Syllabus: for Complex variables

1. **Midterm**, (4/27).
2. Introduction to Numerical PDE (4/30): [Ref.num].
3. Complex variables: [Textbook]Ch.13-Ch.18.
    - Complex numbers and functions, (5/4).
    - Cauchy-Riemann equations, (~~5/7~~, 5/11).
    - Complex integration, (5/14, 5/18).
    - Complex power & Taylor series, (5/21, 5/25).
    - Laurent series & residue, (~~5/28~~, 6/1, 6/4).
    - Conformal mapping, (6/8, 6/11).
    - Applications: real integrals by residual integration, potential theory, (6/15, 6/18).
4. **Final exam**, (6/15).

國立清華大學
National Tsing Hua University

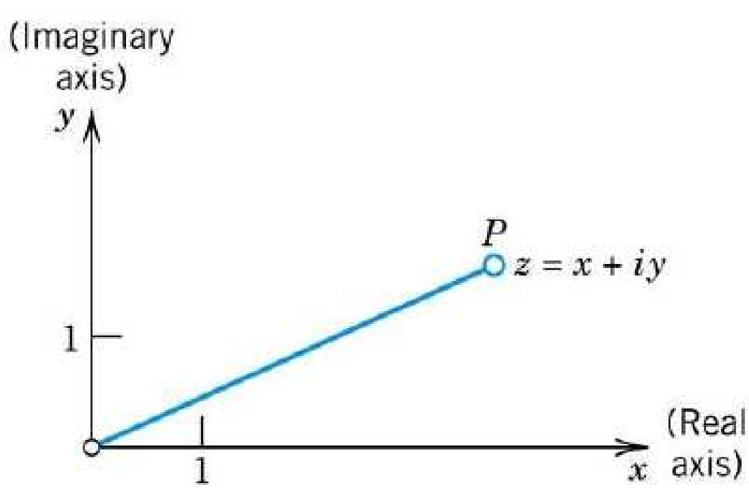
Complex numbers. Complex plane

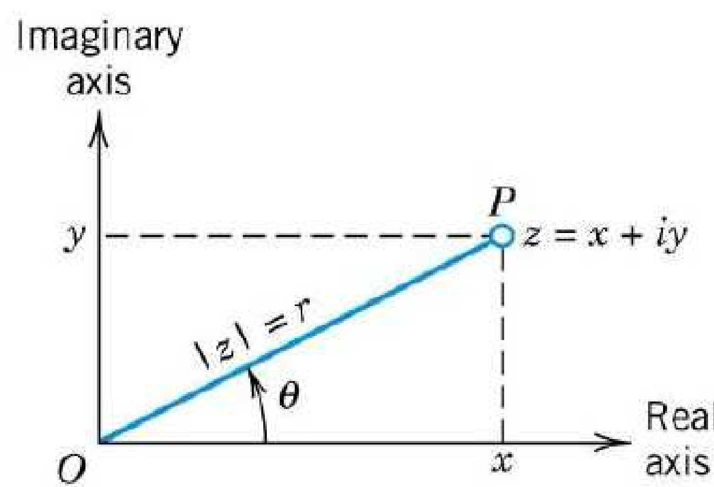
- Cartesian form: $z = x + i\,y$,
  where $x = \text{Re}\{z\}, y = \text{Im}\{z\}, i = \sqrt{-1}$.

- Polar form: $z = r\,e^{i\theta}$,
  where $r = |z| = \sqrt{x^2 + y^2}$ (absolute value or modulus), and
  $\theta = \arg(z) = \tan^{-1}(\frac{y}{x})$ (argument).

- Principle value Arg(z):

$$-\pi < \text{Arg}(z) \le \pi.$$





Complex numbers, properties

- Addition: for two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$

$$z_1 + z_2 = (x_1 + x_2, y_1 + y_2),$$

- Multiplication:

$$z_1\, z_2 = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1),$$

- Quotient:

$$z = \frac{z_1}{z_2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\, \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}$$

- Complex conjugate:

$$\bar{z} = z^* = x - i\, y,$$

- Real part and Imaginary part:

$$\begin{aligned} \text{Re}(z) &= x = \frac{1}{2}(z + \bar{z}), \\ \text{Im}(z) &= y = \frac{1}{2\, i}(z - \bar{z}), \end{aligned}$$

國立清華大學
National Tsing Hua University

Complex numbers, properties, Cont.

- Triangle inequality:

$$|z_1 + z_2| \leq |z_1| + |z_2|,$$

By induction, we can obtain the generalized triangle inequality:

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|,$$

- Cauchy-Schwarz inequality:

$$|z_1\, z_2|^2 \leq |z_1| \cdot |z_2|,$$

- Multiplication in polar form:

$$z_1\, z_2 = r_1\, r_2\, [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)],$$

and

$$|z_1\, z_2| = |z_1||z_2|, \qquad |\frac{z_1}{z_2}| = \frac{|z_1|}{|z_2|},$$

$$\text{arg}(z_1\, z_2) = \text{arg}(z_1) + \text{arg}(z_2), \qquad \text{arg}(\frac{z_1}{z_2}) = \text{arg}(z_1) - \text{arg}(z_2),$$

國立清華大學 National Tsing Hua University

Complex numbers, properties, Cont.

- Integer powers, De Moivre's formula:

$$z^n = r^n\,[\cos(n\theta) + i\sin(n\theta)],$$

- Roots, if $z = w^n$,

$$w = \sqrt[n]{z} = \sqrt[n]{r}[\cos(\frac{\theta + 2k\pi}{n}) + i\sin(\frac{\theta + 2k\pi}{n})],$$

where $k = 0, 1, \ldots, n-1$.

