Photonic analogue of Josephson effect in a dual-species optical-lattice cavity

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Abstract: We extend the idea of quantum phase transitions of light in the photonic Bose-Hubbard model with interactions to two atomic species by a self-consistent mean field theory. The excitation of two-level atoms interacting with a coherent photon field is analyzed with a finite temperature dependence of the order parameters. Four ground states of the system are found, including an isolated Mott-insulator phase and three different superfluid phases. Like two weakly coupled superconductors, our proposed dual-species lattice system shows a photonic analogue of Josephson effect. i.e., the crossovers between two superfluid states. The dynamics of the proposed two species model provides a promising quantum simulator for possible quantum information processes.

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1. Introduction

Quantum phase transitions (QPTs) driven by quantum fluctuations at absolute zero temperature have been intensively studied in the interacting many-body problem [1]. Typically, it is difficult to control and probe such exotic quantum phenomena in a strongly correlated electronic system. Optical lattices (artificial crystals made by interfering laser beams) offer a versatile platform for studying the QPTs of trapped Bose gases [2]. In this situation, a gas of ultracold atoms driven by a periodic potential, the many-body dynamics from a Mott-insulator (MI) to a superfluid (SF) phase can be described by the Bose-Hubbard model that includes on-site two-atom interactions and hopping between adjacent sites [3].

Unlike weakly interacting ultracold atoms, photons are non-interacting bosons and there is no possibility to have any QPT in purely photonic systems. For a pure Bose system, the conducting phase at zero temperature is presumably always superfluid [4]. However, engineered composites of optical cavities, few-level atoms, and laser beams may form a strongly interacting many-body system where the concepts and methods of condensed matter physics might be studied from the viewpoint of quantum optics. In such a case, a photonic condensed matter analogue could be realized with state-of-the-art photonic crystals embedded within high-Q defect cavities. Like in the optical lattices for matter waves case, the QPTs from photonic insulator (excitation localization) to superfluid (excitation delocalization) are predicted by the Bose-Hubbard model with additional photon-atom interactions [5–7]. Related quantum transitions have also been predicted in a Heisenberg spin $1/2$ Hamiltonian [8], two species Bose-Hubbard model [9], and solved exactly in the one dimensional case [10]. The QPTs of light opened the possibilities to study critical quantum phenomena in conventional condensed matter systems by manipulating the interaction between photons and atoms.

Recently, we illustrated the generality of the concept for the QPT of light by constructing the dressed-state basis for an arbitrary number of two-level atoms (TLAs) [11]. As the number of TLAs increases, collective effects due to the interactions of atoms among themselves give rise to intriguing many-body phenomena. With the Dicke-Bose-Hubbard Hamiltonian, we showed that the Mott insulator to superfluid QPTs with photons can be realized in an extended Dicke model for an arbitrary number of two-level atoms. As quantum many-body effects associated with Bose-Einstein condensations in optical lattices have been reported for a long-time [12], it is believed that the realization of strongly correlated many-body photonic cavity quantum electrodynamic (c-QED) systems in experiments is likely to happen soon.

From the point of view of a practical experimental setup, instead of modifying the Q-value of individual optical cavities or changing the overlapping in the adjacent cavities [13], it is more easier to allocate different atomic species in different photonic cavity sites. Recently, based on scanning electron microscopy technologies, the addressability of ultracold atoms in a two-dimensional optical lattice has been demonstrated experimentally down to the single-atom and single-site leve [14]. A natural consequence is to consider the QPT of atom-light interacting systems using different atomic species. In this work, we investigate theoretically QPTs of light

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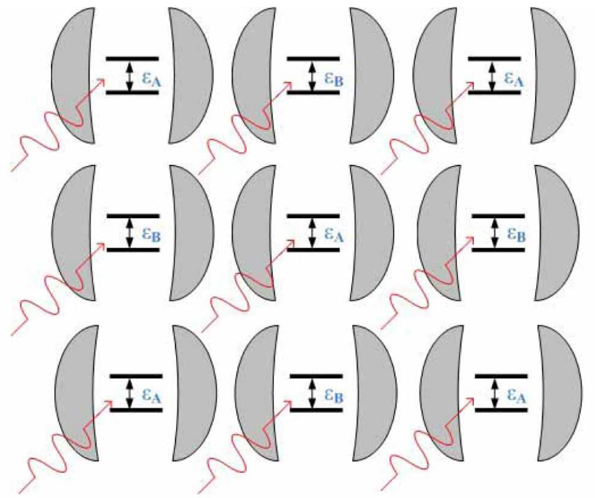

Fig. 1. Illustration schematic for the proposed system. An array of high-Q electromagnetic cavities is formed in the configuration of the square lattice, and each cavity contains a single two-level atom of the type A or B, which is spaced at intervals. In this configuration, the center cavity has four nearest inter-species neighbors (different atomic type) and four next-nearest intra-species neighbors (the same atomic type). ε_A and ε_B are the transition energies for the atomic species A and B, respectively. The incident optical field interacting with the bipartite lattice is also shown in the red color.

from MI to SF quantum phase transitions in high Q-value optical cavities with two species of atoms based on the photonic Bose-Hubbard model embedded with the Jaynes-Cummings interaction Hamiltonian. A rather rich low temperature phase diagram emerges even from a relatively simple structure of the cavity lattice. As in the phase diagram of two-component bosons on an optical lattice [15,16], we show that separated phases corresponding to a MI and three composite SF phases exist. Moreover, these composite SF phases have an analogue to the well known Josephson effect in two weakly coupled superconductors. We expect that future controllable light-wave technologies shall lead to an enhanced understanding of the QPTs of light and the introduction of new applications; such as quantum engineering of the ground-state wave function with distinctive properties, to name an example.

