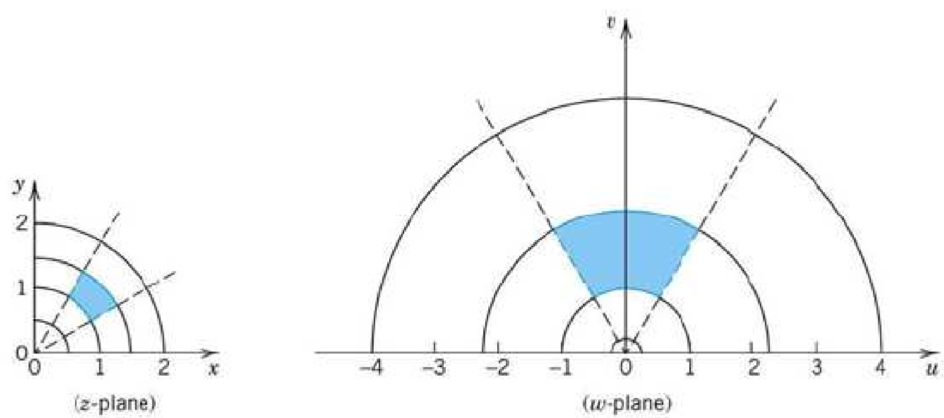
Mapping

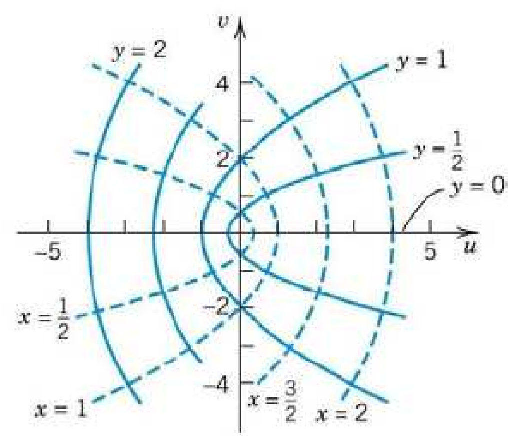
- A complex function $f(z) = u(x, y) + i\,v(x, y)$ gives a mapping of its domain of definition D in the complex z-plane into the complex w-plane.
- Example: $w = f(z) = z^2$,

$$u = \text{Re}(z^2) = x^2 - y^2, \qquad v = \text{Im}(z^2) = 2xy,$$

- The lines $x = c =$ const. and $y = k =$ const. are mapped onto

$$\begin{aligned} v^2 &= 4c^2(c^2 - u), && \text{parabolas open to the left,} \\ v^2 &= 4k^2(k^2 + u), && \text{parabolas open to the right.} \end{aligned}$$