- The principal value of $w = \sqrt[n]{z}$ is obtained for the principal value of arg(z) and $k = 0$.
- Example: $w = \sqrt[3]{1}$,

$$w = 1, \omega, \omega^2,$$

where $\omega = e^{i\frac{2\pi}{3}}$.

國立清華大學
National Tsing Hua University

Sets in the complex plane

- Circle:

$$|z - a| = \rho$$

- Open (closed) circular disk:

$$|z - a| < \rho \quad \text{(open)}, \qquad |z - a| \le \rho \quad \text{(closed)},$$

- An open circular disk $|z - a| < \rho$ is also called a *neighborhood* of a, or a ρ-neighborhood of a.
- Open (closed) annulus (circular ring):

$$\rho_1 < |z - a| < \rho_2 \quad \text{(open)}, \qquad \rho_1 \le |z - a| \le \rho_2 \quad \text{(closed)},$$

- Half-planes:

$$y > 0 \, (y < 0), \qquad \text{upper half-plane (lower half-plane)},$$
$$x > 0 \, (x < 0), \qquad \text{right half-plane (left half-plane)}.$$

Sets in the complex plane, Cont.

- Neighborhood of a:
  An open circular disk $|z - a| < \rho$ is also called a *neighborhood* of a, or a ρ-neighborhood of a.
- Open set S:
  Every point of S has a neighborhood only consisting of points belonging to S.
  E.g. $|z| < 1$ is open, $|z| \leq 1$ is not open.
- Connected set S:
  Any two of its points can be joined by a broken line (linear segments) within S.
  E.g. $\{|z| < 1$ and $|z - 3| < 1\}$ is NOT connected.
- Domain: An open connected set.
- Complement of a set S:
  The set of all points of the complex plane that *do not belong* to S.
- Closed set S:
  A set S is called closed if its complement is open.
- Boundary point:
  A boundary point of a set S is a point every neighborhood of which contains both points that belong to S and points that do not belong to S.

國立清華大學
National Tsing Hua University

Complex function

- For a set of complex numbers, S,
- a function f defined on S is that

$$w = f(z),$$

- z varies in S, and is called a *complex variable*.
- The set S is called the *domain* of definition of f.
  In most cases S will be open and connected.
- The set of all values of a function f is called the *range* of f.
- Example:

$$w = f(z) = z^2 + 3\,z,$$

  is a complex function defined for all z.

國立清華大學
National Tsing Hua University

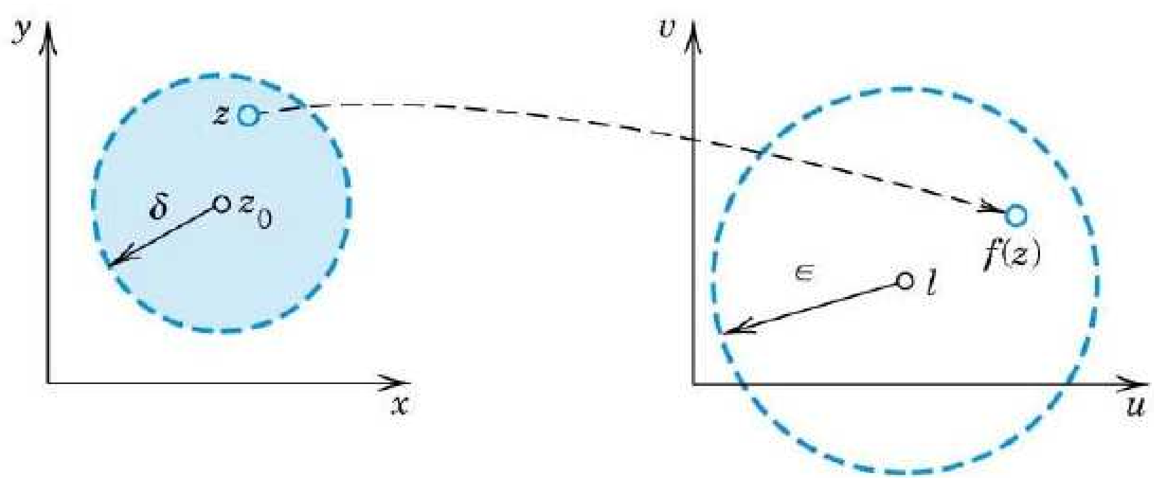
Complex function, limit, continuity

- A function $f(z)$ is said to have the *limit* l as z approaches a point z_0,

$$\lim_{z \to z_0} f(z) = l,$$

- Unlike the calculus, z may approach z_0 *from any direction* in the complex plane.
- Continuous: A function $f(z)$ is said to be continuous at $z = z_0$ if,

$$\lim_{z \to z_0} f(z) = f(z_0).$$



Complex function, Derivative

- The derivative of a complex function f at a point z_0 is,

$$
\begin{aligned}
f'(z_0) &= \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \\
&= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},
\end{aligned}
$$

- Example 1: $f(z) = z^2$

$$
f'(z) = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} = 2\,z
$$

- Example 2: $f(z) = \overline{z}$

$$
f'(z) = \lim_{\Delta z \to 0} \frac{\overline{(z + \Delta z)} - \overline{z}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z} = \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}
$$

not differentiable.

- Analyticity: A function $f(z)$ is said to be analytic in a domain D if $f(z)$ is defined and differentiable at all points of D.