2. Model Hamiltonian

For two atomic species interacting with photons, we consider an array of two-dimensional square photonic bandgap microcavities with two types of two-level atoms (TLAs) labelled as A and B, as illustrated in Fig. 1. The array is made from high-Q electromagnetic cavities. Each cavity contains a single TLA of the type A or B, which is spaced at intervals. The atoms are assumed to interact strongly with photons in the photon-blockade regime so that a coherent composite polaritonic (photon and atom) system is formed in the presence of radiation fields [5,11]. Photons can travel from one cavity to another. We assume that the travelling process for photons is dominated by the hopping between adjacent sites. The value of the hopping matrix elements is a function of the distance between the cavities. The interaction between a two-level atom and the quantized photon mode of an optical cavity is described by the Jaynes-Cummings model [17], with the two-site Hamiltonian for the i-th unit cell (two cavities numbered as $2j$

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and $2j+1$ for A- and B-type atoms, respectively),

$$\begin{aligned} H_i^{two-site} &= \omega \hat{a}_{2j}^{\dagger}\hat{a}_{2j} + \varepsilon_{A,2j}\hat{\sigma}_{A,2j}^{z} + g_{A,2j}(\hat{a}_{2j}\hat{\sigma}_{A,2j}^{\dagger} + \hat{a}_{2j}^{\dagger}\hat{\sigma}_{A,2j}^{-}), \\ &+ \omega \hat{a}_{2j+1}^{\dagger}\hat{a}_{2j+1} + \varepsilon_{B,2j+1}\hat{\sigma}_{B,2j+1}^{z} + g_{B,2j+1}(\hat{a}_{2j+1}\hat{\sigma}_{B,2j+1}^{\dagger} + \hat{a}_{2j+1}^{\dagger}\hat{\sigma}_{B,2j+1}^{-}), \end{aligned} \quad (1)$$

where $\varepsilon_{A(B),j}$ is the transition energy for the TLA in j-th cavity and ω is the radiation field frequency which we have assumed to be the same at all cavities. The atom-photon coupling, $g_{A(B),j}$, is assumed to be real here. $\hat{a}_j^{\dagger}$ and $\hat{a}_j$ are the raising and lowering operators for photons, respectively. The excitation of the A (or B)-type TLA from the low energy state to the high energy state is described by the operator $\hat{\sigma}_{A(B),j}^{\dagger}$, while the operator $\hat{\sigma}_{A(B),j}^{-}$ describes the reverse process. By including the hopping process for photons, the total Hamiltonian describing our proposed system for a configuration of $2N$ cavities is

$$H = -\sum_{\langle i,j\rangle}^{2N} \kappa_{i,j}\hat{a}_i^{\dagger}\hat{a}_j + \sum_{i=1}^{N}\left[-\mu(\hat{n}_{2i} + \hat{n}_{2i+1}) + H_i^{two-site}\right], \quad (2)$$

where $\kappa_{i,j}$ is the hopping matrix element and i, j are the cavity indices. We also introduce a chemical potential term μ for photons which control the overall strength of the photon field, and $\hat{n}_i = \hat{a}_i^{\dagger}\hat{a}_i$ is the photon number operator. For simplicity, in this configuration, each cavity has four nearest inter-species neighbors (different atomic type) with the hopping coefficient $\kappa_{A\leftrightarrow B} = \kappa_{B\leftrightarrow A} \equiv \kappa$, and four next-nearest intra-species neighbors (the same atomic type) with the hopping coefficient $\kappa_{A\leftrightarrow A} = \kappa_{B\leftrightarrow B} \equiv \kappa'$.

To solve the model Hamiltonian, we introduce a self-consistent mean field approach by approximating

$$\hat{a}\hat{\sigma}_{A(B)}^{\pm} \approx \langle\hat{a}\rangle\hat{\sigma}_{A(B)}^{\pm} + \hat{a}\langle\hat{\sigma}_{A(B)}^{\pm}\rangle - \langle\hat{a}\rangle\langle\hat{\sigma}_{A(B)}^{\pm}\rangle. \quad (3)$$

A similar decomposition is also applied to the terms associated with $\hat{a}^{\dagger}\hat{\sigma}_{A(B)}^{\pm}$. It is convenient to introduce two superfluid order parameters $\psi_{A(B)}$ for photons and two TLA order parameters $J_{A(B)}$ for the atomic species A and B, respectively, where

$$\psi_A \equiv \langle\hat{a}_{2j}\rangle \equiv \langle\hat{a}_{2j}^{\dagger}\rangle, \quad (4)$$

$$\psi_B \equiv \langle\hat{a}_{2j+1}\rangle \equiv \langle\hat{a}_{2j+1}^{\dagger}\rangle, \quad (5)$$

$$J_{A(B)} \equiv \langle\hat{\sigma}_{A(B)}^{\dagger}\rangle \equiv \langle\hat{\sigma}_{A(B)}^{-}\rangle. \quad (6)$$

