Syllabus

1. Introduction to modern photonics (Feb. 26),
2. Ray optics (lens, mirrors, prisms, et al.) (Mar. 7, 12, 14, 19),
3. Wave optics (plane waves and interference) (Mar. 26, 28),
4. Beam optics (Gaussian beam and resonators) (Apr. 9, 11, 16),
5. Electromagnetic optics (reflection and refraction) (Apr. 18, 23, 25),
   Midterm (May 7-th),
6. Fourier optics (diffraction and holography) (May 14, May 16),
7. Crystal optics (birefringence and LCDs) (May 21, 23),
8. Waveguide optics (waveguides and optical fibers) (May 28, 30),
9. Photon optics (light quanta and atoms) (June 4),
10. Laser optics (spontaneous and stimulated emissions) (June 6),
11. Semiconductor optics (LEDs and LDs) (June 11),
12. Nonlinear optics (June 13),
13. Quantum optics (June 18),
    Final exam (June 20),
14. Semester oral report (July 25, 27),

Fourier Optics

- Fourier transforms
- Fresnel diffraction and paraxial wave equation
- Near-field region
- Fraunhofer diffraction
- Fourier transformation by a lens

Fourier transforms

- A periodic function of time, f(t), of period T can be represented by a Fourier transform,

$$\mathbf{F}(n) = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}\,dt,$$

$$f(t) = \sum_{-\infty}^{\infty} \mathbf{F}(t)e^{+jn\omega_0 t},$$

where $\mathbf{F}(n)$ is the corresponding Fourier series.

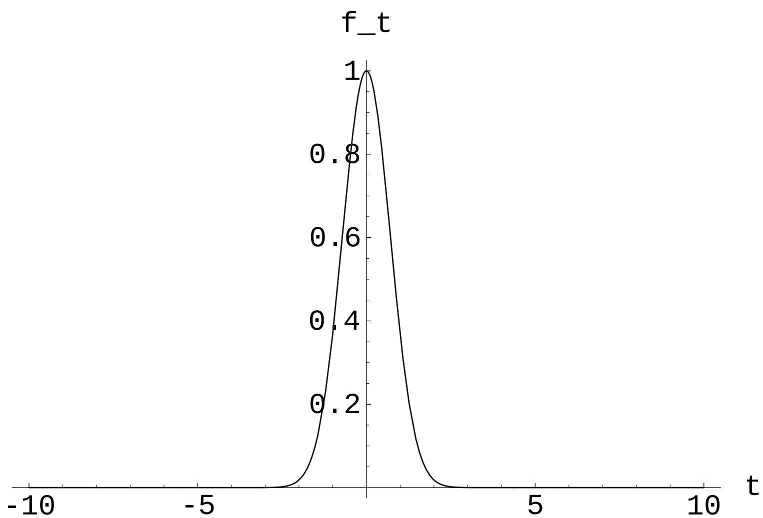
- For an *aperiodic function*,

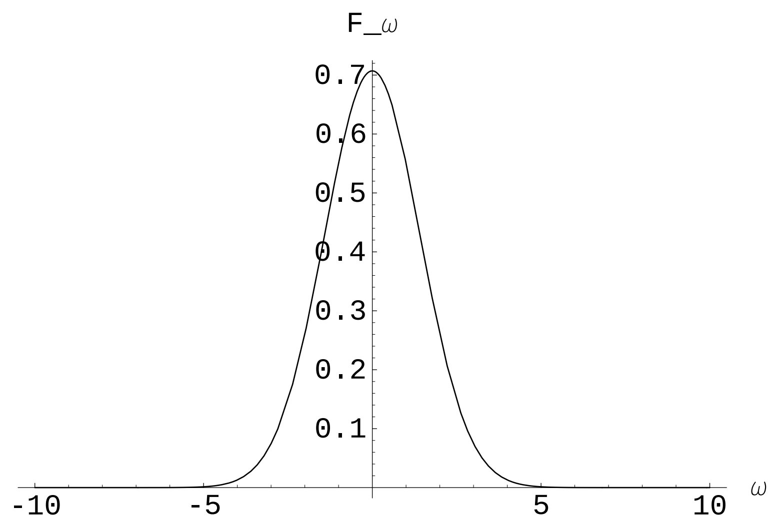
$$\mathbf{F}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt,$$