國立清華大學
National Tsing Hua University

Cauchy-Riemann equations

- For a complex function:

$$w = f(z) = u(x, y) + i\, v(x, y),$$

the criterion (test) for the analyticity is the Cauchy-Riemann equations,

$$u_x = v_y, \qquad \text{and} \qquad u_y = -v_x.$$

- f is analytic in a domain D if and only if the first partial derivatives of u and v satisfy the two Cauchy-Riemann equations.
- Example 1: $f(z) = z^2 = x^2 - y^2 + 2\, i\, x\, y$,

$$u_x = 2x = v_y, \qquad u_y = -2y = -v_x.$$

- Example 2: $f(z) = \bar{z} = x - i\, y$,

$$u_x = 1 \neq v_y = -1, \qquad u_y = -v_x = 0.$$

國立清華大學
National Tsing Hua University

Cauchy-Riemann equations, Proof

- By assumption, the derivative $f'(z)$ at z exists, $f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$.

- Write $\Delta z = \Delta x + i \Delta y$,

$$f'(z) = \lim_{\Delta z \to 0} \frac{[u(x+\Delta x, y+\Delta y) + i\, v(x+\Delta x, y+\Delta y)] - [u(x,y) + i\, v(x,y)]}{\Delta x + i \Delta y},$$

- For the first path I: let $\Delta y \to 0$ first and then $\Delta x \to 0$,

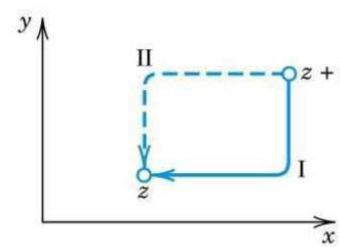
$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x} = u_x + i\, v_x,$$

- For the second path II: let $\Delta x \to 0$ first and then $\Delta y \to 0$,

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\, \Delta y} + i \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{i\, \Delta y} = -i\, u_y + v_y,$$

- the Cauchy-Riemann equations,

$$u_x = v_y, \qquad \text{and} \qquad u_y = -v_x.$$



國立清華大學
National Tsing Hua University

Exponential functions

- the complex exponential function: e^z, or written as $\exp(z)$.
- e^z is analytical for all z, i.e., an *entire function*.
- Proof: by using the Cauchy-Reimanny equations, i.e.,

$$e^z = e^x[\cos(y) + i\,\sin(y)],$$

where

$$u = e^x\,\cos(y), \qquad v = e^x\,\sin(y)$$

- The derivative of e^z is also e^z, i.e., $(e^z)' = e^z$.
- The expansion of $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots$
- $e^{z_1}e^{z_2} = e^{z_1+z_2}$,
- Periodicity of e^z with period $2\pi i$, i.e.,

$$e^{z+2\pi i} = e^z, \qquad \text{for all} z$$

國立清華大學
National Tsing Hua University

Trigonometric functions

- Euler formulas:

$$
\begin{aligned}
e^{i\,z} &= \cos(z) + i\,\sin(z), \\
e^{-i\,z} &= \cos(z) - i\,\sin(z),
\end{aligned}
$$

- the complex trigonometric functions:

$$
\cos z = \frac{e^{i\,z} + e^{-i\,z}}{2}, \qquad \sin z = \frac{e^{i\,z} - e^{-i\,z}}{2\,i},
$$

- $\cos z$, $\sin z$ are analytic for all z, but $\tan z$ is not wherever $\cos z = 0$.
- General formulas of real trigonometric functions remain valid for complex counterparts, i.e.,

$$
\begin{aligned}
\cos(z_1 \pm z_2) &= \cos z_1\ \cos z_2 \mp \sin z_1\ \sin z_2, \\
\sin(z_1 \pm z_2) &= \sin z_1\ \cos z_2 \pm \cos z_1\ \sin z_2.
\end{aligned}
$$

國立清華大學
National Tsing Hua University

Hyperbolic functions

- the complex hyperbolic cosine and sine:

$$\cosh z = \frac{e^z + e^{-z}}{2},$$
$$\sinh z = \frac{e^z - e^{-z}}{2},$$
$$\tanh z = \frac{\sinh z}{\cosh z},$$

- These functions are *entire*, with derivatives

$$(\cosh z)' = \sinh z,$$
$$(\sinh z)' = \cosh z,$$

- Relations between complex trigonometric and hyperbolic functions,

$$\cosh(i\,z) = \cos z, \qquad \sinh(i\,z) = i \sin z,$$
$$\cos(i\,z) = \cosh z, \qquad \sin(i\,z) = i \sinh z,$$

國立清華大學
National Tsing Hua University

Logarithm

- The natural logarithm of $z = x + i\,y$ is denoted by $\ln z$ or $\log z$,
- Define:

$$e^{w} = e^{u+i\,v} = r\,e^{i\theta} = z,$$

then

$$\ln z = \ln r + i\theta, \qquad (r = |z| > 0, \theta = \text{arg} z),$$

- Since the argument of z is determined only determined only up to integer multiples of 2π,
- the complex natural logarithm $\ln z (z \neq 0)$ is infinitely many-valued.
- Principal value of $\ln z$,

$$\text{Ln} z = \ln |z| + i\,\text{Arg}(z)$$

- Other values of $\ln z$ are

$$\ln z = \text{Ln} z \pm 2n\pi i, \qquad n = 1, 2, \ldots$$

國立清華大學
National Tsing Hua University

Logarithm, Cont.