The order parameters introduced here are indeterminate at the phase transition in our photonic Bose-Hubbard Hamiltonian [5–7]. The system is in the photonic SF phase for non-zero photon order parameters, $\psi_{A(B)} \neq 0$, and in the MI phase otherwise, $\psi_{A(B)} = 0$. The parameter $J_{A(B)}$ can be viewed as an order parameter for the atomic coherent states. The physical meaning for $J_{A(B)}$ as a coherent state order parameter for the atoms lies on the coherent superposition of the ground and excited state of the atoms. With these order parameters, the possibility to have a QPT at low temperature would be demonstrated in the right parameter regimes. In the following we assume that both $\psi_{A(B)}$ and $J_{A(B)}$ are real numbers and spatially independent, i.e., there is no spontaneous symmetry-breakings in the translation and rotation for the infinite system considered here. The assumption that $\psi_{A(B)}$ and $J_{A(B)}$ are real numbers will be justified later by the self-consistent mean-field solution.

2.1. *Mean-field solution for H^a, with the atomic operators only*

In general it is very difficult to diagonalize the model Hamiltonian in Eq. (2) and find out all the desired QPTs of the system. However, it may be instructive to diagonalize part of the total Hamiltonian first. With the mean-field approximation, Eq. (2) can be decomposed into two Hamiltonians, labelled as H^a and H^p, where only the atomic and photonic operators are considered separately. For one atom (A or B) inside each cavity, the mean-field Hamiltonian for atomic operators is

$$H^a_{A(B)} = \varepsilon_{A(B)}\hat{\sigma}^z_{A(B)} + g_{A(B)}\psi_{A(B)}\left[\hat{\sigma}^\dagger_{A(B)} + \hat{\sigma}^-_{A(B)}\right]. \tag{7}$$

This Hamiltonian can be diagonalized easily in the form,

$$H^a_{A(B)} = E_{A(B)}\left[\hat{\gamma}^\dagger_{1,A(B)}\hat{\gamma}_{1,A(B)} - \hat{\gamma}^\dagger_{0,A(B)}\hat{\gamma}_{0,A(B)}\right], \tag{8}$$

with the corresponding eigen-energy

$$E_{A(B)} = \left\{\left[g_{A(B)}\psi_{A(B)}\right]^2 + \varepsilon^2_{A(B)}\right\}^{1/2}, \tag{9}$$

and the corresponding dressed atomic eigen-operators,

$$\begin{aligned}\hat{\gamma}_{1,A(B)} &= \sqrt{\frac{1}{2}\left[1+\frac{\varepsilon_{A(B)}}{E_{A(B)}}\right]}\hat{f}_{1,A(B)} + \sqrt{\frac{1}{2}\left[1-\frac{\varepsilon_{A(B)}}{E_{A(B)}}\right]}\hat{f}_{0,A(B)}, \\ \hat{\gamma}_{0,A(B)} &= -\sqrt{\frac{1}{2}\left[1-\frac{\varepsilon_{A(B)}}{E_{A(B)}}\right]}\hat{f}_{1,A(B)} + \sqrt{\frac{1}{2}\left[1+\frac{\varepsilon_{A(B)}}{E_{A(B)}}\right]}\hat{f}_{0,A(B)}.\end{aligned} \tag{10}$$

Here, the new fermion operators $\hat{f}_{0(1)}$ and $\hat{f}^\dagger_{0(1)}$ are introduced to describe the finite temperature for the system. They are related to the pseudo-spin atomic operators,

$$\hat{\sigma}^\dagger_{A(B)} = \hat{f}^\dagger_{1,A(B)}\hat{f}_{0,A(B)}, \tag{11}$$

$$\hat{\sigma}^-_{A(B)} = \hat{f}^\dagger_{0,A(B)}\hat{f}_{1,A(B)}, \tag{12}$$

$$\hat{\sigma}^z_{A(B)} = \hat{f}^\dagger_{1,A(B)}\hat{f}_{1,A(B)} - \hat{f}^\dagger_{0,A(B)}\hat{f}_{0,A(B)}. \tag{13}$$

Then the TLA order parameter $J_{A(B)}$ at zero temperature can be derived as

$$\begin{aligned}J_{A(B)} &= \langle f^\dagger_{1,A(B)}f_{0,A(B)}\rangle, \\ &= \sqrt{\frac{1}{4}\left[1-\frac{\varepsilon^2_{A(B)}}{E^2_{A(B)}}\right]}\left[\langle\hat{\gamma}^\dagger_{1,A(B)}\hat{\gamma}_{1,A(B)}\rangle - \langle\hat{\gamma}^\dagger_{0,A(B)}\hat{\gamma}_{0,A(B)}\rangle\right], \\ &= -\frac{\psi_{A(B)}g_{A(B)}}{2E_{A(B)}}.\end{aligned} \tag{14}$$

