Vibrating-string problem

- Newton's equation of motion,

$$m\,u_{tt} = \quad \text{applied forces to the segment} \quad (x, x, +\Delta x),$$

- Net force due to the tension of the string,

$$T\mathit{Sin}\theta_2 - T\mathit{Sin}\theta_1 \approx T[u_x(x+\delta x, t) - u_x(x,t)],$$

- External force, $F(x,t)$,
- Frictional force against the string, $-\beta u_t$,
- Restoring force, $-\gamma u$,
- For a small segment of string,

$$\Delta x \rho u_{tt} = T[u_x(x+\Delta x, t) - u_x(x,t)] + \Delta x F(x,t) - \Delta x \beta u_t(x,t) - \Delta x \gamma u(x,t),$$
$$\Rightarrow u_{tt} = \alpha^2 u_{xx} - \beta u_t - \gamma u + F(x,t),$$

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Maxwell's equations with total charge and current

(1831-1879)

- Gauss's law for the electric field:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \Longleftrightarrow \oint_S \mathbf{E} \cdot \mathrm{d}\,A = \frac{Q}{\epsilon_0},$$

- Gauss's law for magnetism:

$$\nabla \cdot \mathbf{B} = 0 \Longleftrightarrow \oint_S \mathbf{B} \cdot \mathrm{d}\,A = 0,$$

- Faraday's law of induction:

$$\nabla \times \mathbf{E} = -\kappa \frac{\partial}{\partial\, t}\mathbf{B} \Longleftrightarrow \oint_C \mathbf{E} \cdot \mathrm{d}\,l = -\kappa \frac{\partial}{\partial t}\Phi_B,$$

- Ampére's circuital law:

$$\nabla \times \mathbf{B} = \kappa\mu_0(\mathbf{J} + \epsilon_0 \frac{\partial}{\partial\, t}\mathbf{E}) \Longleftrightarrow \oint_C \mathbf{B} \cdot \mathrm{d}\,l = -\kappa\mu_0(\mathbf{I} + \epsilon_0 \frac{\partial}{\partial\, t}\Phi_E)$$

Helmholtz wave equations

- For a source-free medium, $\rho = \mathbf{J} = 0$,

$$\nabla \times (\nabla \times E) = -\mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} E,$$

$$\Rightarrow \qquad \nabla(\nabla \cdot E) - \nabla^2 E = -\mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} E.$$

- When $\nabla \cdot E = 0$, one has *wave equation*,

$$\nabla^2 E = \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} E$$

- which has following expression of the solutions, in 1D,

$$E = \hat{x}[f_+(z - vt) + f_-(z + vt)],$$

  with $v^2 = \frac{1}{\mu_0 \epsilon_0} = c^2$.

- plane wave solutions: $E_+ = E_0 \cos(kz - \omega t)$, where $\frac{\omega}{k} = c$.

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Wave equation

- The wave equation contains a second-order time derivative, u_{tt}, it requires *two initial conditions*

$$u(x,t=0)=f(x), \quad \text{and} \quad u_t(x,t=0)=g(x),$$

- For electric current along the wire, with the Kirchhoff's laws,

$$\begin{aligned} i_x + C\,v_t + G\,v &= 0, \\ v_x + L\,i_t + R\,i &= 0, \end{aligned}$$

- The transmission-line equations

$$\begin{aligned} i_{xx} &= CL\,i_{tt} + (CR+GL)\,i_t + GR\,i, \\ v_{xx} &= CL\,v_{tt} + (CR+GL)\,v_t + GR\,v, \end{aligned}$$

- If $G=R=0$, and $\alpha^2 = 1/CL$,

$$\begin{aligned} i_{tt} &= \alpha^2, i_{xx}, \\ v_{tt} &= \alpha^2\,v_{xx}. \end{aligned}$$

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The D'Alembert solution of the wave equation

- For the diffusion problems (the parabolic case), we solve the bounded case ($0 \leq x \leq L$) by separation of variables while solve the unbounded case ($-\infty < x < \infty$) by the Fourier transform.
- For the wave problems (the hyperbolic case), we will do the **opposite**.

PDE: $$u_{tt} = \alpha^2 u_{xx}, \qquad -\infty < x < \infty, \quad 0 < t < \infty$$

ICs: $$\begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}, \quad -\infty < x < \infty$$

- Replace (x,t) by new *canonical coordinates* (ξ, η), i.e. the moving-coordinate,

$$\xi = x + \alpha\, t \qquad \eta = x - \alpha\, t$$

- the PDE becomes

$$u_{\xi\eta} = 0$$

with the solution of arbitrary functions of ξ or η, i.e.

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$$u(\xi, \eta) = \phi(\eta) + \psi(\xi),$$

The D'Alembert solution of the wave equation, cont.

- In the original coordinates x and t, we have

$$u(x,t) = \Phi(x - \alpha\, t) + \Psi(x + \alpha\, t),$$

this is the general solution of the wave equation.