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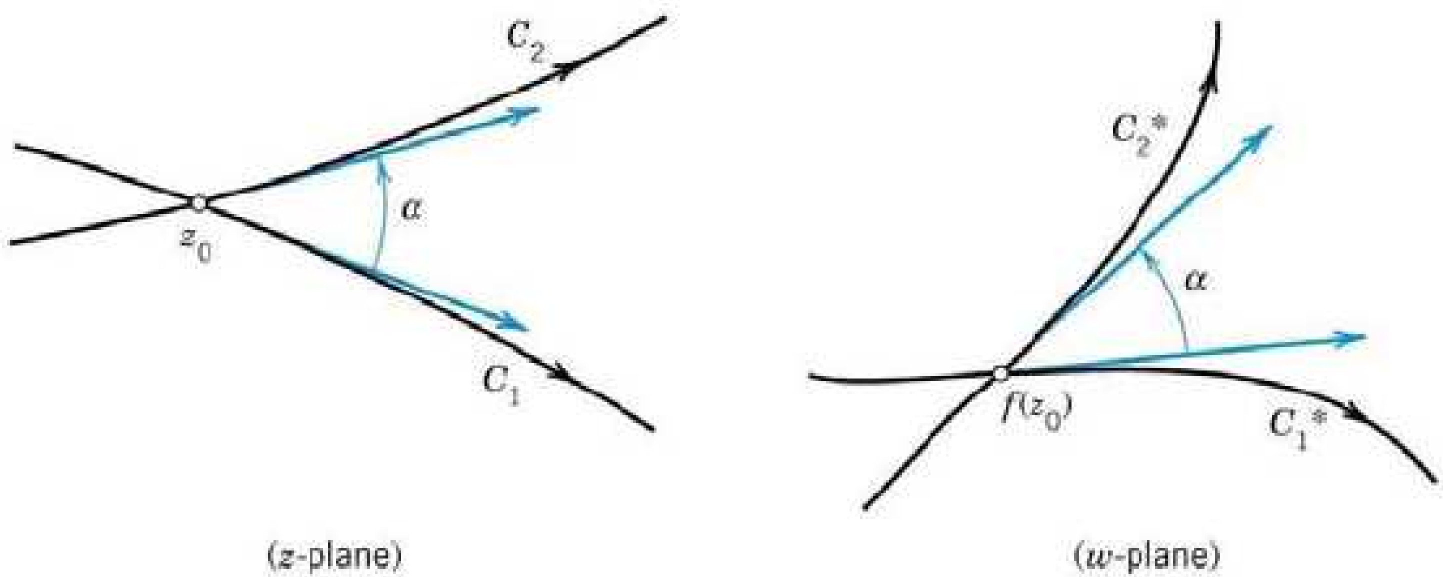
Conformal Mapping

- A mapping $w = f(z)$ is call *conformal* if it preserves angles between oriented curves in magnitude as well as in sense.
- The angle $\alpha (0 \le \alpha \le \pi)$ between two intersecting curves C_1 and C_2 is defined to be the angle between their oriented tangents at the intersection point z_0.
- The *conformality* means that the images C_1^* and C_2^* of C_1 and C_2 make the same angle as the curves themselves in both magnitude and direction.
- The mapping $w = f(z)$ by an analytic function $f(z)$ is conformal, except at *critical points*, that is, points at which the derivative f' is zero.

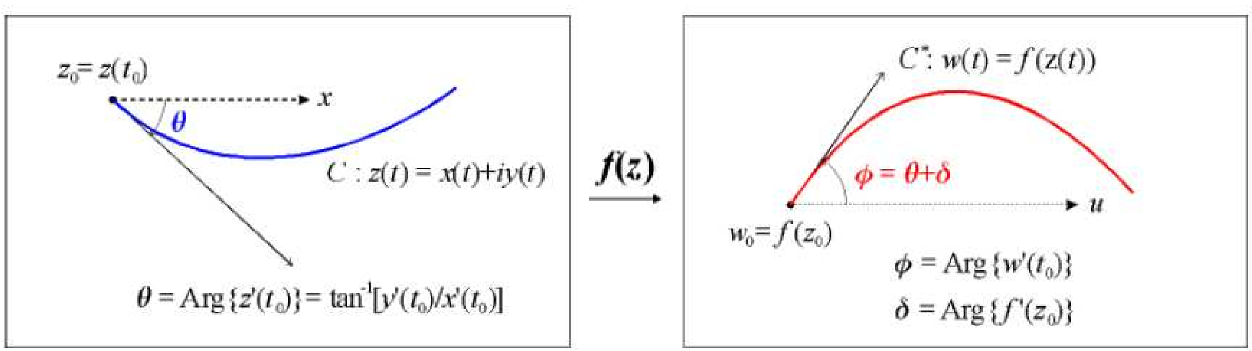


Conformal Mapping, Proof

- Consider a curve, $C : z(t) = x(t) + i\,y(t)$, in the domain of $f(z)$.
- Now $\dot{z}(t) = \mathrm{d}\,z/\mathrm{d}\,t = \dot{x}(t) + i\,\dot{y}(t)$ is tangent to C.
- The image C^* of C is $w = f(z(t))$.
- By the chain rule, $\dot{w}(t) = f'(z(t))\,\dot{z}(t)$.
- Hence the tangent direction of C^* is given by the argument,