$$f(t) = \int_{-\infty}^{\infty} \mathbf{F}(\omega)e^{+j\omega t}\,d\omega.$$

Gaussian function

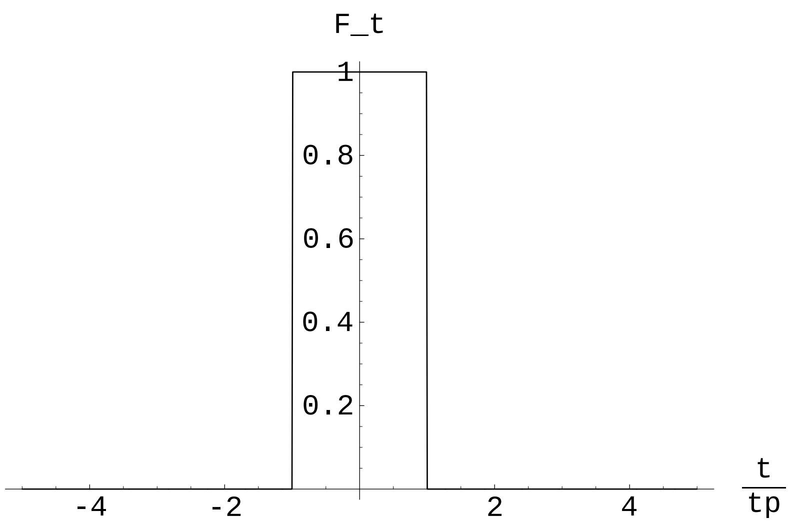
$$\exp\left(-\frac{t^2}{2\tau_p^2}\right) \quad \leftrightarrow \quad \sqrt{\frac{\tau^2}{2\pi}}\exp\left(-\frac{\tau_p^2\omega^2}{2}\right),$$

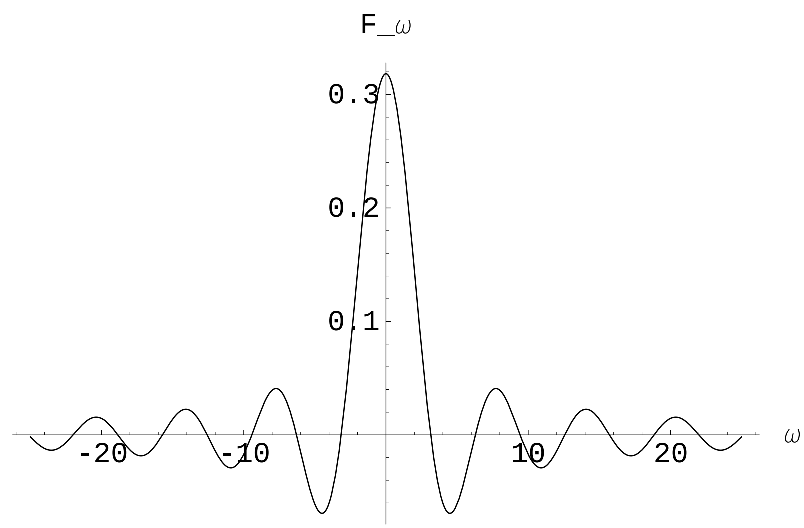




Rectangle function

$$f(t) = \left\{ \begin{array}{ll} 1, & \text{when} |t| \leq \tau_p \\ 0, & \text{when} |t| > \tau_p \end{array} \right\} \quad \leftrightarrow \quad \tau_p \frac{\sin \omega \tau_p}{\pi \omega \tau_p},$$