- Physically it represents the sum of *any two moving waves*, each moving in opposite direction with the velocity α. Eg.

$$\begin{aligned} u(x,t) &= \textit{Sin}(x - \alpha\, t), && \text{(one right-moving wave)} \\ u(x,t) &= (x + \alpha\, t)^2, && \text{(one left-moving wave)} \\ u(x,t) &= \textit{Sin}(x - \alpha\, t) + (x + \alpha\, t)^2, && \text{(two oppositely moving waves)} \end{aligned}$$

spherical waves

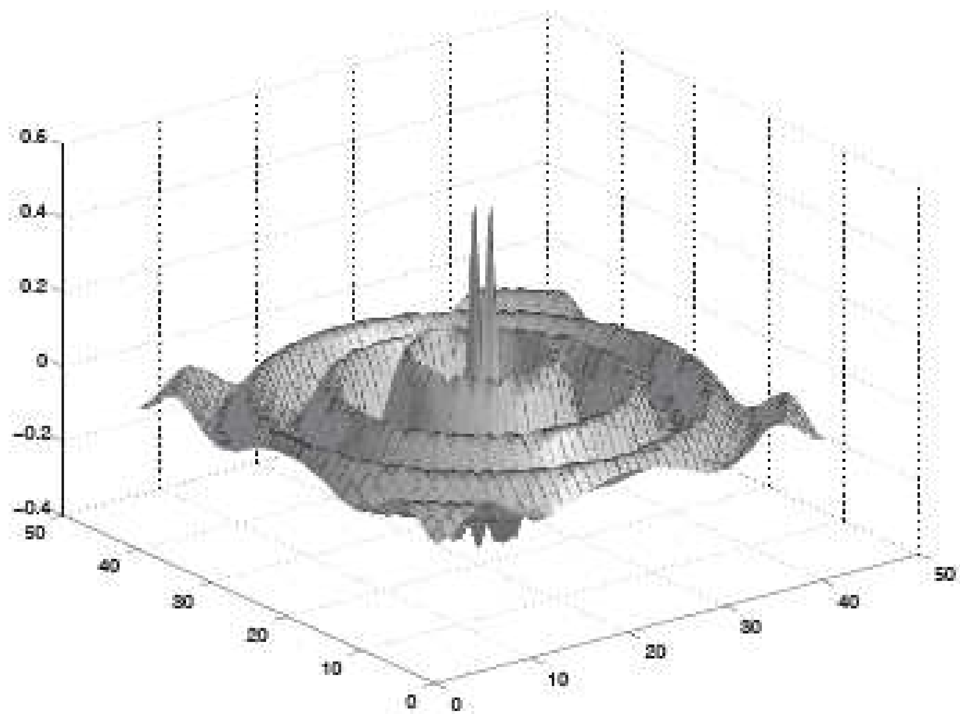
- spherical wave:

$$U(r) = \frac{A}{|r - r_0|} \exp(-ik|r - r_0|),$$

where $k|r - r_0| =$ constant, wavefronts resemble sphere surfaces,

- intensity:

$$I(r) = \frac{|A|^2}{r^2},$$



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ICs

- Substitute the general solution into the two ICs,

$$\text{ICs:} \qquad \begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}, \quad -\infty < x < \infty$$

- for arbitrary functions ϕ and ψ, we have

$$\begin{aligned} \phi(x) + \psi(x) &= f(x), \\ -\alpha\,\phi'(x) + \alpha\,\psi'(x) &= g(x), \end{aligned}$$

- then by integrating from x_0 to x,

$$-\alpha\,\phi(x) + \alpha\,\psi(x) = \int_{x_0}^{x} g(\xi)\mathrm{d}\,\xi + K$$

where K is an integration constant.

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D'Alembert solution

- The solutions for ϕ and ψ are

$$\begin{aligned}\phi(x) &= \frac{1}{2}f(x) - \frac{1}{2\alpha}\int_{x_0}^{x} g(\xi)\mathrm{d}\,\xi, \\ \psi(x) &= \frac{1}{2}f(x) + \frac{1}{2\alpha}\int_{x_0}^{x} g(\xi)\mathrm{d}\,\xi,\end{aligned}$$

- The D'Alembert solution,

$$u(x,t) = \frac{1}{2}[f(x-\alpha,t) + f(x+\alpha\,t)] + \frac{1}{2\alpha}\int_{x-\alpha\,t}^{x+\alpha\,t} g(\xi)\mathrm{d}\,\xi.$$

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Examples of the D'Alembert solution

1. Motion of an initial Sine wave,

PDE: $$u_{tt} = \alpha^2 u_{xx}, \qquad -\infty < x < \infty, \quad 0 < t < \infty$$

ICs: $$\begin{cases} u(x,0) = \mathit{Sin}(x) \\ u_t(x,0) = 0 \end{cases}, \quad -\infty < x < \infty$$

- The solution

$$u(x,t) = \frac{1}{2}[\mathit{Sin}(x - \alpha\, t) + \mathit{Sin}(x + \alpha\, t)].$$

2. Initial velocity given

ICs: $$\begin{cases} u(x,0) = 0 \\ u_t(x,0) = \mathit{Sin}(x) \end{cases}, \quad -\infty < x < \infty$$

- The solution

$$u(x,t) = \frac{1}{2\alpha}\int_{x-\alpha\, t}^{x+\alpha\, t} \mathit{Sin}(\xi)\,\mathrm{d}\,\xi = \frac{1}{2\alpha}[\mathrm{Cos}(x + \alpha\, t) - \mathrm{Cos}(x - \alpha\, t)].$$

