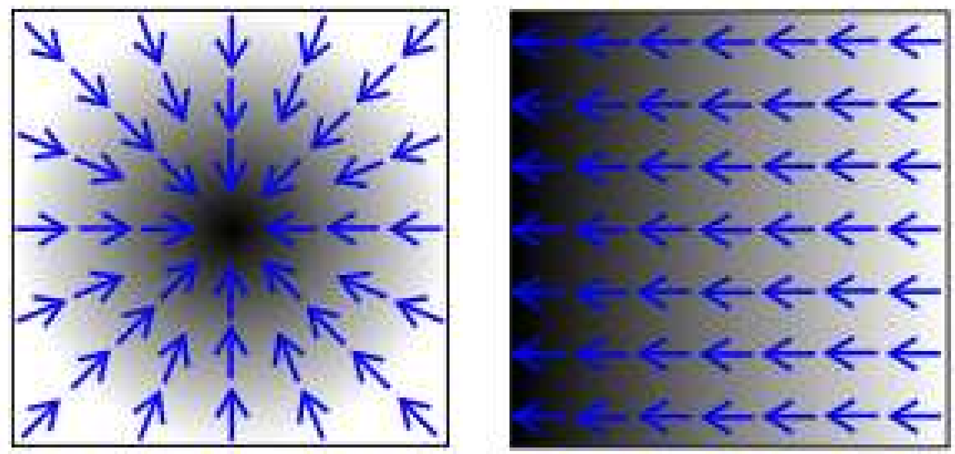
(a) Velocity vector field: $Re = 400$ **(b)** Pressure: $Re = 400$

scalar fields: Ψ, f, V, ρ

vector fields: A, F, E, H, D, B, J

Vector calculus: Gradient ∇

For the measure of steepness of a line, slope.



the gradient of a scalar field is a vector field which points in the direction of the greatest rate of increase of the scalar field, and whose magnitude is the greatest rate of change.

$$\nabla f(x,y,z) = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}, \quad \text{in Cartesian coordinates}$$

$$\nabla f(\rho,\theta,z) = \frac{\partial f}{\partial \rho}\hat{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \theta}\hat{e}_\theta + \frac{\partial f}{\partial z}\hat{e}_z, \quad \text{in cylindrical coordinates}$$

$$\nabla f(r,\theta,\phi) = \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{e}_\phi, \quad \text{in spherical coordinates}$$

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Maxwell's equations with total charge and current

(1831-1879)

- Gauss's law for the electric field:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \Longleftrightarrow \oint_S \mathbf{E} \cdot \mathrm{d}\, A = \frac{Q}{\epsilon_0},$$

- Gauss's law for magnetism:

$$\nabla \cdot \mathbf{B} = 0 \Longleftrightarrow \oint_S \mathbf{B} \cdot \mathrm{d}\, A = 0,$$

- Faraday's law of induction:

$$\nabla \times \mathbf{E} = -\kappa \frac{\partial}{\partial\, t} \mathbf{B} \Longleftrightarrow \oint_C \mathbf{E} \cdot \mathrm{d}\, l = -\kappa \frac{\partial}{\partial t} \Phi_B,$$

- Ampére's circuital law:

$$\nabla \times \mathbf{B} = \kappa \mu_0 (\mathbf{J} + \epsilon_0 \frac{\partial}{\partial\, t} \mathbf{E}) \Longleftrightarrow \oint_C \mathbf{B} \cdot \mathrm{d}\, l = -\kappa \mu_0 (\mathbf{I} + \epsilon_0 \frac{\partial}{\partial\, t} \Phi_E)$$

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Wave equations

- For a source-free medium, $\rho = \mathbf{J} = 0$,

$$\nabla \times (\nabla \times E) = -\mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} E,$$
$$\Rightarrow \qquad \nabla(\nabla \cdot E) - \nabla^2 E = -\mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} E.$$

- When $\nabla \cdot E = 0$, one has *wave equation*,

$$\nabla^2 E = \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} E$$

- which has following expression of the solutions, in 1D,

$$E = \hat{x}[f_+(z - vt) + f_-(z + vt)],$$

with $v^2 = \frac{1}{\mu_0 \epsilon_0} = c^2$.

- plane wave solutions: $E_+ = E_0 \cos(kz - \omega t)$, where $\frac{\omega}{k} = c$.

