Sine and Cosine transforms for finite range

- Fourier sine transform

$$\begin{aligned} S_n &= \frac{2}{L}\int_0^L f(t)\,\text{Sin}(\frac{n\pi}{L}x)\text{d}x, \\ f(x) &= \sum_{n=1}^{\infty} S_n\,\text{Sin}(\frac{n\pi}{L}x). \end{aligned}$$

- Fourier cosine transform

$$\begin{aligned} C_n &= \frac{2}{L}\int_0^L f(t)\,\text{Cos}(\frac{n\pi}{L}x)\text{d}x, \\ f(x) &= \frac{C_0}{2} + \sum_{n=1}^{\infty} C_n\,\text{Cos}(\frac{n\pi}{L}x). \end{aligned}$$

Sine and Cosine transforms of derivatives

- Finite Sine and Cosine transforms:

$$\begin{aligned} \mathcal{F}_s(f) &\equiv F_s(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\,\text{Sin}(\omega t)\text{d}t, \\ \mathcal{F}_c(f) &\equiv F_c(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\,\text{Cos}(\omega t)\text{d}t, \end{aligned}$$

- Sine Transforms of Derivatives

$$\begin{aligned} \mathcal{F}_s(f') &= -\omega\mathcal{F}_c(f), \\ \mathcal{F}_s(f'') &= \frac{2}{\pi}\omega\, f(0) - \omega^2\mathcal{F}_s(f), \end{aligned}$$

- Cosine Transforms of Derivatives

$$\begin{aligned} \mathcal{F}_c(f') &= -\frac{2}{\pi}\, f(0) + \omega\mathcal{F}_s(f), \\ \mathcal{F}_c(f'') &= -\frac{2}{\pi}\omega\, f'(0) - \omega^2\mathcal{F}_c(f). \end{aligned}$$

Diffusion problem in a semi-infinite medium

PDE:	$u_t = \alpha^2 u_{xx},$	$0 < x < \infty, \quad 0 < t < \infty$
BCs:	$u(0,t) = A,$	$0 < t < \infty$
IC:	$u(x,0) = 0,$	$0 \le x < \infty$

- Transform x-variable via the Fourier sine transform, i.e. $\mathcal{F}_s[u(x,t)] = U(\omega,t)$,

$$\mathcal{F}_s[u_t] = \alpha^2 \mathcal{F}_s[u_{xx}],$$

$$\Rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}\,t} U(\omega,t) = \alpha^2 [\frac{2}{\pi}\omega\, u(0,t) - \omega^2\, U(\omega,t)],$$

- For the ODE,

$$\frac{\mathrm{d}}{\mathrm{d}\,t} U(t) = \frac{2A\omega\alpha^2}{\pi} - \alpha^2\omega^2\, U(t)$$

we have the solution for the IC, $U(0) = 0$,

$$U(t) = \frac{2A}{\pi\omega}[1 - e^{-\omega^2\alpha^2 t}],$$

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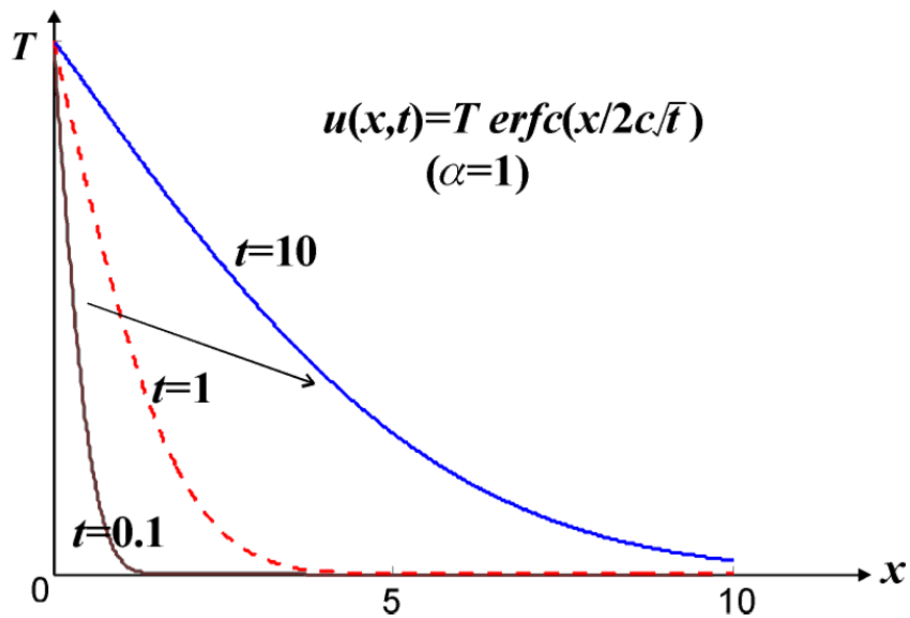
Diffusion problem in a semi-infinite medium