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The finite vibrating string (standing waves)

PDE: $$u_{tt} = \alpha^2 u_{xx}, \qquad 0 < x < L, \quad 0 < t < \infty$$

BCs: $$\begin{cases} u(0,t) = 0 \\ u(L,t) = 0 \end{cases}, \quad 0 < t < \infty$$

ICs: $$\begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}, \quad 0 \le x \le L$$

- Applying separation of variables

$$u(x,t) = X(x)\,T(t),$$

- Substituting this expression into the wave equation, we have two ODEs

$$T'' - \alpha^2 \lambda T = 0$$
$$X'' - \lambda X = 0$$

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The finite vibrating string (standing waves), cont.

- Only negative values of λ give feasible (nonzero and bounded) solutions, i.e., $\lambda \equiv -\beta^2$.

$$u(x,t) = [C\mathit{Sin}(\beta x) + D\mathit{Cos}(\beta x)][A\mathit{Sin}(\alpha\beta t) + B\mathit{Cos}(\alpha\beta t)],$$

- With the BCs,

$$\begin{aligned} u(0,t) &= X(0)\,T(t) = 0, \qquad \Rightarrow D = 0, \\ u(L,t) &= X(L)\,T(t) = 0, \qquad \Rightarrow \beta_n = \frac{n\pi}{L}, \quad n = 1, 2, \ldots \end{aligned}$$

- The fundamental solution

$$\begin{aligned} u_n(x,t) = X_n(x)\,T_n(t) &= \mathit{Sin}(\frac{n\pi x}{L})[a_n \mathit{Sin}(\frac{n\pi\alpha t}{L}) + b_n \mathit{Cos}(\frac{n\pi\alpha t}{L})] \\ &= R_n \mathit{Sin}(\frac{n\pi x}{L})\mathit{Cos}[\frac{n\pi\alpha(t-\delta_n)}{L}] \end{aligned}$$

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The finite vibrating string (standing waves), cont.

- The solution for wave equation

$$u(x,t) = \sum_{n=1}^{\infty} \mathit{Sin}(\frac{n\pi x}{L})[a_n \mathit{Sin}(\frac{n\pi\alpha t}{L}) + b_n \mathit{Cos}(\frac{n\pi\alpha t}{L})]$$

- With ICs,

$$\begin{aligned} u(x,0) &= \sum_{m=1}^{\infty} b_n \mathit{Sin}(\frac{n\pi x}{L}) = f(x), \\ u_t(x,0) &= \sum_{m=1}^{\infty} a_n (\frac{n\pi\alpha}{L}) \mathit{Sin}(\frac{n\pi x}{L}) = g(x), \end{aligned}$$

- Using the orthogonality conditions, $\int_0^L \mathit{Sin}(\frac{m\pi x}{L})\mathit{Sin}(\frac{n\pi x}{L})\mathrm{d}\,x = \frac{L}{2}\delta_{mn}$, we have

$$\begin{aligned} a_n &= \frac{2}{n\pi\alpha}\int_0^L g(x)\mathit{Sin}(\frac{n\pi x}{L})\mathrm{d}\,x, \\ b_n &= \frac{2}{L}\int_0^L f(x)\mathit{Sin}(\frac{n\pi x}{L})\mathrm{d}\,x. \end{aligned}$$

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The finite Fourier transforms

- Finite Sine transform

$$
\begin{aligned}
S\left[f\right] &= S_n = \frac{2}{L}\int_0^L f(x)\,\mathrm{Sin}(\frac{n\pi}{L}x)\mathrm{d}x,\\
f(x) &= \sum_{n=1}^{\infty} S_n\,\mathrm{Sin}(\frac{n\pi}{L}x).
\end{aligned}
$$

- Finite Cosine transform

$$
\begin{aligned}
C\left[f\right] &= C_n = \frac{2}{L}\int_0^L f(x)\,\mathrm{Cos}(\frac{n\pi}{L}x)\mathrm{d}x,\\
f(x) &= \frac{C_0}{2} + \sum_{n=1}^{\infty} C_n\,\mathrm{Cos}(\frac{n\pi}{L}x).
\end{aligned}
$$

- Transform of derivatives

$$
\begin{aligned}
S\left[f_x x\right] &= -[\frac{n\pi}{L}]^2 S\left[f\right] + \frac{2n\pi}{L^2}[f(0,t) + (-1)^{n+1} f(L,t)],\\
C\left[f_x x\right] &= -[\frac{n\pi}{L}]^2 S\left[f\right] - \frac{2}{L}[f_x(0,t) + (-1)^{n+1} f_x(L,t)].
\end{aligned}
$$

A nonhomogeneous BVP via the Finite Sine Transform

PDE: $u_{tt} = u_{xx} + \mathit{Sin}(\pi x), \qquad 0 < x < 1, \quad 0 < t < \infty$

BCs: $\begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases}, \quad 0 < t < \infty$

ICs: $\begin{cases} u(x,0) = 1 \\ u_t(x,0) = 0 \end{cases}, \quad 0 \le x \le 1$

- Transform the PDE by using the finite Sine transform for the variable x, $S_n(t) = S\,[u]$,

$$\frac{\mathrm{d}^2}{\mathrm{d}\,t^2} S_n(t) = -(n\pi)^2 S_n(t) + 2n\pi[u(0,t) + (-1)^{n+1}u(1,t)] + D_n(t),$$

where

$$D_n(t) = S\,[\mathit{Sin}(\pi x)] = \begin{cases} 1 & ; \quad n = 1 \\ 0 & ; \quad n = 2, 3, \ldots \end{cases}$$

A nonhomogeneous BVP via the Finite Sine Transform, Cont.