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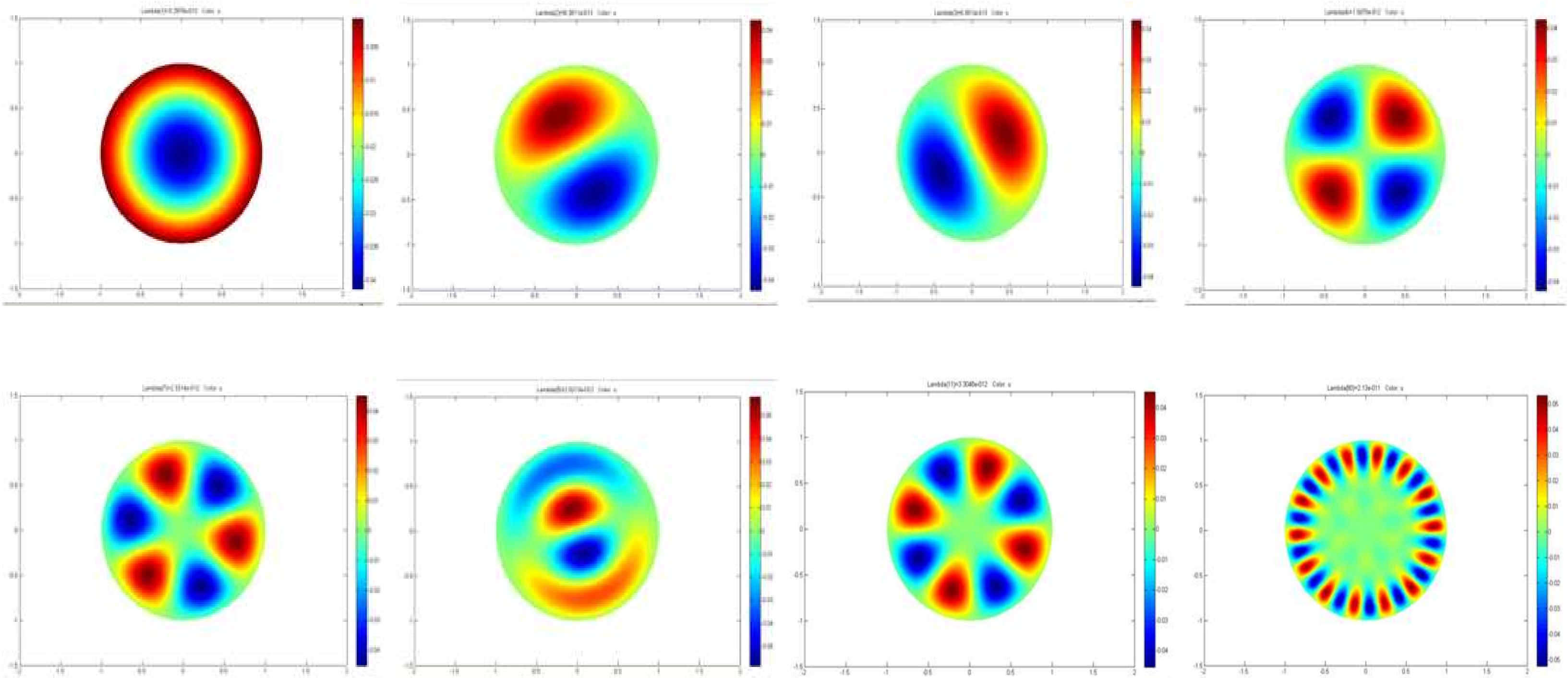
Cavity modes



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More PDEs

Diffusion equation:

$$\frac{\partial}{\partial t}A(x,t) = \kappa\frac{\partial^2}{\partial x^2}A(x,t)$$

Schrödinger equation:

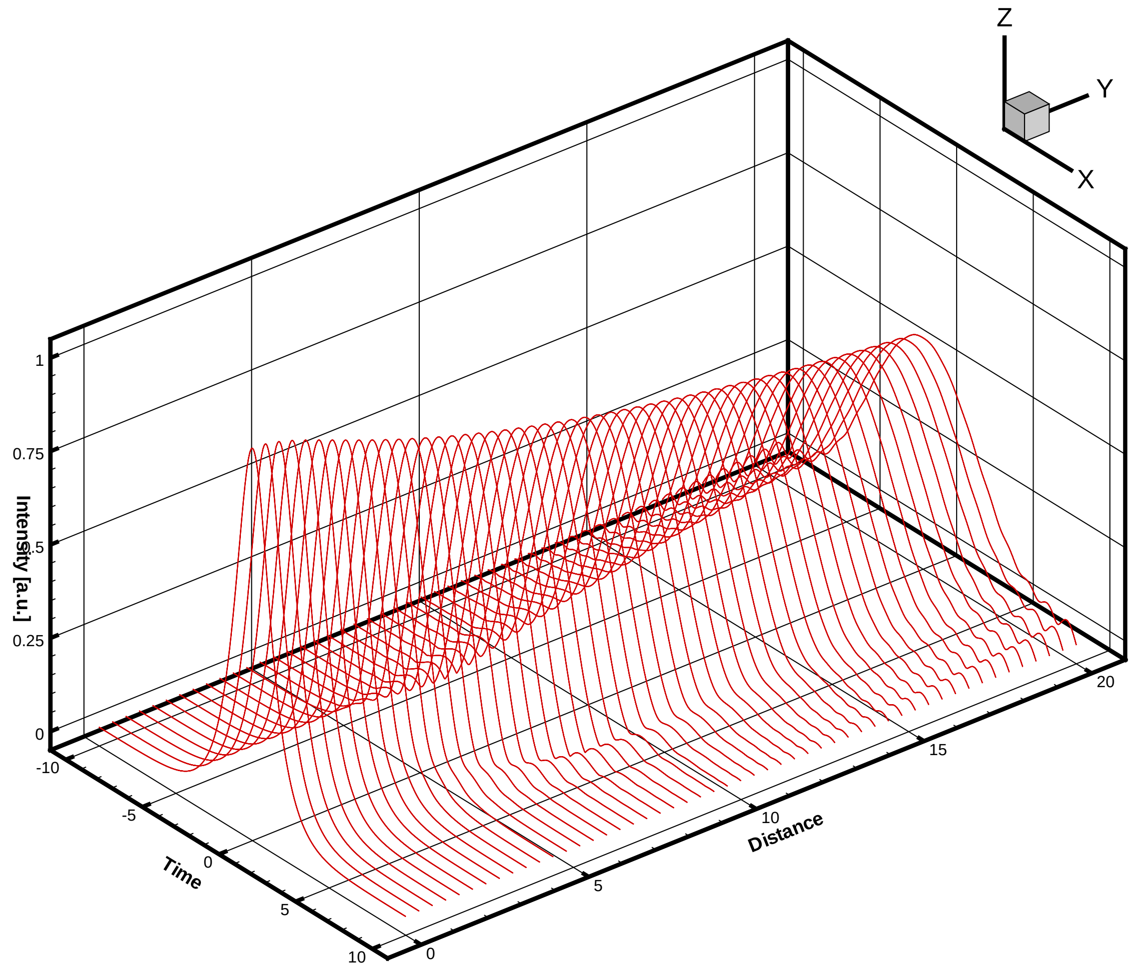
$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + V(x)\Psi(x,t)$$

Nonlinear Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t)+V(x)\Psi(x,t)+\gamma|\Psi(x,t)|^2\Psi(x,t)$$