- By the inverse transform of $\mathcal{F}_s^{-1}[U(\omega,t)] = u(x,t)$, we have

$$u(x,t) = A\,\textit{erfc}(x/2\alpha\sqrt{t}), \quad \text{i.e.} \quad \int_0^\infty \frac{2}{\pi}\left[\frac{1-e^{-a\omega^2}}{\omega}\right]\textit{Sin}(\omega x)\,\mathrm{d}\omega \equiv \textit{erfc}\left(\frac{x}{2\sqrt{a}}\right),$$

where $\textit{erfc}(x)$, $0 < x < \infty$, is called the *complementary-error function* and is given by

$$\textit{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,\mathrm{d}t, \qquad \text{and} \qquad \textit{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,\mathrm{d}t,$$



Fourier transforms

- A periodic function of time, f(t), of period T can be represented by a Fourier transform,

$$\mathbf{F}(n) = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}\,\mathrm{d}t,$$

$$f(t) = \sum_{n=-\infty}^{\infty} \mathbf{F}(n)e^{+jn\omega_0 t},$$

  where $\mathbf{F}(n)$ is the corresponding Fourier series, where $F(n) = \frac{1}{2}(C_n - i\,S_n)$, $F(-n) = \frac{1}{2}(C_n + i\,S_n)$, and $F(0) = \frac{C_0}{2}$.

- For an *aperiodic function*,

$$\mathbf{F}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,\mathrm{d}t,$$

$$f(t) = \int_{-\infty}^{\infty} \mathbf{F}(\omega)e^{+j\omega t}\,\mathrm{d}\omega.$$

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Integral transforms

- Fourier sine transform

$$F(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\,\mathrm{Sin}(\omega t)\mathrm{d}t,$$
$$f(t) = \int_0^\infty F(\omega)\,\mathrm{Sin}(\omega t)\mathrm{d}\omega.$$

- Fourier cosine transform

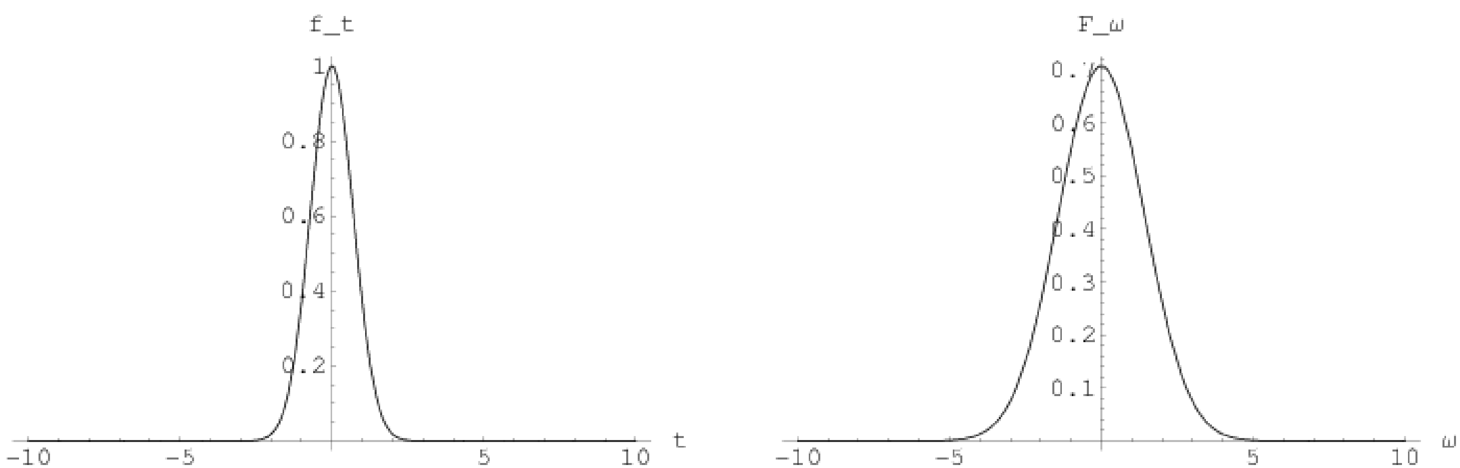
$$F(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\,\mathrm{Cos}(\omega t)\mathrm{d}t,$$
$$f(t) = \int_0^\infty F(\omega)\,\mathrm{Cos}(\omega t)\mathrm{d}\omega.$$