- Solve the new initial-value problems for ODE,

ODE: $$\frac{\mathrm{d}^2}{\mathrm{d}\,t^2}S_n(t)+(n\pi)^2S_n=\begin{cases}1 & ; \quad n=1\\ 0 & ; \quad n=2,3,\ldots\end{cases}$$

ICs: $$\begin{cases}S_n(0)=\begin{cases}\frac{4}{n\pi} & ; \quad n=1,3,\ldots\\ 0 & ; \quad n=2,4,\ldots\end{cases}\\ \frac{\mathrm{d}}{\mathrm{d}\,t}S_n(0)=0\end{cases}$$

- The solutions are

$$S_n(t)=\begin{cases}[\frac{4}{\pi}-\frac{1}{\pi^2}]\text{Cos}(\pi t)+(\frac{1}{\pi})^2 & ; \quad n=1,\ldots\\ 0 & ; \quad n=2,4,\ldots\\ \frac{4}{n\pi}\text{Cos}(n\pi t) & ; \quad n=3,5,7,\ldots\end{cases}$$

- the solution $u(x,t)$ of the problem is

$$u(x,t)=\{[\frac{4}{\pi}-\frac{1}{\pi^2}]\text{Cos}(\pi t)+(\frac{1}{\pi})^2\}\text{Sin}[\pi x]+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n+1}\text{Cos}[(2n+1)\pi t]\text{Sin}[(2n+1)\pi x]$$

Wave equation in polar coordinates

PDE: $$u_{tt} = c^2[u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}], \qquad 0 < r < 1, \quad 0 < t < \infty$$

BCs: $$u(r = 1) = 0, \quad 0 < t < \infty$$

ICs: $$\begin{cases} u(t = 0) = f(r, \theta) \\ u_t(t = 0) = g(r, \theta) \end{cases}, \quad 0 \leq r \leq 1$$

- By separation of variables, $u(r, \theta, t) = U(r, \theta)\, T(t)$, i.e.

$$\nabla^2 U + \lambda^2 U = 0, \qquad \text{Helmholtz equation}$$
$$T'' + \lambda^2 c^2 T = 0, \qquad \text{Simple harmonic motion,}$$

where

$$\nabla^2 U = [U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta}]$$

Helmholtz eigenvalue problem

$$\text{PDE:} \qquad \nabla^2 U + \lambda^2 U = 0,$$

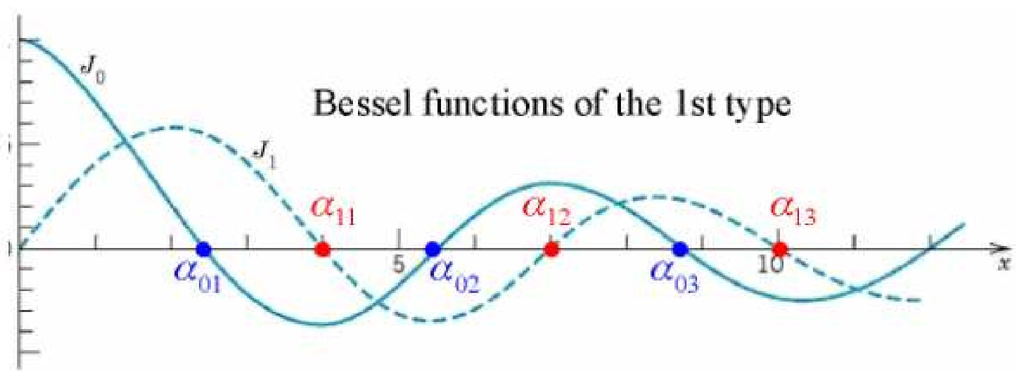
$$\text{BCs:} \qquad U(r=1,\theta) = 0,$$

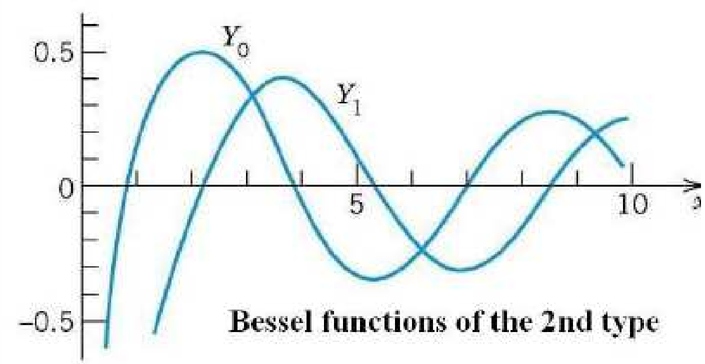
- By separation of variables again, $U(r,\theta) = R(r)\,\Theta(\theta)$, i.e.

$$r^2 R" + rR' + (\lambda^2 r^2 - n^2)R = 0, \qquad \text{Bessel's equation}$$

$$\Theta" + n^2\Theta = 0$$

- The solutions for the Bessel's equation, $J_n(\lambda r)$ and $Y_n(\lambda r)$, n-th-order Bessel function of the first and second kinds.