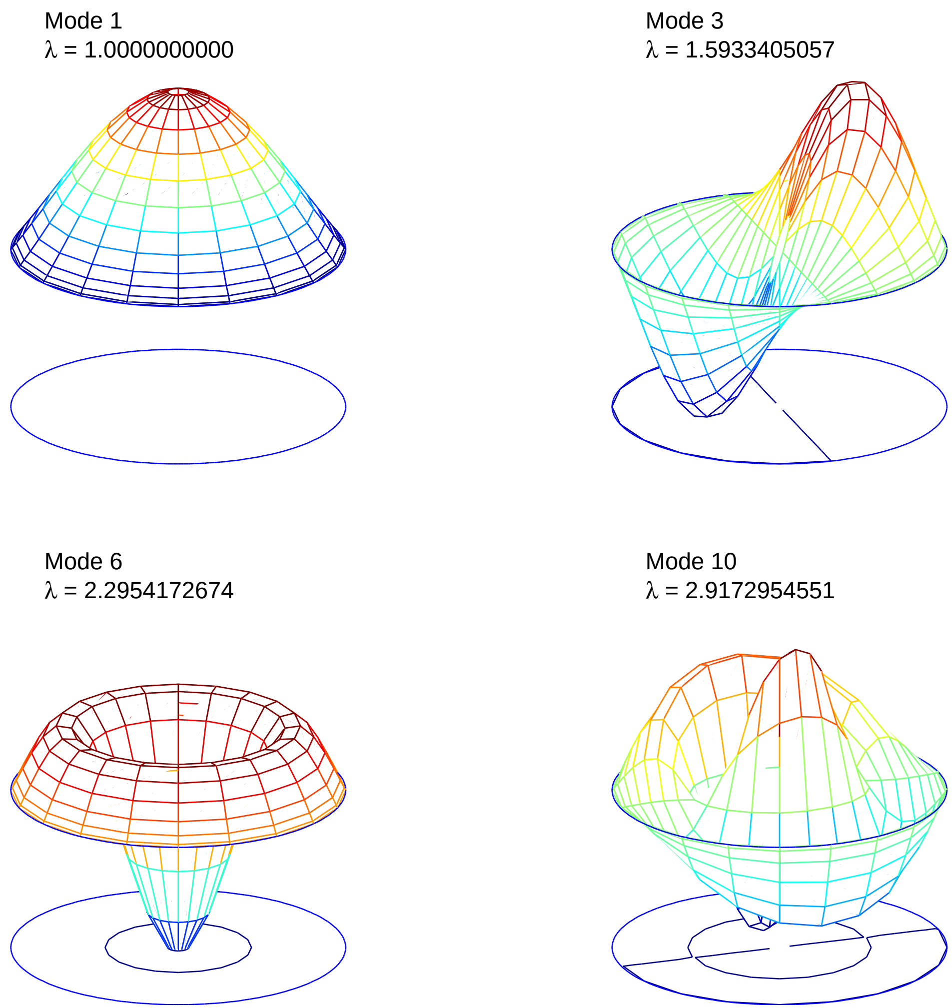
Diffusion equation

$$\frac{\partial}{\partial z}U(z,t) = \frac{i}{2}\frac{\partial^2}{\partial t^2}U(z,t)$$



Laplacian eq. in a disk

Eigenmodes of Laplacian equations, $[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2}]u(x,y) = f(x,y)$.



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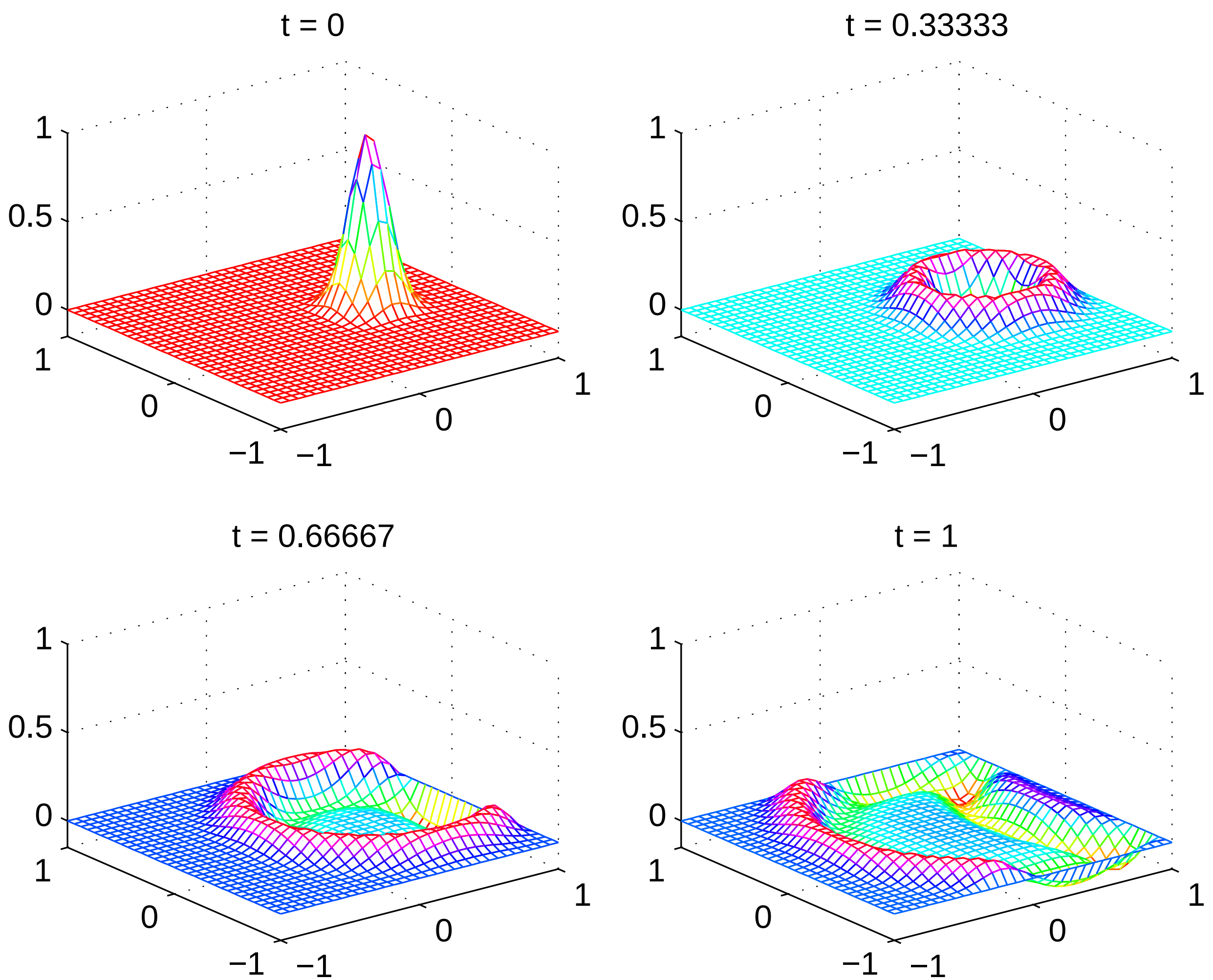
FFT method for wave equation