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Properties of Fourier transform

- if $f(t)$ is real, then $\mathbf{F}(-\omega) = \mathbf{F}^*(\omega)$.
- if $f(t)$ is even, then $\mathbf{F}(-\omega) = \mathbf{F}(\omega)$, i.e., $\mathbf{F}(\omega)$ is even.
- if $f(t)$ is odd, then $\mathbf{F}(-\omega) = -\mathbf{F}(\omega)$, i.e., $\mathbf{F}(\omega)$ is odd.
- Time scaling, $f(a\,t) \leftrightarrow \frac{1}{|a|}\mathbf{F}(\frac{\omega}{a})$.
- Frequency scaling, $\frac{1}{|b|} f(\frac{t}{b}) \leftrightarrow \mathbf{F}(b\,\omega)$.
- Time shifting, $f(t - t_0) \leftrightarrow \mathbf{F}(\omega) e^{-j\omega t_0}$.
- Frequency shifting, $f(t) e^{j\omega_0 t} \leftrightarrow \mathbf{F}(\omega - \omega_0)$.
- Convolution theorem: A convolution of two time functions, $f(t)$ and $g(t)$, is defined,

$$g \otimes f = \int_{-\infty}^{\infty} g(t - t') f(t')\, d\,t',$$

the Fourier transform of the convolution is

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} g \otimes f\, e^{-j\omega t}\, d\,t = 2\pi \mathbf{G}(\omega)\mathbf{F}(\omega).$$

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Optical beams with finite trasverse cross sections

- Any physical optical beam is of finite transverse cross section.
- Beams of finite cross section may be described in terms of a superposition of plane waves, analogous to the representation of a time function of finite duration in terms of a Fourier superposition of sinusoids.
- The Fresnel diffraction integral expresses the amplitude distribution of a scalar wave at any cross section of constant, z, in terms of a given distribution at $z = 0$.
- A general plane-wave solution of the scalar wave equation in Cartesian coordinates is of the form,

$$e^{-jk_x x} e^{-jk_y y} e^{-jk_z z},$$

with

$$k_x^2 + k_y^2 + k_z^2 = k^2.$$

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Paraxial wave equation

- If the propagation vector k is inclined by a small angle with respect to the z axis, then the wave vector is *paraxial*, and

$$k_z = \sqrt{k^2 - k_x^2 - k_y^2} \simeq k - \frac{k_x^2 + k_y^2}{2k}.$$

This is the paraxial approximation for the z component of k.

- Let us build up an amplitude distribution $u(x, y, z)$ by superposition of plane waves,

$$u(x, y, z) = \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y U_0(k_x, k_y) e^{-j(k_x x + k_y y)} e^{[j(k_x^2 + k_y^2)/2k]z},$$

$U_0(k_x, k_y)$ is the amplitude of the plane wave solution with particular transverse components of k, k_x, and k_y.

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Fourier decompostions of plane waves

- At $z = 0$, the amplitude distribution $u_0(x, y)$ is,

$$u_0(x, y) = \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y U_0(k_x, k_y) e^{-j(k_x x + k_y y)}.$$

- The wave amplitude function $U_0(k_x, k_y)$ is the Fourier transform of the amplitude distribution at $z = 0$, $u_0(x, y)$.
- Expressing $U_0(k_x, k_y)$ in terms of $u_0(x, y)$ by inverse Fourier transform,

$$U_0(k_x, k_y) = (\frac{1}{2\pi})^2 \int_{-\infty}^{\infty} dx_0 \int_{-\infty}^{\infty} dy_0 u_0(x_0, y_0) e^{j(k_x x_0 + k_y y_0)}.$$