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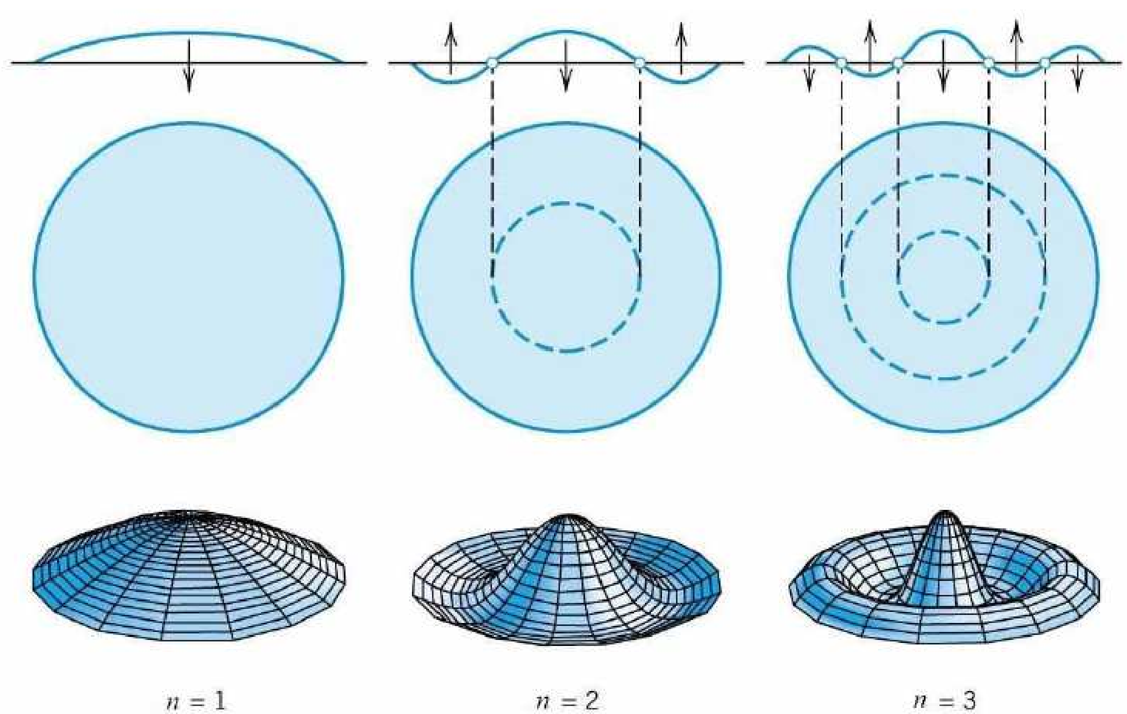
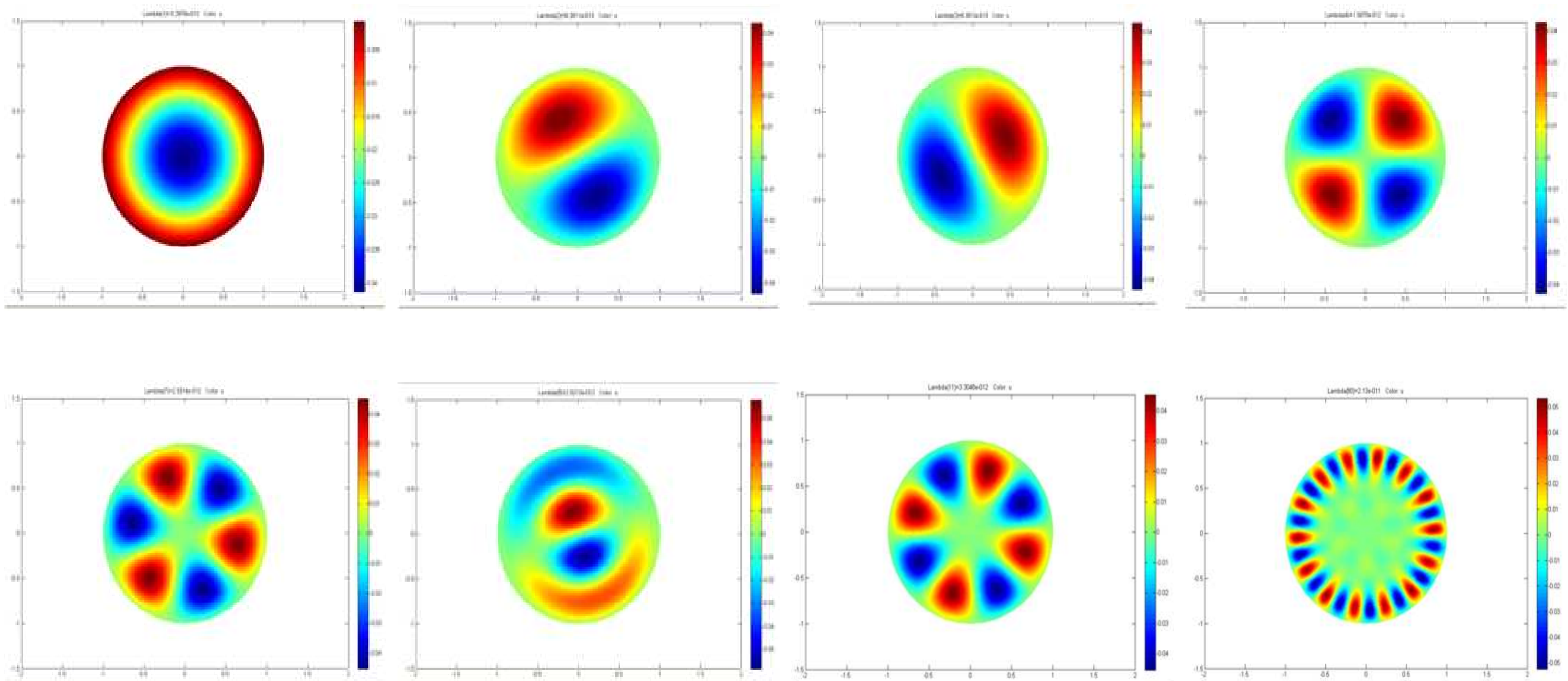
Helmholtz eigenvalue problem, Cont.