- Fourier transform

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(t)\,e^{-i\omega t}\mathrm{d}t,$$
$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty F(\omega)\,e^{i\omega t}\mathrm{d}\omega,$$

Gaussian function

$$\exp\left(-\frac{t^2}{2\tau_p^2}\right) \quad \leftrightarrow \quad \tau_p \exp\left(-\frac{\tau_p^2\omega^2}{2}\right),$$



where we use the identity $\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}$. And

$$\Delta t \cdot \Delta \omega = \tau_p \cdot \frac{1}{\tau_p} = 1$$

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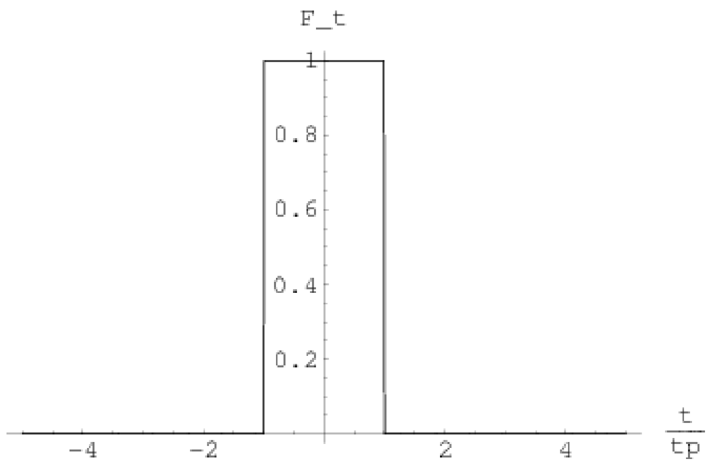
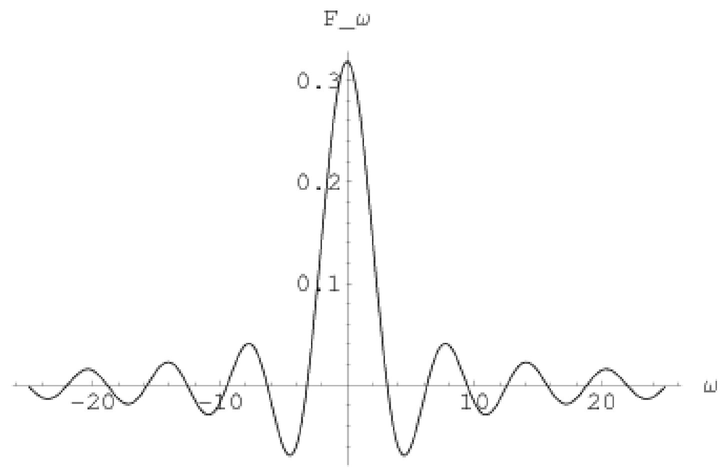
Rectangle function

$$f(t) = \begin{Bmatrix} 1, & \text{when} |t| \le \tau_p \\ 0, & \text{when} |t| > \tau_p \end{Bmatrix} \quad \leftrightarrow \quad \sqrt{\frac{2}{\pi}} \tau_p \frac{\text{sin} \omega\tau_p}{\omega\tau_p} \equiv \sqrt{\frac{2}{\pi}} \tau_p \mathit{Sinc}(\omega\tau_p),$$





Some common Fourier transforms

- Fourier transform

$$\mathbf{F}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t} \mathrm{d}t,$$
$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbf{F}(\omega)\, e^{i\omega t} \mathrm{d}\omega,$$

- Common Fourier transforms

$$f(x) = e^{-x^2} \quad \leftrightarrow \quad \mathbf{F}(\omega) = \frac{1}{\sqrt{2}} e^{-(\omega/2)^2}, \quad \text{Gaussian}$$

$$f(x) = \begin{cases} 1 & ; \quad \text{for} \quad -1 \le x \le 1 \\ 0 & ; \quad \text{elsewhere} \end{cases} \quad \leftrightarrow \quad \mathbf{F}(\omega) = \sqrt{\frac{2}{\pi}} \frac{\mathit{Sin}(\omega)}{\omega}, \quad \text{rectangle}$$

$$f(x) = \begin{cases} e^{-x} & ; \quad \text{for} \quad x > 0 \\ 0 & ; \quad \text{for} \quad x \le 0 \end{cases} \quad \leftrightarrow \quad \mathbf{F}(\omega) = \frac{1}{\sqrt{2\pi}} \frac{1 - i\omega}{1 + \omega^2},$$