- In the paraxial approximation, one is able to express the solution to the scalar wave equation $u(x, y, z)$ in terms of the know distribution $u_0(x, y)$, at $z = 0$,

$$\begin{aligned} u(x, y, z) \quad &= \quad \int_{-\infty}^{\infty} dx_0 \int_{-\infty}^{\infty} dy_0 u_0(x_0, y_0) \cdot \\ & (\frac{1}{2\pi})^2 \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y e^{-j[k_x(x-x_0)+k_y(y-y_0)]} e^{[j(k_x^2+k_y^2)/2k]z}. \end{aligned}$$

Fresnel diffraction and paraxial wave equation

- This expression is the convolution of $u_0(x,y)$ with the *Fresnel kernel*,

$$h(x,y,z) = (\frac{1}{2\pi})^2 \int_{\infty}^{\infty} dk_x \int_{\infty}^{\infty} dk_y e^{-j(k_x x + k_y y)} e^{[j(k_x^2+k_y^2)/2k]z}$$
$$= \frac{j}{\lambda z} e^{-jk[(x^2+y^2)/2z]}.$$

- Then the *Fresnel diffraction integral* in the paraxial approximation,

$$u(x,y,z) = \frac{j}{\lambda z} \int_{-\infty}^{\infty} dx_0 \int_{-\infty}^{\infty} dy_0 u_0(x_0,y_0) e^{-j(k/2z)[(x-x_0)^2+(y-y_0)^2)]}$$
$$= h \otimes u_0.$$

i.e.

$$\int_{-\infty}^{\infty} e^{-u^2} \mathrm{d}u = \sqrt{\pi},$$

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Near-field region

- The wave of finite transverse extent retains its profile as it propagates in the "*near-field region*" defined by,

$$z \ll \frac{d_x^2}{\lambda}, \qquad z \ll \frac{d_y^2}{\lambda}.$$

- For a initial profile with an inclined phase front, $u_0(x_0, y_0) = f_0(x_0, y_0)e^{-jk_x x_0}$, here $f_0(x_0, y_0)$ is assumed to vary with x_0 much less rapidly than $exp(-jk_x x_0)$.
- Then the amplitude distribution at z is,

$$\begin{aligned} h \otimes u_0 &= \frac{j}{\lambda z}\int_{-\infty}^{\infty} dx_0 \int_{-\infty}^{\infty} dy_0 f_0(x_0, y_0) e^{-jk_x x_0} e^{-j(k/2z)[(x-x_0)^2+(y-y_0)^2)]} \\ &\simeq \sqrt{\frac{j}{\lambda z}}\int_{-\infty}^{\infty} dx_0 f_0(x_0, y) e^{-j(k/2z)[x-(k_x/k)z-x_0)]^2} e^{-jk_x x} e^{j(k/2)(k_x/k)^2 z} \\ &\simeq f_0(x - \frac{k_x}{k}z, y) e^{-jk_x x} e^{j(k/2)(kx/k)^2 z}, \end{aligned}$$

the profile is undistorted but shifts in the transverse direction as it propagates along z.

Fraunhofer diffraction

- Fraunhofer diffraction is the limit of Fresnel diffraction for large distances between the input plane at $z = 0$, at which $u_0(x_0, y_0)$ is specified, and the observation plane at z.

- In this limit, the *far-field limit*, one approximates the argument of the exponential,

$$\frac{k}{z}[(x - x_0)^2 + (y - y_0)^2] \simeq \frac{k}{z}[(x^2 + y^2) - 2xx_0 - 2yy_0],$$

  and ignores the term $k(x_0^2 + y_0^2)/z$.