- For $R(r=0) < \infty$, we have $R(r) = A\,J_n(\lambda r)$.
- The BC, $R(1) = 0$, we have $J_n(\lambda) = 0 \Rightarrow \lambda = k_{n,}$, where k_{nm} is the m-th root of $J_n(\lambda)$.
- The corresponding eigenfunctions $U_{nm}(r, \theta)$

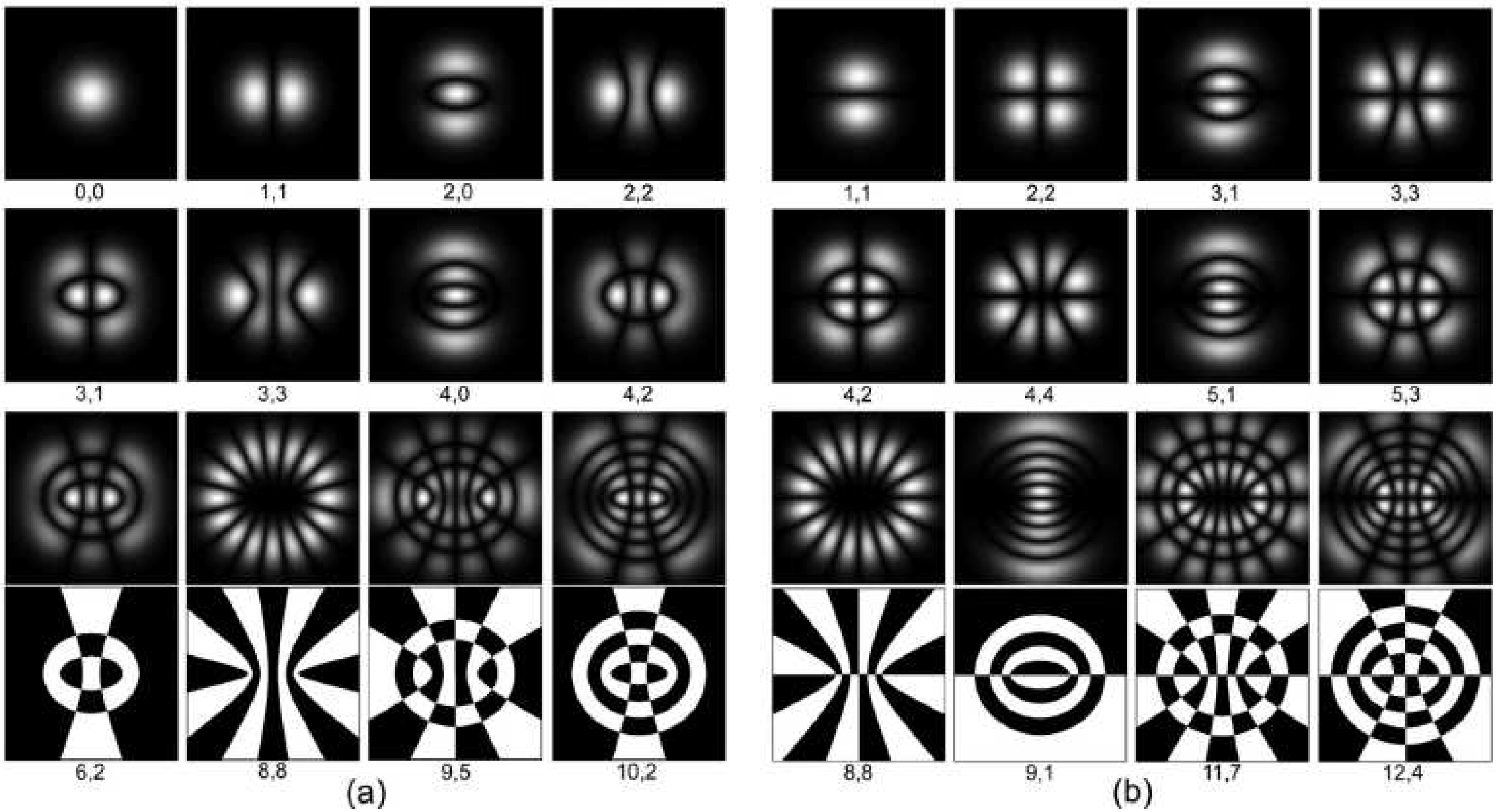
$$U_{nm} = J_n(k_{nm} r)[A\mathit{Sin}(n\theta) + B\mathit{Cos}(n\theta)],$$