$$u_{tt} = u_{xx} + u_{yy}, \qquad -1 < x, y < 1, \quad t > 0, \quad u = 0 \quad \text{on the boundary}$$



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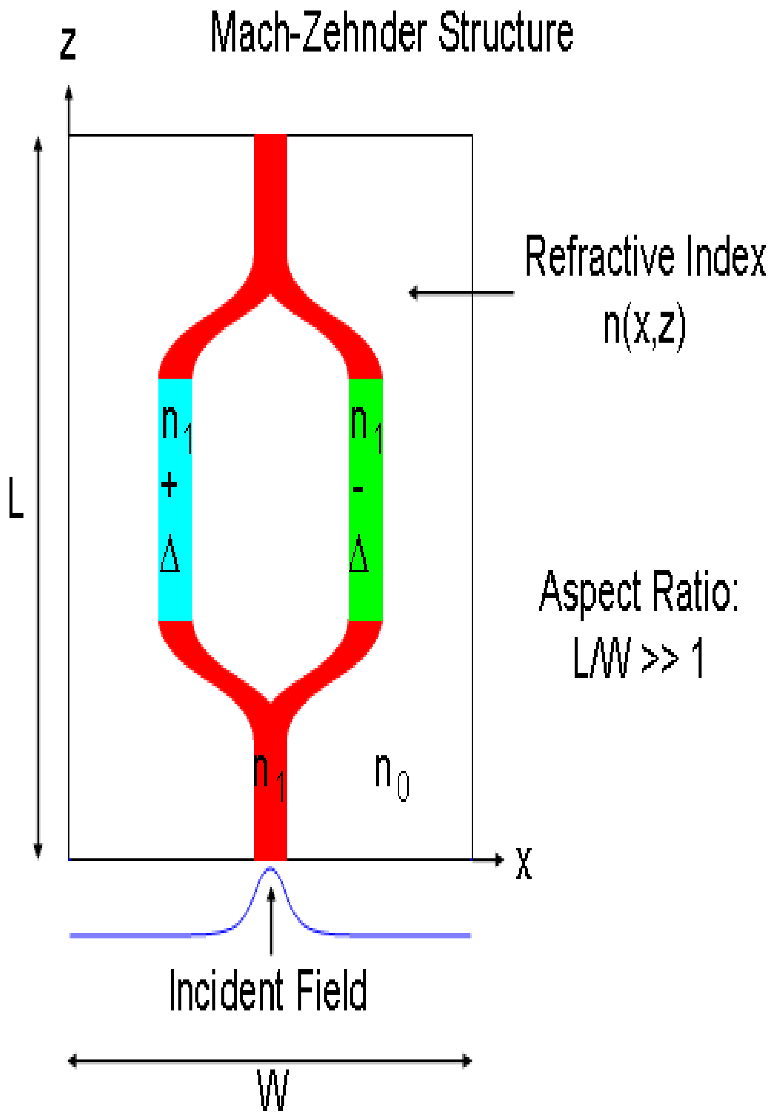
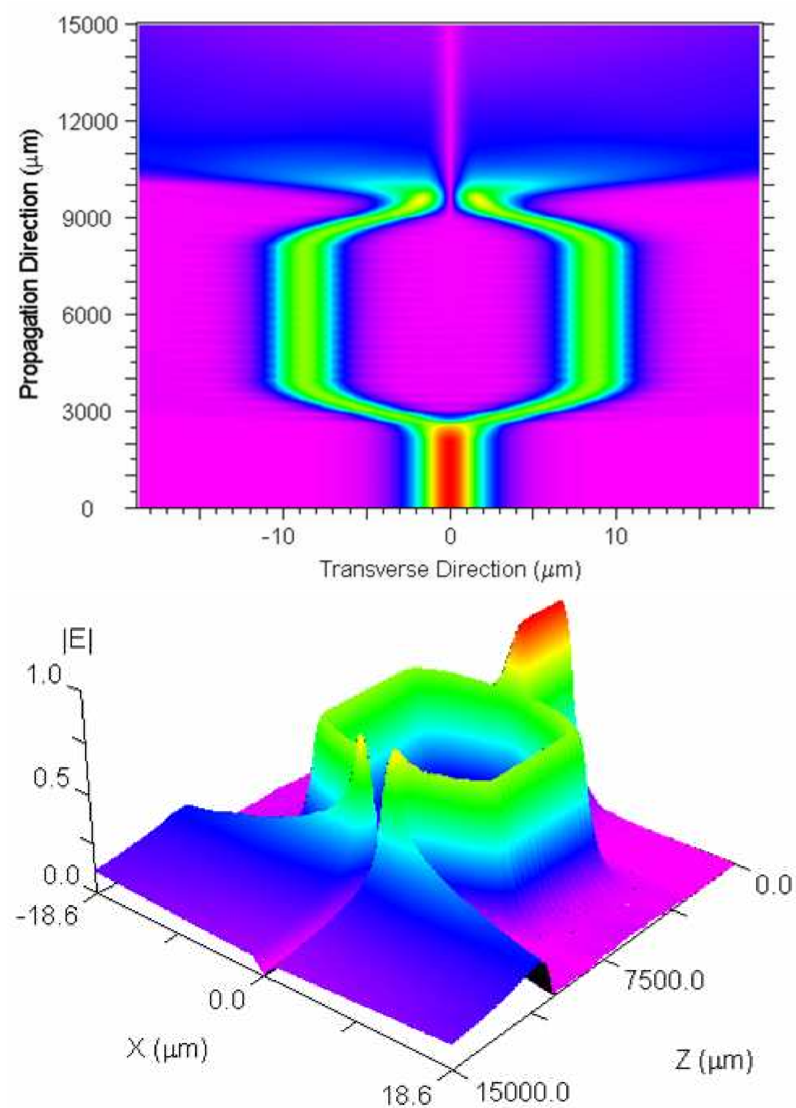
Mach-Zehnder structure