- Thus the Fraunhofer approximation is valid if the amplitude distribution in the input plane extends over a transverse dimension d such that

$$d \ll \sqrt{\frac{z}{k}}.$$

- In this limit

$$u(x, y, z) = \frac{j}{\lambda z} e^{-j[k(x^2+y^2)/2z]} \int_{-\infty}^{\infty} dx_0 \int_{-\infty}^{\infty} dy_0 u_0(x_0, y_0) exp[\frac{jk}{z}(xx_0 + yy_0)].$$

Fraunhofer diffraction

- The integral over x_0 and y_0 produces the Fourier transform of $u_0(x_0, y_0)$, denoted by U_0,

$$u(x,y,z) = j\frac{(2\pi)^2}{\lambda z} e^{-j[k(x^2+y^2)/2z]} U_0(\frac{kx}{z}, \frac{ky}{z}).$$

- Note that there is a phase factor multiplying the amplitude distribution: $\psi(x,y,z) = u(x,y,z)exp(-jkz)$, where

$$\text{phase factor} = exp\{-j[\frac{k(x^2+y^2)}{2z} + kz]\},$$

- indicating that the phase front is curved, with the equation of constant phase,

$$\frac{k^2(x^2+y^2)}{2\phi} + kz = \phi.$$

- This is the equation of a paraboloid, which has the radius of curvature R at x =0,

$$\frac{1}{R} = \frac{-d^2z/dx^2}{\sqrt{1+(dz/dx)^2}^3} = \frac{k}{\phi} = \frac{1}{z}.$$

Single rectangular slit

- Consider as an example the uniform illumination of a slit at $z = 0$,

$$u_0(x_0, y_0) = \begin{cases} 1, & |x_0| < d_x/2; \quad 0 < |y_0| < d_y/2 \\ 0, & d_x/2 \leq |x_0|; \quad\quad d_y/2 \leq |y_0| \end{cases}$$

  Then

$$u(x, y, z) = \frac{j}{\lambda z} exp[-\frac{jk(x^2 + y^2)}{2z}] d_x d_y \frac{\sin(kd_x x/2z)\sin(kd_y y/2z)}{(kd_x x/2z)(kd_x x/2z)}.$$

- The widths of the diffraction patterns in the x and y directions are characterized by the first null at