Cavity modes



Hermite/Laguerre/Ince-Gaussian beams



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http://www.uwgb.edu/dutchs/petrolgy/intfig1.htm

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Helmholtz eigenvalue problem, Cont.

- For the time equation, we have

$$T" + k_{nm}^2 c^2 T = 0, \qquad \text{Simple harmonic motion,}$$
$$\Rightarrow T_{nm}(t) = A\mathrm{Sin}(k_{nm}ct) + B\mathrm{Cos}(k_{nm}ct).$$

- The total solution of our problem is

$$u(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n(k_{nm}r)\mathrm{Cos}(n\theta)[A_{nm}\mathrm{Sin}(k_{nm}ct) + B_{nm}\mathrm{Cos}(k_{nm}ct)]$$

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Wave equation in polar coordinates, θ-independent

- For the special case independent of θ, i.e. $n = 0$

$$u(r,t) = \sum_{m=1}^{\infty} J_0(k_{0m}r)[A_m \text{Sin}(k_{0m}ct) + B_m \text{Cos}(k_{0m}ct)]$$

- In particular, we consider the ICs,

$$\text{ICs:} \quad \left\{ \begin{array}{l} u(t=0) = f(r) = \sum_{m=1}^{\infty} A_m J_0(k_{0m}r) \\ u_t(t=0) = 0 = B_m \end{array} \right. , \quad 0 \leq r \leq 1$$

- Using the orthogonality condition of the Bessel functions, i.e.

$$\int_0^1 r\, J_0(k_{0m}r)\, J_0(k_{0n}r) \mathrm{d}\, r = \delta_{nm} \frac{1}{2} J_1^2(k_{0m}),$$

we have

$$A_m = \frac{2}{J_1^2(k0m)} \int_0^1 r\, f(r)\, J_0(k_{0m}r) \mathrm{d}\, r, \qquad m = 1, 2, \ldots,$$

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Laplace's equation

$$\nabla^2 u = 0,$$

- in Cartesian coordinates: x, y, z

$$\nabla^2 u(x, y, z) = u_{xx} + u_{yy} + u_{zz}$$

- in Cylindrical coordinates: $x = r\ \cos\theta, y = r\ \sin\theta, z$

$$\nabla^2 u(r, \theta, z) = [u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz}]$$

- in Spherical coordinates: $x = r\ \cos\theta\sin\phi, y = r\ \sin\theta\sin\phi, z = r\cos\phi$

$$\begin{aligned}\nabla^2 u(r, \theta, \phi) &= [u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}u_{\phi\phi} + \frac{\cot\phi}{r^2}u_\phi + \frac{1}{r^2\sin^2\phi}u_{\theta\theta}] \\ &= \frac{1}{r^2}[\frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r}) + \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi\frac{\partial u}{\partial\phi}) + \frac{1}{\sin^2\phi}\frac{\partial^2 u}{\partial\theta^2}]\end{aligned}$$

Laplace equation in spherical coordinates

PDE: $$\nabla^2 u = \frac{1}{r^2}\left[\frac{\partial}{\partial r}(r^2 \frac{\partial u}{\partial r}) + \frac{1}{\sin\phi}\frac{\partial}{\partial \phi}(\sin\phi \frac{\partial u}{\partial \phi})\right] = 0,$$

BCs: $$\begin{cases} u(r = R, \phi) = f(\phi) \\ \lim_{r\to\infty} u(r,\phi) = 0 \end{cases}$$

- By separation of variables, $u(r,\phi) = R(r)\,\Phi(\phi)$, i.e.

$$r^2 \frac{\mathrm{d}^2 R}{\mathrm{d}\, r^2} + 2r\frac{\mathrm{d}\, R}{\mathrm{d}\, r} = k\,R \equiv n(n+1)R, \qquad \textit{Euler-Cauchy equation}$$

$$\frac{1}{\sin\phi}\frac{\mathrm{d}}{\mathrm{d}\,\phi}(\sin\phi \frac{\mathrm{d}\,\Phi}{\mathrm{d}\,\phi}) = -k\Phi,$$

- Assume $R(r) = r^a$, then $a(a-1) + 2a - n(n+1) = 0 = (a-n)(a+n+1)$, with roots $a = n$ and $a = -n-1$.
- The solutions for the Euler-Cauchy equation are

$$R_n(r) = A_n r^n + B_n \frac{1}{r^{n+1}},$$

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Legendre polynomials

- Set $w = \cos\phi$, then $\sin^2\phi = 1 - w^2$, then

$$\frac{\mathrm{d}}{\mathrm{d}\,\phi} = -\sin\phi \frac{\mathrm{d}}{\mathrm{d}\,w},$$