Properties of Fourier transform

- if $f(t)$ is real, then $\mathbf{F}(-\omega) = \mathbf{F}^*(\omega)$.
- if $f(t)$ is even, then $\mathbf{F}(-\omega) = \mathbf{F}(\omega)$, i.e., $\mathbf{F}(\omega)$ is even.
- if $f(t)$ is odd, then $\mathbf{F}(-\omega) = -\mathbf{F}(\omega)$, i.e., $\mathbf{F}(\omega)$ is odd.
- Time scaling, $f(a\,t) \leftrightarrow \frac{1}{|a|}\mathbf{F}(\frac{\omega}{a})$.
- Frequency scaling, $\frac{1}{|b|} f(\frac{t}{b}) \leftrightarrow \mathbf{F}(b\,\omega)$.
- Time shifting, $f(t - t_0) \leftrightarrow \mathbf{F}(\omega)e^{-j\omega t_0}$.
- Frequency shifting, $f(t)e^{j\omega_0 t} \leftrightarrow \mathbf{F}(\omega - \omega_0)$.
- Convolution theorem: A convolution of two time functions, $f(t)$ and $g(t)$, is defined,

$$g \otimes f \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(t - t')f(t')\,dt',$$

the Fourier transform of the convolution is

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g \otimes f\, e^{-j\omega t}\,dt = \mathbf{G}(\omega)\mathbf{F}(\omega) = \mathcal{F}[g]\mathcal{F}[f].$$

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Transformation of Partial Derivatives

- Fourier transform

$$\begin{aligned}\mathcal{F}[f] = \mathbf{F}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t} \mathrm{d}t, \\ \mathcal{F}^{-1}\{\mathcal{F}[f]\} = f(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbf{F}(\omega)\, e^{i\omega t} \mathrm{d}\omega,\end{aligned}$$

- Transformation of partial derivatives

$$\begin{aligned}\mathcal{F}[u_x] &= i\omega \mathcal{F}[u], \\ \mathcal{F}[u_{xx}] &= -\omega^2 \mathcal{F}[u],\end{aligned}$$

- Useful identities

$$\int_{-\infty}^{\infty} e^{-x^2}\, \mathrm{d}x = \sqrt{\pi}$$

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Diffusion problem in a infinite medium

PDE:	$u_t = \alpha^2 u_{xx}, \qquad -\infty < x < \infty, \quad 0 < t < \infty$
BCs:	no boundary condition
IC:	$u(x,0) = \phi(x), \quad -\infty \leq x < \infty$

- Transform x-variable via the Fourier sine transform, i.e. $\mathcal{F}[u(x,t)] = U(\omega,t)$,

$$\mathcal{F}[u_t] = \alpha^2 \mathcal{F}[u_{xx}],$$

$$\Rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}\,t} U(\omega,t) = -\alpha^2\omega^2\, U(\omega,t),$$

- With the IC, we can solve the ODE to have,

$$U(t) = U_0(\omega)\, e^{-\alpha^2\omega^2 t},$$

where $U_0(\omega) \equiv \mathcal{F}[\phi(x)]$

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Diffusion problem in a infinite medium

- By the inverse Fourier transformation,

$$u(x,t) = \mathcal{F}^{-1}[U(t)] = \mathcal{F}^{-1}[U_0(\omega)\, e^{-\alpha^2\omega^2 t}]$$

- With the *convolution theorem*,

$$\begin{aligned} u(x,t) &= \mathcal{F}^{-1}[U_0(\omega)] \otimes \mathcal{F}^{-1}[e^{-\alpha^2\omega^2 t}] \\ &= \phi(x) \otimes [\frac{1}{\alpha\sqrt{2t}} e^{-\frac{x^2}{4\alpha^2 t}}] \\ &= \frac{1}{2\alpha\sqrt{\pi t}} \int_{-\infty}^{\infty} \phi(\xi) e^{-(x-\xi)^2/4\alpha^2 t}\, \mathrm{d}\xi \end{aligned}$$

- The integrand is made up of two terms:
  1. The initial temperature $\phi(x)$.
  2. The *Green function* or *impulse-response function* $G(x,t) = [\frac{1}{2\alpha\sqrt{\pi t}} e^{-\frac{x^2}{4\alpha^2 t}}]$
- The initial temperature $u(x,0) = \phi(x)$ is *decomposed* into a continuum of impulses.