$$x = \frac{2\pi z}{kd_x} \qquad \text{and} \qquad y = \frac{2\pi z}{kd_y}.$$

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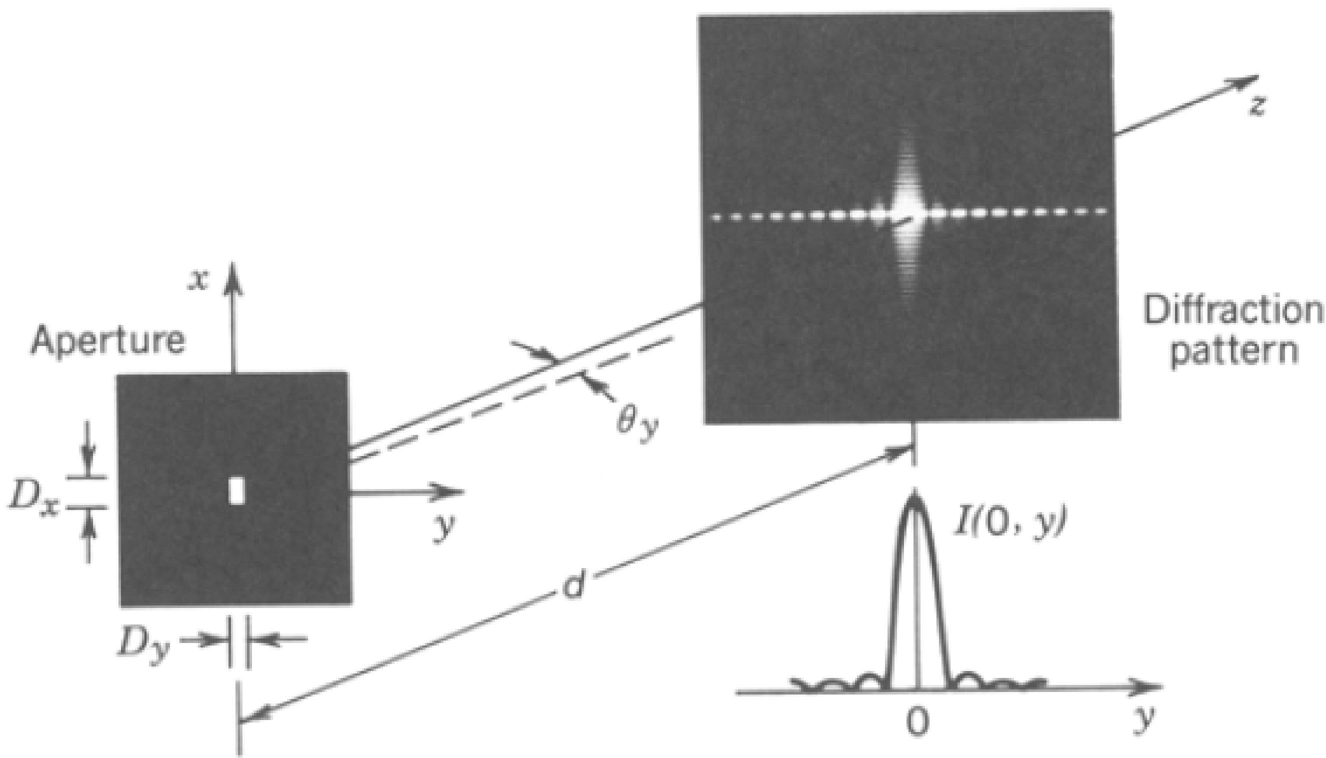
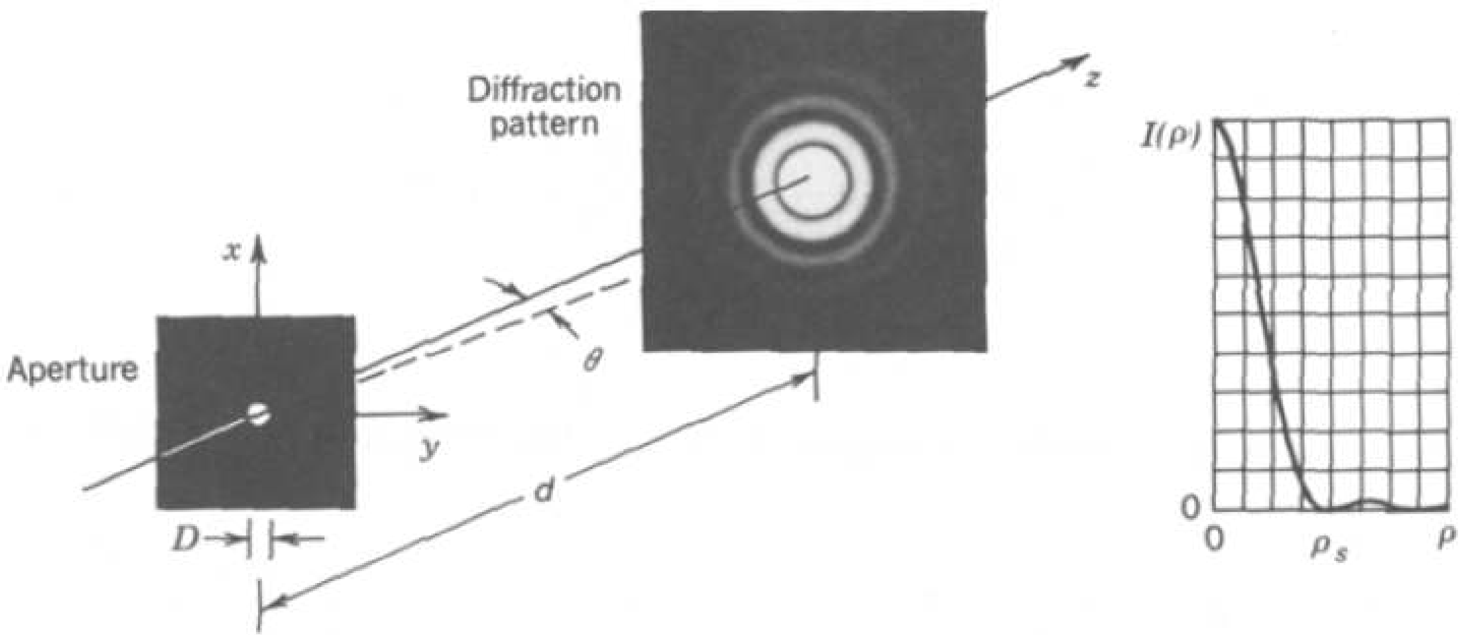
Single-slit