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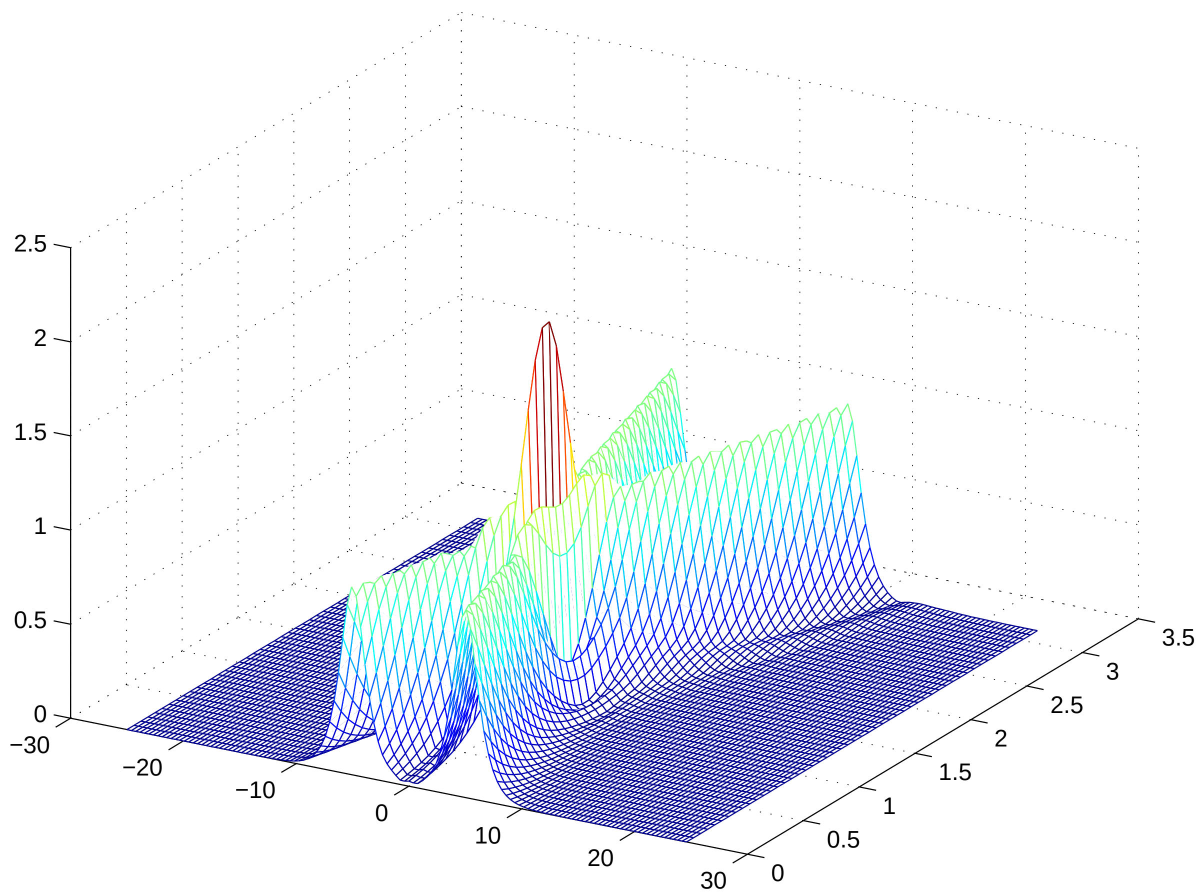
by BeamProp