Furthermore, we can assume that the occupation number of these dressed atomic operators follows Bose-Einstein statistics at a given temperature as

$$\begin{aligned}\langle\hat{\gamma}^\dagger_{1,A(B)}\hat{\gamma}_{1,A(B)}\rangle &= \frac{1}{e^{\beta E_{A(B)}}+1}, \\ \langle\hat{\gamma}^\dagger_{0,A(B)}\hat{\gamma}_{0,A(B)}\rangle &= \frac{1}{e^{-\beta E_{A(B)}}+1},\end{aligned} \tag{15}$$

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where $\beta = 1/k_B T$ and we set the Boltzmann constant $k_B = 1$ for the sake of convenience. For a finite temperature, T, the TLA order parameter $J_{A(B)}$ becomes,

$$J_{A(B)} = -\frac{\psi_{A(B)} g_{A(B)}}{2E_{A(B)}} Tanh[\frac{E_{A(B)}}{2T}]. \tag{16}$$

Note that here $J_{A(B)}$ is a real number and consistent with our initial assumption. From Eq. (16), it can be clearly seen that only at zero temperature, the TLA order parameter $J_{A(B)}$ has a maximum value. The atomic order parameter $J_{A(B)}$ is a linear function of the photonic order parameter $\psi_{A(B)}$, with the coefficient defined by the coupling strength $g_{A(B)}$ and the atomic eigen-energy $E_{A(B)}$. Whenever the atom-photon coupling strength $g_{A(B)}$ is zero, the atomic order parameter $J_{A(B)}$ goes to zero as there is no atom-photon interaction. The larger the coupling strength, the easier to have a non-zero atomic order parameter.

2.2. *Mean-field solution for H^p, with the photonic operators only*

For the radiation fields, the related photonic mean-field Hamiltonian in Eq. (2) can be separated for the two sites $2j$ and $2j+1$ as

$$H^p_{2j} = -\sum_{\langle i,j\rangle}^{N} \kappa_{i,j}\hat{a}_i^\dagger a_j + \sum_{j=1}^{N}\left[(\omega-\mu)\hat{n}_{2j} + g_{A,2j}J_A(\hat{a}_{2j} + \hat{a}_{2j}^\dagger)\right], \tag{17}$$

$$H^p_{2j+1} = -\sum_{\langle i,j\rangle}^{N} \kappa_{i,j}\hat{a}_i^\dagger \hat{a}_j + \sum_{j=1}^{N}\left[(\omega-\mu)\hat{n}_{2j+1} + g_{B,2j+1}J_B(\hat{a}_{2j+1} + \hat{a}_{2j+1}^\dagger)\right]. \tag{18}$$

These Hamiltonians can be diagonalized by the Fourier transforms for the even- and odd-numbered sites, i.e.,

$$\hat{a}_{A,k} = \sum_{j}^{N} e^{i\vec{k}\cdot\vec{r}_{2j}}\,\hat{a}_{A,2j}, \tag{19}$$

$$\hat{a}_{B,k} = \sum_{j}^{N} e^{i\vec{k}\cdot\vec{r}_{2j+1}}\,\hat{a}_{B,2j+1}. \tag{20}$$

In terms of $\hat{a}_{A(B),k}$, the combined photonic mean-field Hamiltonian of two sites in a unit cell becomes

$$\begin{aligned} H^p &= H^p_{2j} + H^p_{2j+1}, \qquad (21)\\ &= [g_A J_A(\hat{a}^\dagger_{A,k=0} + \hat{a}_{A,k=0}) + g_B J_B(\hat{a}^\dagger_{B,k=0} + \hat{a}_{B,k=0})],\\ &+ \sum_k \Omega_0(\vec{k})\left[\hat{a}^\dagger_{A,k}\hat{a}_{A,k} + \hat{a}^\dagger_{B,k}\hat{a}_{B,k}\right] + \sum_k \Omega_1(\vec{k})\left[\hat{a}^\dagger_{A,k}\hat{a}_{B,k} + \hat{a}^\dagger_{B,k}\hat{a}_{A,k}\right], \end{aligned}$$

where

$$\Omega_0(\vec{k}) = -4\kappa' Cos(k_x)Cos(k_y) + \omega - \mu, \tag{22}$$

$$\Omega_1(\vec{k}) = -2\kappa\left[Cos(k_x) + Cos(k_y)\right], \tag{23}$$

with the Fourier vector $\vec{k} = k_x\hat{x} + k_y\hat{y}$.

The photonic mean-field Hamiltonian H^p in Eq. (21) can be diagonalized in the following form

$$\begin{aligned} H^p &= \sum_k \left[\Omega_{sym}(\vec{k})\hat{a}^\dagger_{sym,k}\hat{a}_{sym,k} + \Omega_{asym}(\vec{k})\hat{a}^\dagger_{asym,k}\hat{a}_{asym,k}\right] \qquad (24)\\ &+ g_{sym}J_{sym}(\hat{a}^\dagger_{sym,k=0} + \hat{a}_{sym,k=0}) + g_{asym}J_{asym}(\hat{a}^\dagger_{asym,k=0} + \hat{a}_{asym,k=0}), \end{aligned}$$