- Examples:

$$\text{Ln}(1) = 0, \qquad \ln 1 = 0, \pm 2\pi i, \pm 4\pi i, \ldots$$
$$\text{Ln}(-1) = \pi i, \qquad \ln 1 = \pm \pi i, \pm 3\pi i, \pm 5\pi i, \ldots$$
$$\text{Ln}(i) = \pi i/2, \qquad \ln i = \pi i/2, -3\pi i/2, 5\pi i/2, \ldots$$

- Relations:

$$\ln(z_1\, z_2) = \ln z_1 + \ln z_2,$$
$$\ln(\frac{z_1}{z_2}) = \ln z_1 - \ln z_2,$$

- Analyticity of the Logarithm: for every $n = 0, \pm 1, \pm 2, \ldots,$

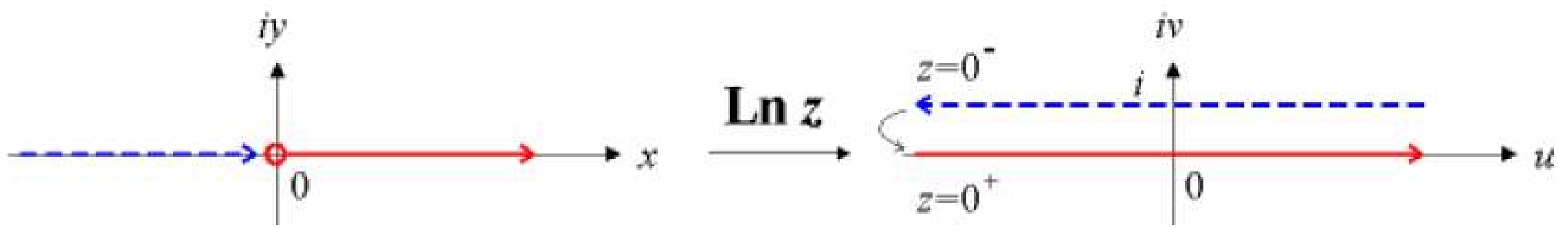
$$\ln z = \text{Ln} z \pm 2n\pi i, \qquad n = 1, 2, \ldots$$

is analytic, *except at* 0 *and on the negative real axis.*

- Each of the infinitely n is call a *brach* of the logarithm.

國立清華大學
National Tsing Hua University

Logarithm, Branch cut and Derivative



- If z is negatively real (where real logarithm is undefined), Ln$z = \ln|z| + i\pi$.
- The derivative:

$$(\ln z)' = \frac{1}{z}, \qquad (z\text{not 0 or negative real}).$$

- Proof:

$$\ln z = \ln r + i(\theta + c) = \frac{1}{2}\ln(x^2 + y^2) + i[\tan^{-1}\frac{y}{x} + c],$$

where the constant c is a multiple of 2π. By the Cauchy-Riemann equations, i.e., $u_x = v_y, \qquad u_y = -v_x$, and

$$(\ln z)' = u_x + i\,v_x = \frac{x - i\,y}{x^2 + y^2} = \frac{1}{z}.$$

國立清華大學
National Tsing Hua University

General Powers

- The general powers of a complex number,

$$z^c = e^{c \ln z}, \qquad c \text{ complex and } z \neq 0.$$

- Principle value:

$$z^c = e^{c\,\mathrm{Ln} z},$$

- Example 1:

$$i^i = e^{i \ln i} = \exp[i(\frac{\pi}{2} i \pm 2\,n\pi i)] = e^{-\pi/2 \mp 2npi},$$

the principal value (n = 0) is $e^{-\pi/2}$.

- Example 2:

$$\begin{aligned}(1+i)^{2-i} &= \exp[(2-i)(\ln\sqrt{2} + \frac{1}{4}\pi i \pm 2\,n\pi i)] \\ &= 2e^{\pi/4 \pm 2\,n\pi}[\sin(\frac{1}{2}\ln 2) + i\,\cos(\frac{1}{2}\ln 2)]\end{aligned}$$

國立清華大學
National Tsing Hua University

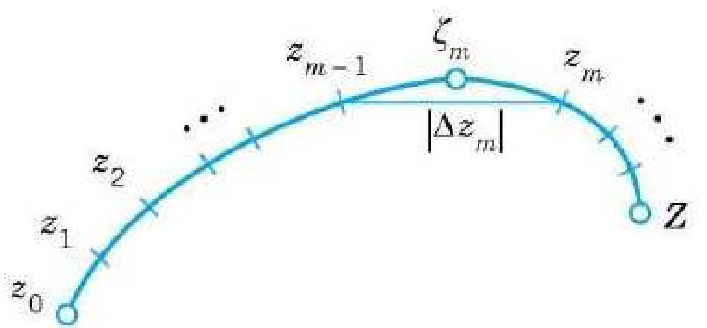
Line integral in the complex plane

- An *indefinite* integral is a function whose derivative equals a given analytic function in a region.
- Complex definite integral are called *line integrals*,

$$\int_C f(z) \mathrm{d}\, z,$$

- The line integral over curve $C : \{z(t) = x(t) + i\, y(t)\}$ in the complex plane is defined as the limit of partial sum,

$$\int_C f(z) \mathrm{d}\, z \approx \sum_{m=1}^{n} f(\xi_m)\, \Delta z_m$$



國立清華大學
National Tsing Hua University

Indefinite integration of analytic functions

$$\int_{z_0}^{z_1} f(z)\mathrm{d}\,z = F(z_1) - F(z_2), \qquad F'(z) = f(z)$$

- Examples:

$$\int_0^{1+i} z^2\mathrm{d}\,z = \frac{-2}{3} + \frac{2}{3}i,$$

$$\int_{-\pi i}^{\pi i} \cos z\mathrm{d}\,z = 2\sin\pi i,$$

$$\int_{-i}^{i} \frac{1}{z}\mathrm{d}\,z = \mathrm{Ln}z|_{-i}^{i} = i\pi,$$

國立清華大學
National Tsing Hua University

Use of a representation of a Path

- Let C be a piecewise smooth path, represented by $z = z(t)$, where $a \le t \le b$,

$$\int_C f(z)\mathrm{d}\,z = \int_a^b f[z(t)]\dot{z}(t)\mathrm{d}\,t, \qquad \dot{z} = \frac{\mathrm{d}\,z}{\mathrm{d}\,t}$$

- Example 1: around the Unit Circle

$$\oint_C \frac{\mathrm{d}\,z}{z},$$

- Use $z(t) = \cos t + i \sin t = e^{it}$, $0 \le t \le 2\pi$, i.e.,

$$\oint_C \frac{\mathrm{d}\,z}{z} = \int_0^{2\pi} e^{-it} i\, e^{it}\mathrm{d}\,t = i\int_0^{2\pi} \mathrm{d}\,t = 2\pi i,$$

- Compare with $\mathrm{Ln} z|_{z_1}^{z_1} = 0$.
- Now $1/z$ is not analytic at $z = 0$. But any simply connected domain containing the unit circle must contain $z = 0$.