- We have the Legendre's equation for $\Phi(\phi)$,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\,w}[(1-w^2)\frac{\mathrm{d}\,\Phi}{\mathrm{d}\,w}] + n(n+1)\Phi, \\
&= (1-w^2)\frac{\mathrm{d}^2\,\Phi}{\mathrm{d}\,w^2} - 2w\frac{\mathrm{d}\,\Phi}{\mathrm{d}\,w} + n(n+1)\Phi = 0.
\end{aligned}$$

- For integer $n = 0, 1, 2, \ldots$, the solutions are the Legendre polynomials

$$\Phi = P_n(w) = P_n(\cos\phi),$$

$$P_0(w) = 1, \qquad P_1(w) = w, \qquad P_2(w) = \frac{1}{2}(3w^2 - 1),$$

$$P_3(w) = \frac{1}{2}(5w^3 - 3w), \qquad P_n(w) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}\,w^n}[(w^2-1)^n]$$

Laplace equation in spherical coordinates, Cont.

- The fundamental solutions for the Laplace equation is

$$u_n(r,\phi) = [A_n r^n + B_n \frac{1}{r^{n+1}}]\, P_n(\cos\phi),$$

- Interior problem: potential within the sphere,

$$u(r,\phi) = \sum_{n=0}^{\infty} A_n r^n \, P_n(\cos\phi),$$

- BC: $u(R,\phi) = \sum_{n=0}^{\infty} A_n R^n \, P_n(\cos\phi) = f(\phi)$,
- The orthogonality condition for the Legendre polynomials

$$\int_{-1}^{1} P_n(w)\, P_m(w) \mathrm{d}\, w = \frac{2}{2n+1}\delta_{mn}$$

then the coefficients are

$$A_n = \frac{2n+1}{2R^n}\int_0^{\pi} f(\phi)\, P_n(\cos\phi)\sin\phi \mathrm{d}\,\phi.$$

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Dimensionless problem, heat equation

PDE: $$u_t = \alpha^2 u_{xx}, \qquad 0 < x < L, \quad 0 < t < \infty$$

BCs: $$\begin{cases} u(0,t) = T_1 \\ u(L,t) = T_2 \end{cases}, \quad 0 < t < \infty$$

IC: $$u(x,0) = \sin(\pi x/L), \quad 0 \le x \le L$$

- Define $U(x,t) = \frac{u(x,t)-T_1}{T_2-T_1}$,

PDE: $$U_t = \alpha^2 U_{xx}, \qquad 0 < x < L, \quad 0 < t < \infty$$

BCs: $$\begin{cases} U(0,t) = 0 \\ U(L,t) = 1 \end{cases}, \quad 0 < t < \infty$$

IC: $$U(x,0) = \frac{\sin(\pi x/L) - T_1}{T_2 - T_1}, \quad 0 \le x \le L$$

Dimensionless problem, heat equation, Cont.

- Define dimensionless-space variable $\xi = x/L$,

PDE: $$U_t = (\alpha/L)^2 U_{\xi\xi}, \qquad 0 < \xi < 1, \quad 0 < t < \infty$$

BCs: $$\begin{cases} U(0,t) = 0 \\ U(1,t) = 1 \end{cases}, \quad 0 < t < \infty$$

IC: $$U(\xi, 0) = \frac{\sin(\pi\xi) - T_1}{T_2 - T_1}, \quad 0 \le \xi \le 1$$

- Define dimensionless-time variable $\tau = c\,t = [\alpha/L]^2 t$,

PDE: $$U_\tau = U_{\xi\xi}, \qquad 0 < \xi < 1, \quad 0 < \tau < \infty$$

BCs: $$\begin{cases} U(0,\tau) = 0 \\ U(1,\tau) = 1 \end{cases}, \quad 0 < \tau < \infty$$

IC: $$U(\xi, 0) = \frac{\sin(\pi\xi) - T_1}{T_2 - T_1} = \phi(\xi), \quad 0 \le \xi \le 1$$

And the solution to the original problem is

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$$u(x,t) = T_1 + (T_2 - T_1) U(x/l, \alpha^2\, t/L^2)$$

Normalization of coefficients

Nonlinear Schrödinger equation:

$$i\frac{\partial U}{\partial z} + D_2\frac{\partial^2 U}{\partial t^2} + \chi_3|U|^2U = 0$$

set $\eta = az$ and $\tau = bt$, then NLSE becomes

$$ia\frac{\partial U}{\partial \eta} + D_2 b^2\frac{\partial^2 U}{\partial \tau^2} + \chi_3|U|^2U = 0$$

then by choosing

$$\begin{aligned} a &= \chi_3, \\ b &= \sqrt{\frac{\chi_3}{2D_2}}, \end{aligned}$$

NLSE is reduced to scaleless form,

$$i\frac{\partial U}{\partial \eta} + \frac{1}{2}\frac{\partial^2 U}{\partial \tau^2} + |U|^2U = 0$$

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