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where we introduce symmetric and antisymmetric photon field operators $\hat{a}_{sym,k}$ and $\hat{a}_{asym,k}$ as,

$$\hat{a}_{sym,k} \equiv (\hat{a}_{A,k} + \hat{a}_{B,k})/\sqrt{2}, \tag{25}$$

$$\hat{a}_{asym,k} \equiv (\hat{a}_{A,k} - \hat{a}_{B,k})/\sqrt{2}, \tag{26}$$

with the related photonic dispersions

$$\Omega_{sym}(\vec{k}) = \Omega_0(\vec{k}) + \Omega_1(\vec{k}), \tag{27}$$

$$\Omega_{asym}(\vec{k}) = \Omega_0(\vec{k}) - \Omega_1(\vec{k}). \tag{28}$$

Note that from Eq. (24), we can find the stability of photon fields by requiring a non-zero energy (frequency), i.e., $\Omega_{sym(asym)}(\vec{k}) \geq 0$. Equivalently, we have the condition $(\omega - \mu) - 4\kappa' > 4\kappa$ required for photon fields. The assumption that $J_{A(B)}$ is spatially independent implies that the atoms couple only to the $\vec{k} = 0$ photon mode. In other words, we only search for the spatially homogeneous solutions in our mean-field theory. From the coupling of photon field to atoms, one may imply that the $\vec{k} = 0$ mode of photon develops a ground state, with the expectation values

$$\langle \hat{a}_{sym,k=0} \rangle = -\frac{(g_A J_A + g_B J_B)}{\sqrt{2}\,\Omega_{sym}(\vec{k} = 0)}, \tag{29}$$

$$\langle \hat{a}_{asym,k=0} \rangle = -\frac{(g_A J_A - g_B J_B)}{\sqrt{2}\,\Omega_{asym}(\vec{k} = 0)}. \tag{30}$$

Again, the photonic order parameter $\langle \hat{a}_{sym(asym)} \rangle$ is a linear function of the atomic order parameter $J_{A(B)}$, with the coefficient defined by the coupling strength $g_{A(B)}$ and the atomic eigen-energy $\Omega(\vec{k} = 0)$. Whenever the atom-photon coupling strength $g_{A(B)}$ is zero, the photonic order parameter goes to zero as there is no atom-photon interactions.

Now, the order parameters for photons can be found as

$$\psi_A = \frac{1}{\sqrt{2}} \left[\langle \hat{a}_{sym,k=0} \rangle + \langle \hat{a}_{asym,k=0} \rangle \right], \tag{31}$$

$$\psi_B = \frac{1}{\sqrt{2}} \left[\langle \hat{a}_{sym,k=0} \rangle - \langle \hat{a}_{asym,k=0} \rangle \right]. \tag{32}$$

Together with the order parameters for a two level atom in Eq. (16), we have a self-consistent set of equations to determine the photonic superfluid order parameters $\psi_{A(B)}$ and the atomic coherent state order parameter for the TLA $J_{A(B)}$. Based on these results, the proposed two-site Hamiltonian will be used to study the QPT of light analytically and numerically in the following.

3. Analysis of the mean field equations

In general, there are more than one solution to the self-consistent mean-field equations. But, by calculating the corresponding free energy, one can determined the equilibrium state which occupies the lowest free energy. The free energy density of our system F_s can be written as $F_s = F_a + F_p - E_m$, where F_a, F_p, and E_m are the free energy densities associated with the atomic-only Hamiltonian in Eq. (8), photonic-only Hamiltonian in Eq. (24), and the correction term for the double counting in our mean-field decomposition in Eq. (3), respectively, i.e.,

$$F_f = -E_A - E_B - \frac{2}{\beta} \left[\ln(1 + e^{-\beta E_A}) + \ln(1 + e^{-\beta E_B}) \right], \tag{33}$$

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$$F_p = \frac{1}{V}\sum_k \frac{1}{\beta} ln\left[(1-e^{-\beta\Omega_A(\vec{k})})(1-e^{-\beta\Omega_B(\vec{k})})\right] - \frac{(g_A J_A)^2}{\Omega_A} - \frac{(g_B J_B)^2}{\Omega_B}, \quad (34)$$

$$E_m = 2\left[\frac{(g_A J_A)^2}{\Omega_A} + \frac{(g_B J_B)^2}{\Omega_B}\right], \quad (35)$$

where V is the volume of the system.

At zero temperature, one can find the corresponding ground state energy density per cavity for the system as

$$E_g = \frac{1}{2}\left[\frac{(g_A J_A)^2}{\Omega_A} + \frac{(g_B J_B)^2}{\Omega_B} - E_A - E_B\right]. \quad (36)$$

3.1. *Solutions for the special case of identical atoms*

The combination of the mean-field equations in Eqs. (31-32) and Eq. (16) yields the photon order parameter, $\psi_{A(B)}$, in the following matrix form,

$$\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \begin{pmatrix} \Omega_+^{-1} & \Omega_-^{-1} \\ \Omega_-^{-1} & \Omega_+^{-1} \end{pmatrix} \times \begin{pmatrix} \frac{g_A^2}{4E_A} Tanh(\frac{E_A}{2T})\psi_A \\ \frac{g_B^2}{4E_B} Tanh(\frac{E_B}{2T})\psi_B \end{pmatrix} \quad (37)$$

where $\Omega_+^{-1} = \Omega_{sym}^{-1} + \Omega_{asym}^{-1}$ and $\Omega_-^{-1} = \Omega_{sym}^{-1} - \Omega_{asym}^{-1}$ are the related photon dispersions. Eq. (37) supports a trivial solution to the mean-field equations, i.e., $\psi_A = \psi_B = 0$. Non-trivial solutions can only be obtained numerically for the reason that $E_{A(B)}$ is a function of $\psi_{A(B)}$ in Eq. (9).