國立清華大學 National Tsing Hua University

Use of a representation of a Path, Example 2

- Example 2:

$$\oint_C (z - z_0)^m \mathsf{d}\, z$$

- Represent C in the form,

$$z(t) = z_0 + \rho(\cos t + i\ \sin t) = z_0 + \rho\, e^{i\,t},$$

then

$$\begin{aligned} \oint_C (z - z_0)^m \mathsf{d}\, z &= \int_0^{2\pi} \rho^m e^{i\,mt} i\rho e^{i\,t} \mathsf{d}\, t, \\ &= i\rho^{m+1} \int_0^{2\pi} e^{i(m+1)t} \mathsf{d}\, t, \\ &= \begin{cases} 2\pi i & ; \quad m = -1 \\ 0 & ; \quad m \neq -1 \text{ and integer} \end{cases} \end{aligned}$$

國立清華大學
National Tsing Hua University

Use of a representation of a Path, Example 3

- Example 3:

$$\int_C \text{Re}(z)\text{d}\,z$$

- Path 1: C^*, where $z(t) = t + 2i\,t$, $(0 \le t \le 1)$,

$$\int_C \text{Re}z\text{d}\,z = \int_0^1 t(1+2i)\text{d}\,t = \frac{1}{2} + i,$$

- Path 2: C_1 and C_2, where $z(t) = t$ and $z(t) = 1 + i\,t$,

$$\begin{aligned}\int_C \text{Re}z\text{d}\,z &= \int_{C_1} \text{Re}z\text{d}\,z + \int_{C_2} \text{Re}z\text{d}\,z, \\ &= \int_0^1 t\text{d}\,t + \int_0^2 i\text{d}\,t = \frac{1}{2} + 2i.\end{aligned}$$

國立清華大學
National Tsing Hua University

Bounds for integrals, ML-Inequality

- ML-inequality:

$$\left| \int_C f(z)\,\mathrm{d}\,z \right| \le ML$$

where L is the length of C and M is a constant such that $|f(z)| \le M$ everywhere on C.

- Example: estimation of an integral (find an upper bound),

$$\int_C z^2\,\mathrm{d}\,z, \qquad C \text{ the straight-line segment from 0 to } 1+i$$

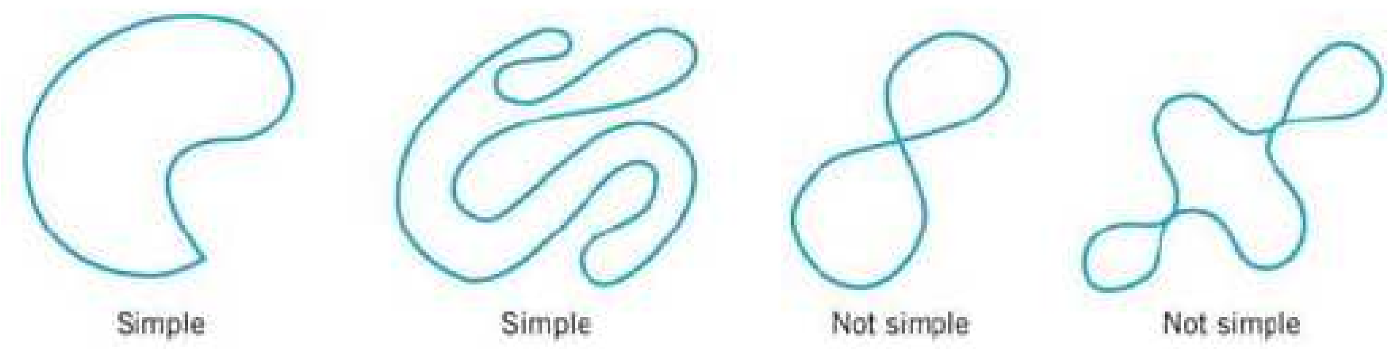
- Solution: $L = \sqrt{2}$ and $|f(z)| = |z^2| \le 2$ on C,