$$I(x,y) = I_0 \text{sinc}^2 \frac{D_x x}{\lambda d} \text{sinc}^2 \frac{D_x x}{\lambda d},$$



Cicular aperture

$$I(x,y) = I_0[\frac{2J_1(\pi D\rho/\lambda d)}{\pi D\rho/\lambda d}]^2, \quad \rho = (x^2+y^2)^{1/2},$$



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The radius of the central **Airy disk**: $\rho_s = 1.22\lambda d/D$, and the angle $\theta = 1.22\lambda/D$,

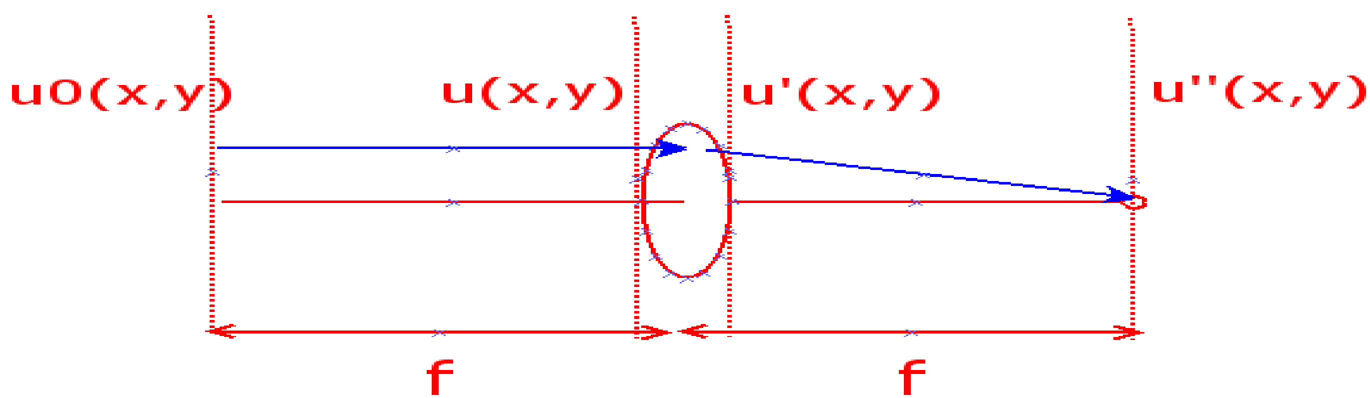
Fourier transformation by a lens

- Consider a general illumination over the input plane, $u_0(x_0, y_0)$. If the lens is in the far-field, then the excitation at the front face reference plane of the lens is ,