In order to give a simple picture of the QPTs of light in our system, we first consider the special case when the atoms A and B are identical. For this case $\psi_A = \psi_B = \psi$, $E_A = E_B = E$, $\varepsilon_A = \varepsilon_B = \varepsilon$, and $g_A = g_B = g$, we can derive the photonic superfluid order parameter explicitly,

$$\psi = \frac{g^2}{4E\Omega_{sym}} Tanh(\frac{E}{2T})\,\psi. \quad (38)$$

In particular, for a non-trivial order parameter solution, $\psi \neq 0$, to exist at zero temperature one can find

$$\psi = \sqrt{(\frac{g}{4\Omega_{sym}})^2 - (\frac{\varepsilon}{g})^2}, \quad (39)$$

which indicates a zero-temperature phase transition from a Mott-insulation to the superfluid phases at the condition

$$\frac{g^2}{4\Omega_{sym}} > \varepsilon. \quad (40)$$

In terms of the atom-photon coupling strength g, our composite system is in the Mott-insulation phase $\psi = 0$ or in the superfluid phase $\psi \neq 0$ depending the coupling strength is smaller or larger than the critical value, $g_c = \sqrt{4\varepsilon\Omega_{sym}}$. Compared to the known results in the literature [5, 7, 11, 13], where the phase transition of light are demonstrated in the parameter space defined by the hopping coefficient κ and the chemical potential μ, in our approach we absorb these two effects into the parameter $\Omega_{sym}(\vec{k} = 0)$ in Eq. (27), i.e.,

$$\Omega_{sym}(\vec{k} = 0) = -4\kappa' - 4\kappa + \omega - \mu. \quad (41)$$

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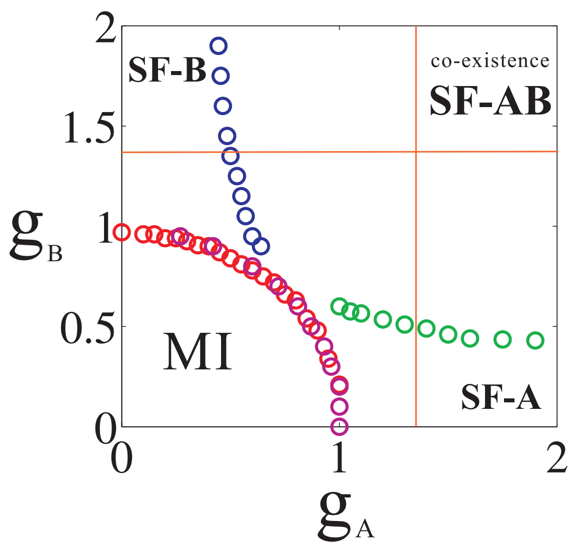

Fig. 2. Phase diagram on the parameter plane (g_A and g_B) at zero temperature, i.e., for different atom-photon coupling strengths. Four different phases are indicated in the plot without ($\kappa = 0$ in the solid lines) and with ($\kappa = 0.4$ in the circle markers) inter-atomic species hopping effects, respectively. Other parameters used in the simulations are the same as $\kappa' = 0.2$, $\omega = 2.7$, $\mu = 0.2$, $\varepsilon_A = 0.27$, and $\varepsilon_B = 0.25$.

For the case of identical atoms, $\kappa' = \kappa$, as expected, when the hopping term becomes larger, then Ω_{sym} becomes smaller, and our system approaches the superfluid state. On the other hand, when the hopping coefficient is smaller, our system would be in the Mott-insulator state.

For a corresponding finite temperature, the phase transition exists at the critical temperature T_c given by

$$\frac{4\varepsilon\Omega_{sym}}{g^2} = Tanh(\frac{\varepsilon}{2T_c}). \tag{42}$$

3.2. The phase diagram for non-identical atoms

To illustrate what happens when the atoms A and B are different, we consider the limit case without inter-atomic species hopping effects, i.e., $\kappa = 0$ (or $\Omega_-^{-1} = 0$). In this case, at $T = 0$ the atoms A and B couple separately to the radiation fields. Now each of the two species has a MI to SF phase transition at $g_A = \sqrt{4\Omega_{sym}\varepsilon_A}$ and $g_B = \sqrt{4\Omega_{sym}\varepsilon_B}$, respectively. In such a way, there would be four possible phases in the parameter plane (g_A, g_B), as the regions defined by the solid lines shown in Fig. 2. These four phases correspond to (1) both of the two-species atoms are in the MI state, $g_{A(B)} < \sqrt{4\Omega_{sym}\varepsilon_{A(B)}}$; (2) only A-type atoms are in the SF state, named as the SF-A state for $g_A > \sqrt{4\Omega_{sym}\varepsilon_A}$ and $g_B > \sqrt{4\Omega_{sym}\varepsilon_B}$; (3) the SF-B state with the case for only B-type atoms are in the SF state; and (4) all the atoms are in the SF state, i.e., the co-existence SF-AB state for $g_{A(B)} > \sqrt{4\Omega_{sym}\varepsilon_{A(B)}}$.