$$\left| \int_C z^2\,\mathrm{d}\,z \right| \le 2\sqrt{2}.$$

國立清華大學
National Tsing Hua University

Simple closed path

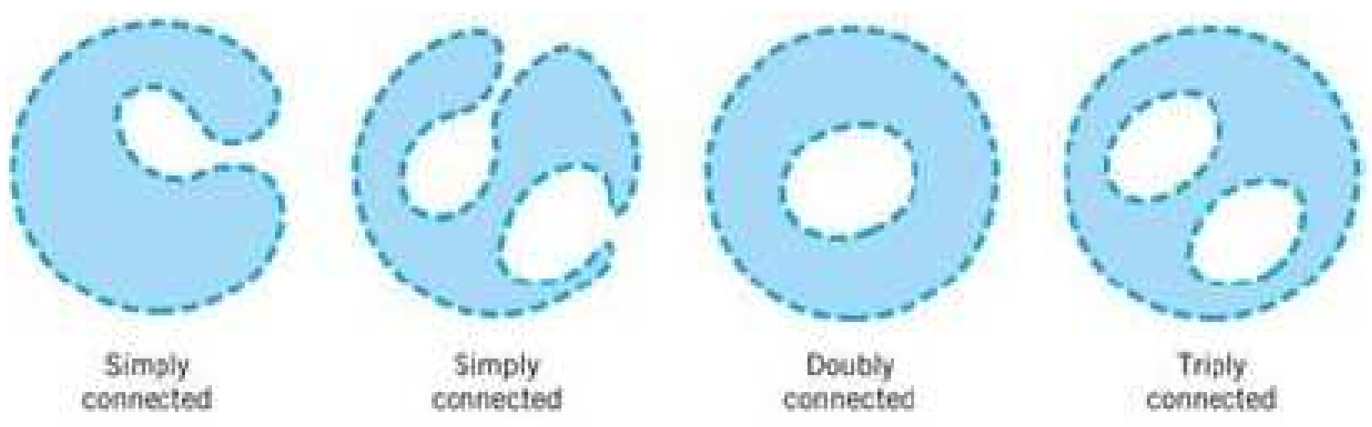
- Simple closed path: a closed path that does not intersect or touch itself.
- Examples:



Closed path

Simply connected domain

- Simply connected domain D: a domain such that every simple closed path in D enclosed only points of D.
- A domain that is not simply connected is called *multiply connected*.
- Examples:



Simply and multiply connected domains

Cauchy's integral theorem

- If $f(z)$ is analytic in a simply connected domain D, then for every simple closed path C in D,

$$\oint_C f(z)\,\mathrm{d}\,z = 0.$$

- Proof:

$$\oint_C f(z)\,\mathrm{d}\,z = \oint_C [u\,\mathrm{d}\,x - v\,\mathrm{d}\,y] + i\oint_C [u\,\mathrm{d}\,y + v\,\mathrm{d}\,x].$$

Since $f(z)$ is analytic in D, its derivative $f'(z)$ exists in D.

國立清華大學
National Tsing Hua University

Cauchy's integral theorem, Proof

- Green's theorem:

$$\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = \iint_R \nabla \times \vec{F} \cdot \vec{k} \, \mathrm{d}\,x \mathrm{d}\,y$$

$$\oint_C [F_1 \, \mathrm{d}\,x + F_2 \, \mathrm{d}\,y] = \iint_R [\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}] \mathrm{d}\,x \mathrm{d}\,y$$

- Replace F_1 and F_2 in the Green's theorem by u and $-v$,

$$\oint_C [u \, \mathrm{d}\,x - v \, \mathrm{d}\,y] = \iint_R [-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}] \mathrm{d}\,x \mathrm{d}\,y = 0.$$

  by using the Cauchy-Riemann equations.

- Replace F_1 and F_2 in the Green's theorem by v and u,

$$\oint_C [u \, \mathrm{d}\,y + v \, \mathrm{d}\,x] = \iint_R [\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}] \mathrm{d}\,x \mathrm{d}\,y = 0.$$

  by using the Cauchy-Riemann equations again.

Cauchy's integral theorem, Examples

- Example 1: No singularities (entire functions),

$$\oint_C e^z \,\mathsf{d}\, z = 0, \qquad \oint_C \cos z \,\mathsf{d}\, z = 0, \qquad \oint_C z^n \,\mathsf{d}\, z = 0, \qquad (n = 0, 1, \dots).$$

- Example 2: Singularities outside the contour,

$$\oint_C \sec(z) \,\mathsf{d}\, z = 0, \qquad C \text{ is the unit circle.}$$

$$\oint_C \frac{1}{z^2+4}, \mathsf{d}\, z = 0.$$

- Example 3: nonanalytic function, C is the unit circle,

$$\oint_C \mathrm{Re}(z) \,\mathsf{d}\, z = \int_0^{2\pi} e^{-it}\, i\, e^{it} \,\mathsf{d}\, t = 2\pi i.$$

- Example 4: Simple connectedness essential,

國立清華大學
National Tsing Hua University

$$\oint_C \frac{1}{z} \,\mathsf{d}\, z = 2\pi i.$$

Cauchy's integral theorem, 2

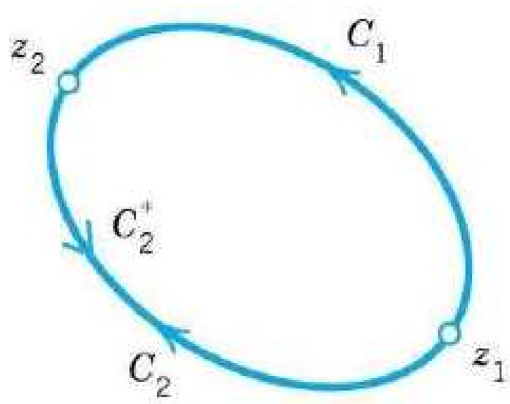
- Independence of Path,
  If $f(z)$ is analytic in a simply connected domain D, then the integral of $f(z)$ is independent of path in D.

- Proof:

$$\int_{C_1} f\,\mathrm{d}\,z + \int_{C_2^*} f\,\mathrm{d}\,z = 0$$

thus,

$$\int_{C_1} f\,\mathrm{d}\,z = -\int_{C_2^*} f\,\mathrm{d}\,z = \int_{C_2} f\,\mathrm{d}\,z$$



Principle of deformation of path

- As long as our deforming path always contains only points at which $f(z)$ is analytic, the integral retains the same value.

$$\oint (z - z_0)^m \,\mathrm{d}\, z = \begin{cases} 2\pi i & ; \quad m = -1 \\ 0 & ; \quad m \neq -1 \text{ and integer} \end{cases}$$

- If $f(z)$ is analytic in a simply connected domain D, then there exists an indefinite integral $F(z)$ of $f(z)$ in D, thus, $F'(z) = f(z)$, which is analytic in D.
- For all paths in D joining any two points z_0 and z_1 in D, the integral of $f(z)$ from z_0 to z_1 can be evaluated by

$$\int_{z_0}^{z_1} f(z) \mathrm{d}\, z = F(z_1) - F(z_2), \qquad F'(z) = f(z)$$

- Proof: see p.p. 650-651 in the textbook.