Soliton collisions

$$U(t=0,x) = sech(x+x_0) + sech(x-x_0)$$



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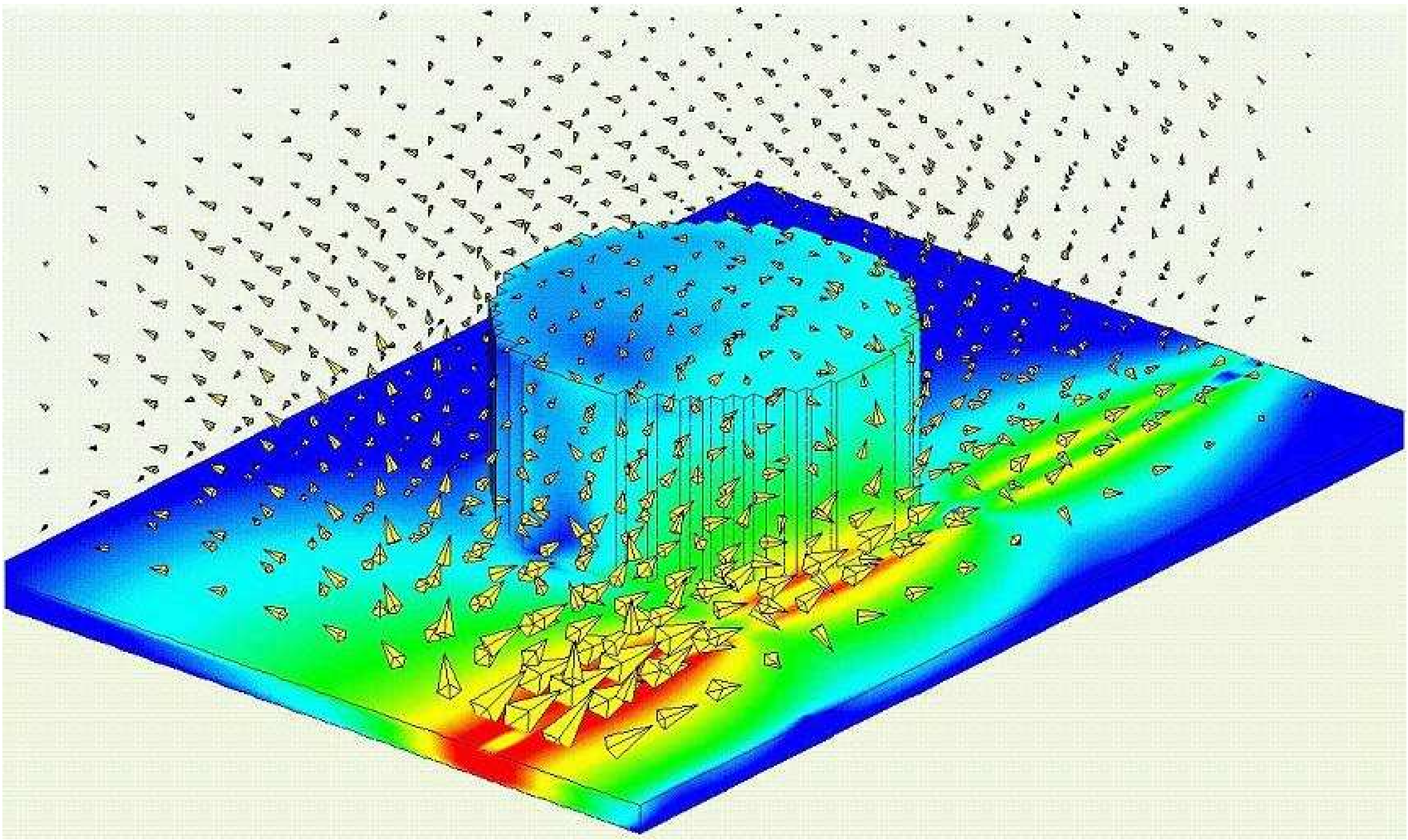
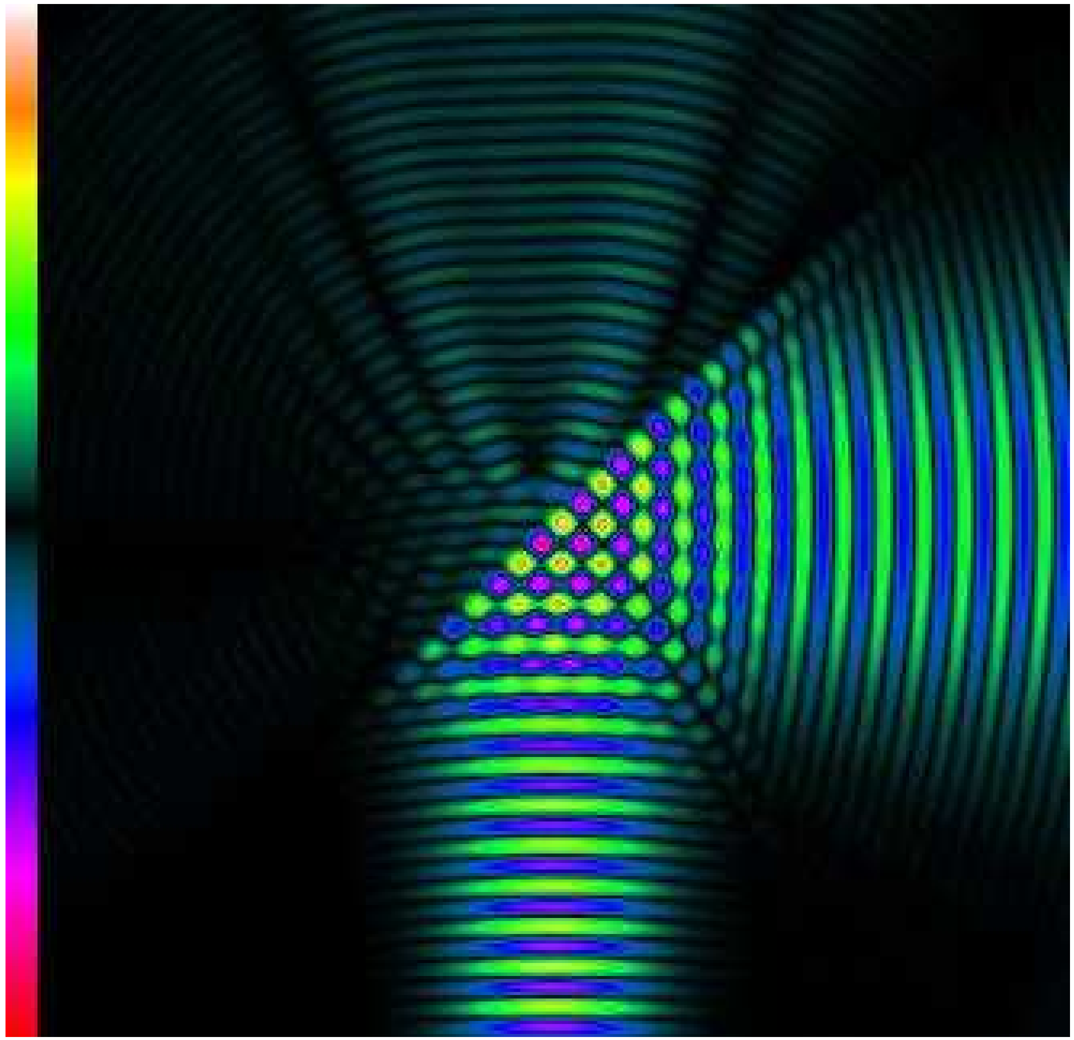
FDTD: example