When we turn on the inter-atomic species hopping term, $\kappa \neq 0$, the four phase states mentioned above should be modified. To give a qualitative analysis on the possible four phase states, we assume Ω_-^{-1} to be small and perform a perturbative expansion on the limit case $\Omega_-^{-1} = 0$ for the solutions of the mean-field Hamiltonian in Eq. (37) at zero temperature $T = 0$. By expanding the superfluid order parameter $\psi_{A(B)} = \psi_{A(B)}^{(0)} + \delta\psi_{A(B)}$, to the zero-th order one can again

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obtain,

$$\psi_{A(B)}^{(0)} = \begin{cases} 0 & ; \quad \text{for} \quad \frac{g_{A(B)}^2}{4\Omega_+} < \varepsilon_{A(B)} \\ \sqrt{\left[\frac{g_{A(B)}}{4\Omega_+}\right]^2 - \left[\frac{\varepsilon_{A(B)}}{g_{A(B)}}\right]^2} & ; \quad \text{for} \quad \frac{g_{A(B)}^2}{4\Omega_+} > \varepsilon_{A(B)} \end{cases}, \tag{43}$$

and to the first order expansion in Ω_-^{-1},

$$\delta\psi_{A(B)} = \begin{cases} \frac{g_{B(A)}^2}{4\left[1 - \frac{g_{A(B)}^2}{4\varepsilon_{A(B)}\Omega_+}\right]\Omega_- E_{B(A)}^{(0)}} \psi_{B(A)}^{(0)} & ; \quad \text{for} \quad \psi_{A(B)}^{(0)} = 0 \\ \frac{\Omega_+}{\Omega_-} \frac{g_{B(A)}^2 E_{A(B)}^{(0)2}}{g_{A(B))}^4 E_{B(A)}^{(0)2}} \frac{\psi_{B(A)}^{(0)}}{\psi_{A(B)}^{(0)3}} & ; \quad \text{for} \quad \psi_{A(B)}^{(0)} \neq 0 \end{cases}, \tag{44}$$

with $E_{A(B)}^{(0)} = \sqrt{g_{A(B)}^2 \psi_{A(B)}^{(0)2} + \varepsilon_{A(B)}^2}$.

With the above perturbative results, one can have a clear physical interpretation for the QPT in our proposed system. When all the atoms are in the MI phase, $\psi_A^{(0)} = \psi_B^{(0)} = 0$, both of the perturbed superfluid order parameters $\delta\psi_A$ and $\delta\psi_B$ are zero as expected. The MI phase of the system is not modified by the perturbation in Ω_-^{-1}. On the contrary, the properties of SF states are strongly modified. From the second line in Eq. (44), one can easily find that $\delta\psi_{A(B)}$ is non-zero as long as $\psi_{B(A)}^{(0)} \neq 0$, which is independent of whether $\psi_{A(B)}^{(0)}$ is zero or not. In such a scenario, one of the atomic species in the superfluid phase can induce a non-zero superfluid order parameter on the other species of atoms, which is originally in the MI state. Unlike the case of only one atomic species, in our system both species of atoms develop nonzero order parameters with the superfluidity driven by the other type of atomic species.

It is well known that, in the two weakly coupled superconductors, two separated superfluid states interact through a tunneling current, i.e., Josephson effect [18]. As pointed out by Gerace *et al*. [19], it is possible to have a quantum-optical Josephson interferometer in a three coupled cavities system. In our system, the interesting crossovers between two superfluid phases of light can be viewed as a photonic analogue of Josephson effect. In our system, we have two photonic superfluid states, due to the chessboard arrangement of the cavity array, and it is this photonic hopping interaction of a common radiation field that modifies the original phase state.

The coupling between two atomic species through the radiation fields smears out the difference between the three superfluid phases mentioned above, i.e., SF-A, SF-B, and the co-existence SF-AB phases, and turns the phase transitions between these different phases into crossovers. Nevertheless, the qualitative properties of these three SF states are quite different. For example, turning on an additional laser with frequency $\sim \varepsilon_B$ in our system, that is the new input light resonates strongly with the B-type atoms, modifies sequentially the superfluidity of A-type atoms depending on the original state of the system. For an original SF-A phase, the superfluidity of A-type atoms is destroyed due to the decoupling from B-type atoms; while an original co-existence SF-AB phase is driven into a SF-B phase.