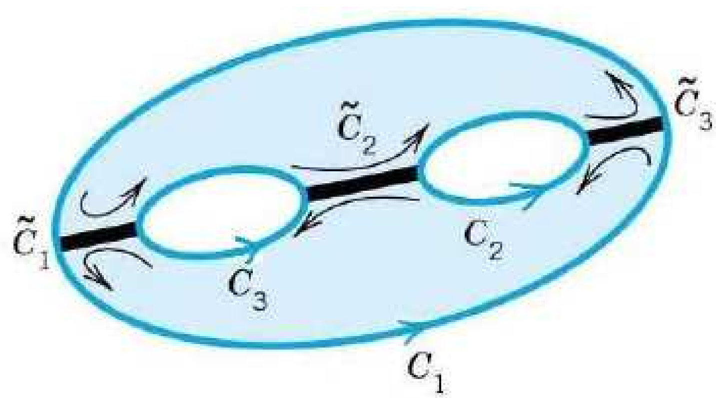
國立清華大學
National Tsing Hua University

Cauchy's integral theorem for multiply connected domains

- Cauchy's theorem applies to multiply connected domains.
- If $f(z)$ is analytic in a multiply connected domain D defined by an outer contour C_1 and multiple inner contours $C_i, i = 2, 3, \ldots, n$ (all are in counterclockwise sense),

$$\oint_{C_1} f \, \mathrm{d}\, z = \sum_{i=2}^{n} \oint_{C_i} f \, \mathrm{d}\, z$$

- Proof: Introducing three inner cuts $\tilde{C}_1, \tilde{C}_2, \tilde{C}_3$ to divide the domain D into two simply connected domains. Apply Theorem 1 to them, integral over cuts will be canceled.



Cauchy's integral formula

- Let $f(z)$ be analytic in a simply connected domain D. Then for any points z_0 in D and any simple closed path C in D that enclose z_0,

$$\oint_C \frac{f(z)}{z - z_0} \mathrm{d}\, z = 2\pi\, i\, f(z_0).$$

  where the integration being taken *counterclockwise*.

- Alternatively for $f(z_0)$,

$$f(z_0) = \frac{1}{2\pi\, i} \oint_C \frac{f(z)}{z - z_0} \mathrm{d}\, z,$$

- Proof: by replacing $f(z) = f(z_0) + [f(z) - f(z_0)]$,

$$\begin{aligned} \oint_C \frac{f(z)}{z - z_0} \mathrm{d}\, z &= f(z_0) \oint_C \frac{1}{z - z_0} \mathrm{d}\, z + \oint_C \frac{f(z) - f(z_0)}{z - z_0} \mathrm{d}\, z \\ &= 2\pi\, i\, f(z_0) + \frac{\epsilon}{\rho} 2\pi\rho, \\ &= 2\pi\, i\, f(z_0) \end{aligned}$$

國立清華大學
National Tsing Hua University

Cauchy's integral formula, examples

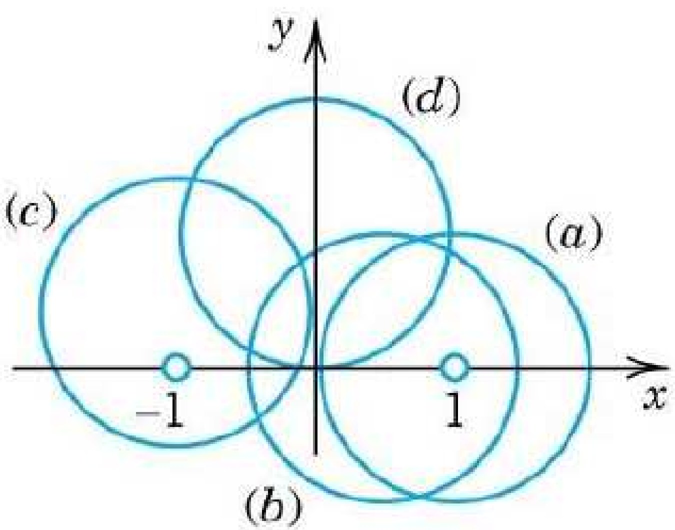
- Example 1:

$$\oint_C \frac{e^z}{z-2}\,\mathsf{d}\,z = 2\pi\,i\,e^z|_{z=2} = 2\pi\,i\,e^2.$$

- Example 2:

$$\oint_C \frac{z^3-6}{2z-i}\,\mathsf{d}\,z = 2\pi\,i\,[\frac{1}{2}z^3 - 3]|_{z=i/2} = \frac{\pi}{8} - 6\pi\,i.$$

- Example 3: integrate $\oint_C \frac{z^2+1}{z^2-1}\,\mathsf{d}\,z$ counterclockwise around each of the four circles,



国立清華大學
National Tsing Hua University

Cauchy's integral formula, Example 3

- (a) The circle $|z-1|=1$, enclosing the point $z_0=-1$

$$\oint_C \frac{z^2+1}{z^2-1}\mathrm{d}\,z = \oint_C [\frac{z^2+1}{z+1}][\frac{1}{z-1}]\mathrm{d}\,z = 2\pi\,i\,[\frac{z^2+1}{z+1}]|_{z=1} = 2\pi\,i.$$

- (b), enclosing the point $z_0=-1$, gives the same as (a) by the principle of deformation of path.