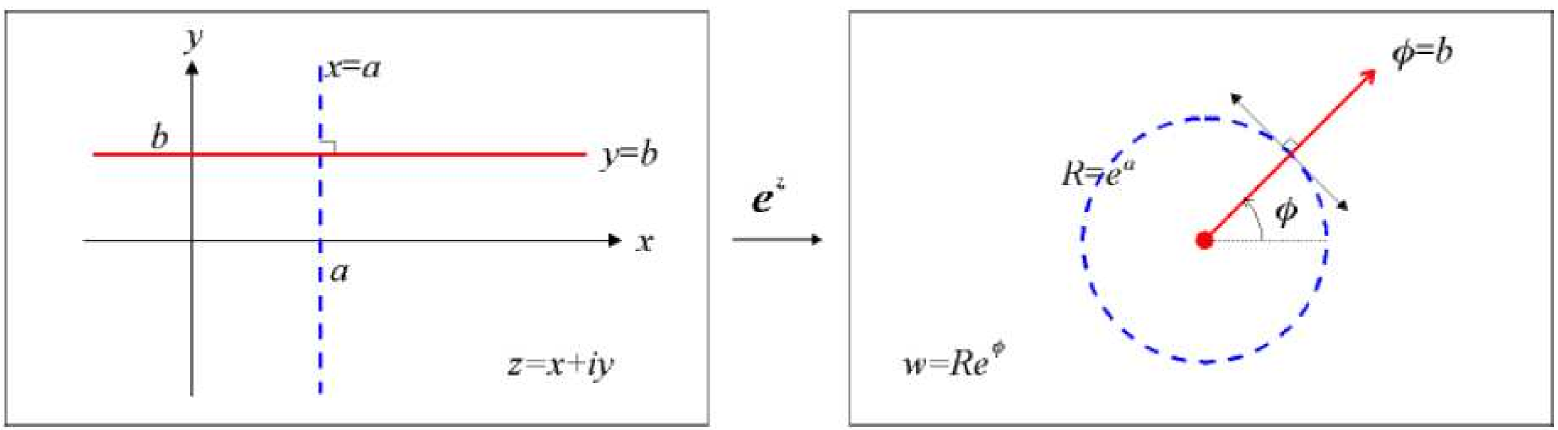
$$\arg\dot{w} = \arg f' + \arg\dot{z}$$

  where $\arg\dot{z}$ gives the tangent direction of C.
- This shows that the mapping rotates all direction at a point z_0 in the domain of analyticity of f through the same angle $\arg f'$, which exists as long as $f'(z_0)\ neq 0$.



Conformal Mapping, Example

- $w = z^n$
- the mapping $w = z^n, n = 2, 3, \ldots,$ is conformal, except at $z = 0$, where $w' = nz^{n-1} = 0$.
- For general n the angles at 0 are multiplied by a factor n under the mapping.
- Hence the section $0 \leq \theta \leq \pi/n$ is mapped by z^n onto the upper half-plane $v \geq 0$.



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Linear Fractional Transformations

- Linear fractional transformations (or Möbius transformations) are mappings

$$w = \frac{a\,z + b}{c\,z + d}, \qquad (ad - bc \neq 0),$$

where a, b, c, d are complex or real numbers.

- The differentiation gives,

$$w' = \frac{ad - bc}{(c\,z + d)^2},$$

which motivates the requirement $ad - bc \neq 0$.

- Special linear fractional transformation,

$$\begin{aligned} w &= z + b, && \text{(Translations)}, \\ w &= a\,z, \text{ with } |a| = 1 && \text{(Rotations)}, \\ w &= a\,z + b, && \text{(Linear transformation)}, \\ w &= 1/z, && \text{(Inversion int he unit circle)}. \end{aligned}$$

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Conformal Mapping and the Potential Theory

- Conformality is the most important geometric property of analytic functions and gives the possibility of a geometric approach to complex analysis.
- A more important application of the conformal mapping is connected with potential problems.
- Conformal mapping yields a standard method for solving *boundary value problems* in 2D potential theory by transforming a complicated region into a simple one.

- See Chap. 17 and Chap. 18 in the textbook.