4. Numerical results and discussions

In order to verify the perturbative analyses shown above, we solve the mean-field Hamiltonian in Eq. (37) by using direct numerical simulations. For zero temperature, we solve the equations for different values of g_A and g_B by fixing the value of transition energy $\varepsilon_{A(B)}$ and photon dispersion $\Omega_{+(-)}$. Figure 2 demonstrates the phase diagram of our system in the parameter plane g_A and g_B at zero temperature. As conjectured by the perturbative analysis, we have the

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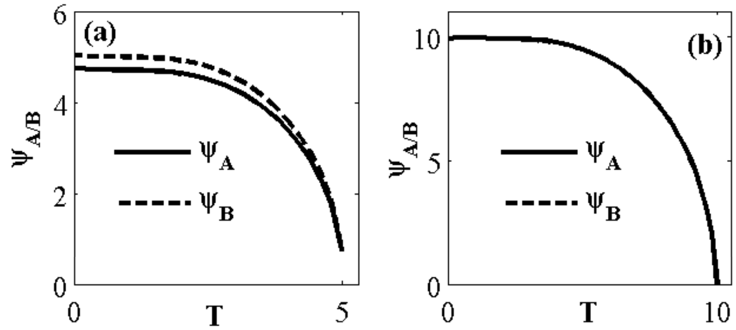

Fig. 3. Superfluid order parameters ψ_A and ψ_B versus the temperature T with different values of (a) $g_A = 0.1, g_B = 2.0$; and (b) $g_A = 2.0, g_B = 2.0$. Other parameters used are the same as those in Fig. 2. Here the hopping constants are $\kappa = 0.4$ and $\kappa' = 0.2$.

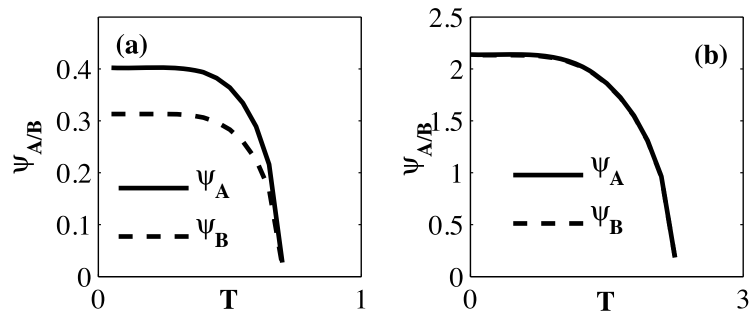

Fig. 4. Superfluid order parameters ψ_A and ψ_B versus the temperature T. All the parameters used are the same as those in Fig. 3, but with different hopping constants, $\kappa = 0.35$ and $\kappa' = 0.175$.

phase diagram for the QPT of light in our dual-species configuration, where the original well-defined boundaries for SF-A/B to SF-AB phases for $\kappa = 0$ is now turned into crossovers.

It can be clearly seen in Fig. 2 that our system prefers a SF phase due to the introduction of hoping effects, which gives the same tendency as those known results in the literature. Moreover, due to the Josephson-like coupling effect mentioned above, the co-existence SF-AB state in Fig. 2 also occupies a broader area in the direct numerical simulations compared to the analytical results.

For nonzero temperature, we plot in Fig. 3(a) and (b) the superfluid order parameters ψ_A and ψ_B as a function of the temperature, T, with different values of g_A and g_B, respectively. As shown in Fig. 3(a), for the parameters $g_A = 0.1$ and $g_B = 2.0$, the system is in the SF-B phase at zero temperature. A clear finite temperature insulator to superfluid transition is found at the critical temperature $T_c \approx 5$. Notice that $\psi_B > \psi_A$ but the differences between ψ_A and ψ_B remains small throughout the whole temperature range in the SF-B phase, indicating the importance of Josephson coupling effect. On the other hand, for the co-existence SF-AB state at zero temperature, the two curves for ψ_A and ψ_B stay close to each other for the whole temperature range. The overall magnitudes of ψ_A and ψ_B are larger by a ratio of factor 2 when compared with the case with $g_A = 0.1$ and $g_B = 2.0$.

In Fig. 4, we show the temperature dependence of $\psi_{A(B)}$ in order to carefully examine the

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effect of Josephson-like coupling. All the parameters are kept the same to those studied before except for the hopping constants changed to $\kappa = 0.35$ and $\kappa' = 0.175$. Comparing to the cases in Fig. 3, we find that the magnitudes of ψ_A and ψ_B become both smaller due to the reduced Josepshon coupling effect, and the difference between ψ_A and ψ_B becomes larger. Notice that the values of Ω_{sym} and Ω_{asym} are increased by decreasing κ' and κ and the system is driven towards the MI phase. In fact we find that $\psi_A = \psi_B = 0$ and the system is already in the MI regime for $\kappa = 0.3$ and $\kappa' = 0.2$ with our chosen set of parameters.

5. Conclusion

We show the phase diagrams for the quantum phase transitions of light in the two-site two-atomic species system modelled by the Bose-Hubbard plus the Jaynes-Cummings Hamiltonians for photon and photon-atom interactions, respectively. Via a self-consistent mean-field approximation, we analyze the equations for the superfluid order parameters analytically and numerically. Four different phases are found, including a Mott-insulator phase and three superfluid phases that we label as SF-A, SF-B and co-existence SF-AB states. The transitions between the different superfluid phases are found to be smeared out by the Josephson-like coupling effect between different types of atoms. Our results demonstrate the possibility to implement a photonic cavity system as a quantum simulator based on different atomic species. As studies in condensed matter physics suggest, more exotic phases and richer phase diagrams for the quantum phase transition of light are expected with more complicated configurations and multiple atomic species in the proposed model.

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