# Linear Algebra, EE 10810/EECS 205004

**Note** 2.1

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- **Office Hours:**
  1. 1st Exam on Oct. 30th, (10:10AM - 1:00 PM, Friday), covering Chap. 1 and Chap. 2.

- **Integrity**: Next Quiz on October 14th, Wednesday, 10:10 AM - 10:30 AM.

- **Assignment**: for the Quiz on Oct. 21st
  1. Let $\mathcal{V}$ be a vector space having dimension $n$, and let $S$ be a subset of $\mathcal{V}$ that generate $\mathcal{V}$.
     (a) Prove that there is a subset of $S$ that is a basis for $\mathcal{V}$.
     (b) Prove that $S$ contains at least $n$ vectors.

  2. Let $W_1$ and $W_2$ be subspaces of a finite-dimensional vector space $\mathcal{V}$. Determine necessary and sufficient conditions on $W_1$ and $W_2$ so that $\dim(W_1 \cap W_2) = \dim(W_1)$.

  3. Prove that $\mathcal{T}$ is a linear transformation, find bases for both $N(\mathcal{T})$ and $R(\mathcal{T})$, and calculate the nullity and rank of $\mathcal{T}$.
     (a) $\mathcal{T}$: $\mathcal{R}^3 \to \mathcal{R}^2$ defined by $\mathcal{T}(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$.
     (b) $\mathcal{T}$: $P_2(\mathcal{R}) \to P_3(\mathcal{R})$ defined by $\mathcal{T}(f(x)) = x\, f(x) + f'(x)$.

  4. Let $\mathcal{V}$ and $\mathcal{W}$ be finite-dimensional vector spaces and $\mathcal{T} : \mathcal{V} \to \mathcal{W}$ be linear.
     (a) Prove that if $\dim(\mathcal{V}) < \dim(\mathcal{W})$, then $\mathcal{T}$ cannot be *onto*.
     (b) Prove that if $\dim(\mathcal{V}) > \dim(\mathcal{W})$, then $\mathcal{T}$ cannot be *one-to-one*.

**From Scratch !!**

- Summary of Chap. 1: Vector Space:
- Vector Space $\mathcal{V}$: Thms 1.1 - 1.2
- Subspace $\mathcal{W}$: Thms 1.3 - 1.4
- Generation Set, Span: Thm 1.5
- Linearly Dependence, Linearly Independence: Thms 1.6 - 1.7
- Basis: Thms 1.8, 1.9, 1.10 (Replacement Theorem), 1.11
- Every vector space has a basis: Thms 1.12 - 1.13

- Chapter 2: Linear Transformations and Matrices
- Definition: Linear Transformation $\mathcal{T}$
- Identity Transformation $\mathcal{I}$, Zero Transformation $\mathcal{T}_0$
- Definition: Null space (Kernel) $N(\mathcal{T})$
- Definition: Range (image) $R(\mathcal{T})$

- Theorem 2.1: $N(\mathcal{T})$ and $R(\mathcal{T})$ are subspaces of $\mathcal{V}$ and $\mathcal{W}$, respectively.

- Theorem 2.2: If $\beta = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a basis of $\mathcal{V}$, then

$$R(\mathcal{T}) = \mathrm{span}(\mathcal{T}(\beta)) = \mathrm{span}(\{\mathcal{T}(\vec{v}_1), \mathcal{T}(\vec{v}_2), \ldots, \mathcal{T}(\vec{v}_n)\})$$

- Definition: $\mathrm{Nullity}(\mathcal{T}) = \dim(N(\mathcal{T}))$
- Definition: $\mathrm{Rank}(\mathcal{T}) = \dim(R(\mathcal{T}))$

- Theorem 2.3 (Dimension Theorem):

$$\mathrm{nullity}(\mathcal{T}) + \mathrm{rank}(\mathcal{T}) = \dim(\mathcal{V})$$

- Theorem 2.4: $\mathcal{T}$ is *one-to-one* iff $N(\mathcal{T}) = \{\vec{0}\}$

- Theorem 2.5: The following are equivalent,
  1. $\mathcal{T}$ is one-to-one.
  2. $\mathcal{T}$ is onto.
  3. $\mathrm{rank}(\mathcal{T}) = \dim(\mathcal{V})$.