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Integral transforms, cont.

- Laplace transform

$$\begin{aligned} F(s) &= \int_0^\infty f(t)\, e^{-s\,t} \mathrm{d}t, \\ f(t) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s)\, e^{st} \mathrm{d}s, \end{aligned}$$

- Hankel transform (Fourier-Bessel)

$$\begin{aligned} F_n(\xi) &= \int_0^\infty r\, J_n(\xi r)\, f(r) \mathrm{d}r, \\ f(r) &= \int_0^\infty J_n(\xi r)\, F_n(\xi) \mathrm{d}\xi, \end{aligned}$$

- Mellin transform

$$\begin{aligned} F(s) &= \int_0^\infty f(t)\, t^{s-1} \mathrm{d}t, \\ f(t) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s)\, x^{-s} \mathrm{d}s, \end{aligned}$$

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Some common Laplace transforms

- Laplace transform

$$
\begin{aligned}
F(s) &= \int_0^\infty f(t)\, e^{-s\,t} \mathrm{d}t, \\
f(t) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s)\, e^{st} \mathrm{d}s,
\end{aligned}
$$

- Common Laplace transforms, for $0 < t < \infty$,

$$
\begin{aligned}
f(t) = 1 \quad &\leftrightarrow \quad \mathbf{F}(s) = \frac{1}{s}, \\
f(t) = e^{2t} \quad &\leftrightarrow \quad \mathbf{F}(s) = \frac{1}{s-2} \quad s > 2 \\
f(t) = \mathit{Sin}(\omega t) \quad &\leftrightarrow \quad \mathbf{F}(s) = \frac{\omega}{s^2+\omega^2}, \\
f(t) = e^{t^2} \quad &\leftrightarrow \quad \mathbf{F}(s) \quad \text{don't exist.}
\end{aligned}
$$

Transformation of Partial Derivatives

- Laplace Transformation of partial derivatives, i.e. $\mathcal{L}[u(x,t)] \equiv U(x,s)$

$$\begin{aligned} \mathcal{L}[u_t] &= s\,U(x,s) - u(x,0), \\ \mathcal{L}[u_{tt}] &= s^2\,U(x,s) - s\,u(x,0) - u_t(x,0), \end{aligned}$$

- Definition of finite convolution

$$[g \otimes f](t) \equiv \int_0^t g(t')f(t-t')\,\mathrm{d}t' = \int_0^t g(t-t')f(t')\,\mathrm{d}t',$$

- Convolution theorem for Laplace transform

$$\begin{aligned} \mathcal{L}[g \otimes f] &= \mathcal{L}[g]\,\mathcal{L}[f], \\ \mathcal{L}^{-1}\{\mathcal{L}[g]\,\mathcal{L}[f]\} &= g \otimes f. \end{aligned}$$

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Heat conduction in a semi-infinite medium

PDE: $u_t = u_{xx}, \quad 0 < x < \infty, \quad 0 < t < \infty$

BC: $u_x(0,t) - u(0,t) = 0, \quad 0 < t < \infty$

IC: $u(x,0) = u_0, \quad 0 \le x < \infty$

- Transform t-variable via the Laplace transform, i.e. $\mathcal{L}[u(x,t)] = U(x)$,

$$\mathcal{L}[u_t] = s\,U(x,s) - u(x,0),$$

- For the ODE,

$$s\,U(x) - u_0 = \frac{\mathrm{d}^2}{\mathrm{d}x^2} U(x),$$

we have the general solution (homogeneous + a particular solution),

$$U(x) = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x} + \frac{u_0}{s},$$

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Heat conduction in a semi-infinite medium, cont.

- For the BC,

$$\frac{\mathrm{d}}{\mathrm{d}\,x}U(0) - U(0) = 0,$$

we have the coefficients,

$$U(x) = -u_0 \frac{e^{-\sqrt{s}x}}{s(\sqrt{s}+1)} + \frac{u_0}{s},$$

- By the inverse Laplace transform,

$$\begin{aligned} u(x,t) &= \mathcal{L}^{-1}[U(x,s)], \\ &= u_0 - u_0[\mathit{erfc}(x/2\sqrt{t}) - \mathit{erfc}(\sqrt{t} + x/2\sqrt{t})e^{x+t}], \end{aligned}$$

where the complementary-error function is

$$\mathit{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-\xi^2}\, \mathrm{d}\xi.$$

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