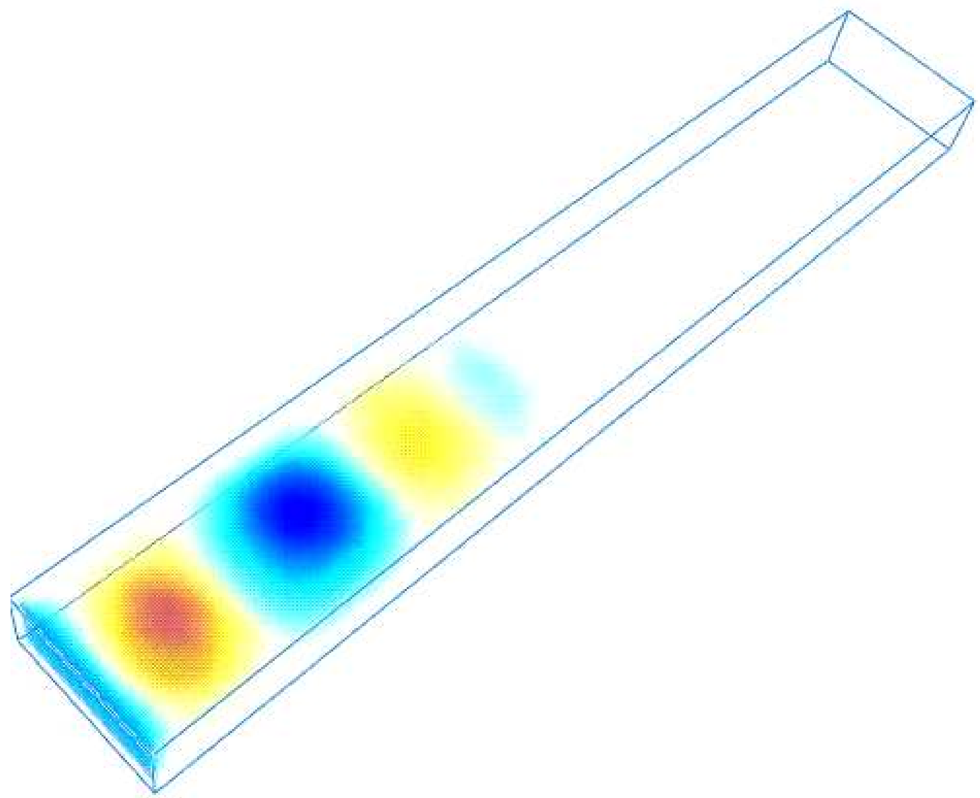
From: http://www.bay-technology.com

FDTD: example



From: http://www.fdtd.org

Metallic Waveguide



Examples in "Field and Wave Electromagnetics," 2nd ed., by David K. Cheng, pp. 554-555; simulated by ToyFDTD