- (c) the path encloses the point $z_0=-1$,

$$\oint_C \frac{z^2+1}{z^2-1}\mathrm{d}\,z = \oint_C [\frac{z^2+1}{z-1}][\frac{1}{z+1}]\mathrm{d}\,z = 2\pi\,i\,[\frac{z^2+1}{z-1}]|_{z=-1} = -2\pi\,i.$$

- (d) don't enclose any singular point,

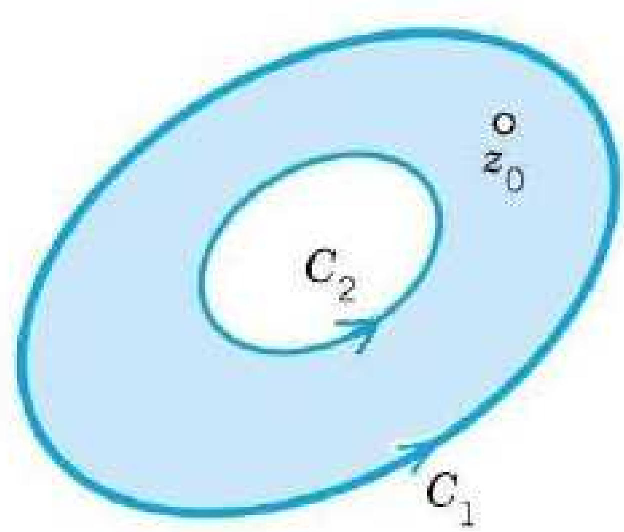
$$\oint_C \frac{z^2+1}{z^2-1}\mathrm{d}\,z = 0.$$

國立清華大學
National Tsing Hua University

Cauchy's integral formula, multiply connected domains

- If $f(z)$ is analytic in a doubly connected domain D bounded by two *counterclockwise* contours C_1, C_2,

$$f(z_0) = \frac{1}{2\pi i}[\oint_{C_1} \frac{f(z)}{z - z_0} \mathrm{d}\, z - \oint_{C_2} \frac{f(z)}{z - z_0} \mathrm{d}\, z].$$



National Tsing Hua University

Derivatives of an analytic function

- If $f(z)$ is analytic in a domain D, then it has derivatives of all orders in D, which are then also analytic functions in D.
- The values of these derivatives at a point z_0 in D are given by

$$f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^2} \mathrm{d}\, z,$$
$$f''(z_0) = \frac{2!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^3} \mathrm{d}\, z,$$

- and in general

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} \mathrm{d}\, z,$$

where C is any simple closed path in D that encloses z_0 and whose full interior belongs to D;

- We integrate *counterclockwise* around C.

國立清華大學
National Tsing Hua University

Derivatives of an analytic function, Proof

- The definition of the Derivative,

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z},$$

- By Cauchy's integral formula,

$$\begin{aligned} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} &= \frac{1}{2\pi i \, \Delta z} [\oint_C \frac{f(z)}{z - (z_0 + \Delta z)} \mathrm{d}\, z - \oint_C \frac{f(z)}{z - z_0} \mathrm{d}\, z], \\ &= \frac{1}{2\pi i \, \Delta z} \oint_C \frac{f(z)}{(z - z_0 - \Delta z)(z - z_0)} \mathrm{d}\, z \end{aligned}$$

- as $\Delta z \to 0$,

$$f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2} \mathrm{d}\, z,$$

國立清華大學
National Tsing Hua University

Evaluation of line integrals

- Example 1: for any contour enclosing the point πi (counterclockwise)

$$\oint_C \frac{\cos z}{(z-\pi i)^2} \mathrm{d}\, z = 2\pi\, i\, (\cos z)'|_{z=\pi i} = 2\pi \sinh \pi.$$

- Example 2: for any contour enclosing the point $-i$ (counterclockwise)

$$\oint_C \frac{z^4 - 3z^2 + 6}{(z+i)^3} \mathrm{d}\, z = \pi i (z^4 - 3z^2 + 6)''|_{z=-i} = -18\pi i.$$

- Example 3: for any contour for which 1 lies inside and $\pm 2i$ lie outside (counterclockwise)

$$\oint_C \frac{e^z}{(z-1)^2(z^2+4)} \mathrm{d}\, z = 2\pi\, i\, (\frac{e^z}{z^2+4})'|_{z=1} = \frac{6\pi e}{25} i.$$

國立清華大學
National Tsing Hua University

Cauchy's inequality

- For $|f(z)| \leq M$ on C,

$$|f^{(n)}(z_0)| = \frac{n!}{2\pi\, i} | \oint_C \frac{f(z)}{(z - z_0)^{n+1}} \mathrm{d}\, z| \leq \frac{n!}{2\pi\, i} M \frac{1}{r^{n+1}} 2\pi r$$

- Cauchy's inequality

$$|f^{(n)}(z_0)| \leq \frac{n!M}{r^2}.$$

國立清華大學
National Tsing Hua University

Liouville's and Morera's theorems

- Liouville's theorem:
  If an entire function is bounded in absolute value in the whole complex plane, then this function must be a constant.
- Proof: By assumption, $|f(z)|$ is bounded, say, $|f(z)| < K$ for all z. Using Cauchy's inequality, wee see that $|f'(z_0) < K/r$.
- Since $f(z)$ is entire, this holds for every r, so that we can take r as large as we please and conclude that $f'(z_0) = 0$. Since z_0 is arbitrary, $f'(z) = 0$ for all z, then $f(z)$ is a constant.
- Morera's theorem:
  If $f(z)$ is continuous in a simply connected domain D and if

$$\oint_C f(z)\,\mathrm{d}\,z = 0,$$

for every closed path in D, then $f(z)$ is analytic in D.

國立清華大學 National Tsing Hua University