Fields profile in 2D, $H_x(x,z)$, $H_z(x,z)$, and $E_y(x,z)$

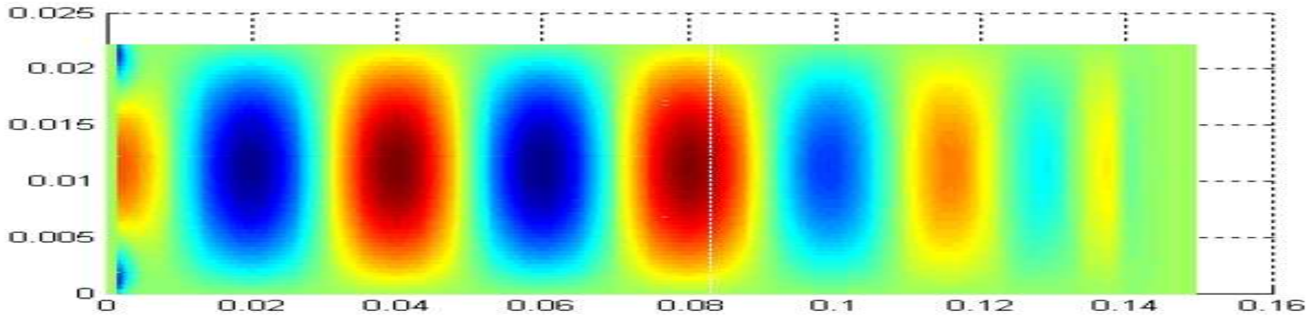


Figure 1: Hx 在 xz 平面

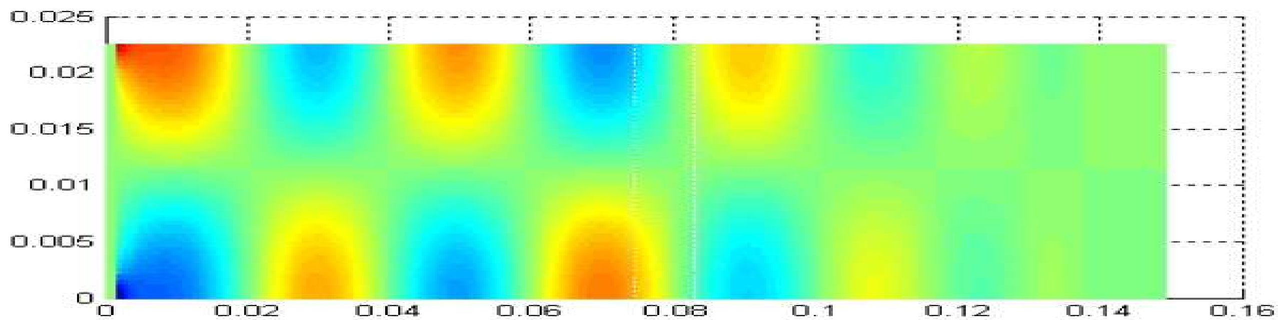


Figure 2: Hz 在 xz 平面

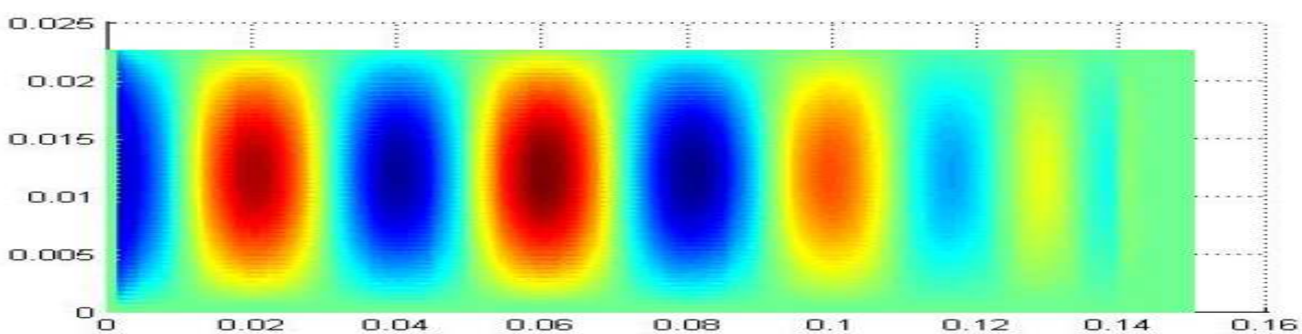


Figure 3: Ey 在 xz 平面

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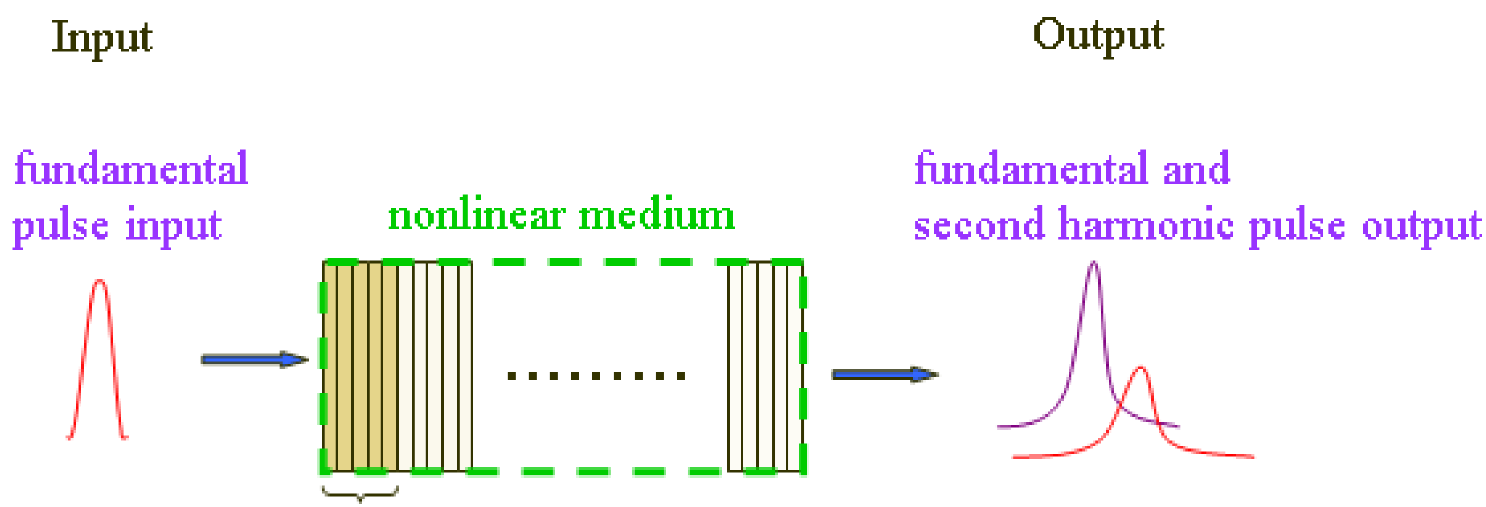
Optimization of SHG pulse

$$\frac{\partial A}{\partial z} = \frac{\eta}{2}\frac{\partial A}{\partial T} + i\xi_1\frac{\partial^2 A}{\partial T^2} - i\rho_1 A^* B,$$

$$\frac{\partial B}{\partial z} = -\frac{\eta}{2}\frac{\partial B}{\partial T} + i\xi_2\frac{\partial^2 A}{\partial T^2} - i\Delta k B - i\rho_1 A^2,$$



Divide a coherence length into 5 segments,
with each segment have nonlinear coefficient +1 or -1

1. **Office hours:**
   3:00-5:00PM, Thursday at Room 523, EECS bldg.
2. e-mail: rklee@ee.nthu.edu.tw:

   I should reply every e-mail.
3. **Website:** For more information and course slides:
   http://mx.nthu.edu.tw/∼rklee
4. **TA hours:**
   at Room 521, EECS bldg. (2×2 hours per week to be confirmed)
   (a) **I-Hong Chen**, 2nd-year PhD student, e-mail: ehome4829@gmail.com
   (b) **Chih-Yao Chen**, 2nd-year Master student, e-mail: hihittttt@yahoo.com.tw

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