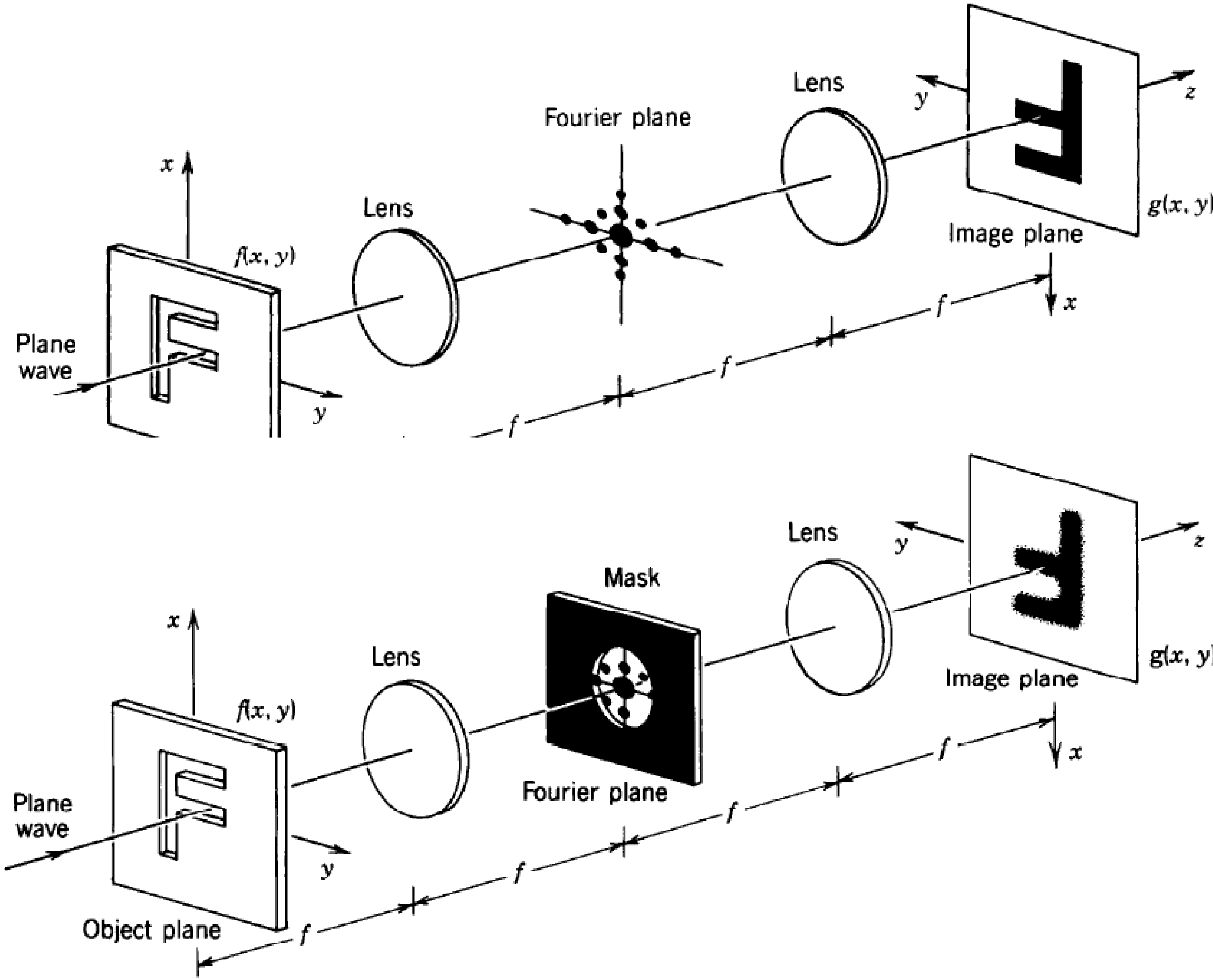
$$u(x,y) = \frac{j(2\pi)^2}{\lambda f} e^{-[jk(x^2+y^2)/2f]} U_0(kx/f, ky/f).$$

- The transmission through the lens removes the exponential factor, then at the output plane,

$$u''(x,y) = \frac{j(2\pi)^2}{\lambda f} U_0(kx/f, ky/f),$$



$4f$-system


Lens
Fourier plane
Lens
x
f(x, y)
Plane wave
y
y
z
g(x, y)
Image plane
x
f
f
f
Lens
Mask
Lens
x
f(x, y)
Plane wave
y
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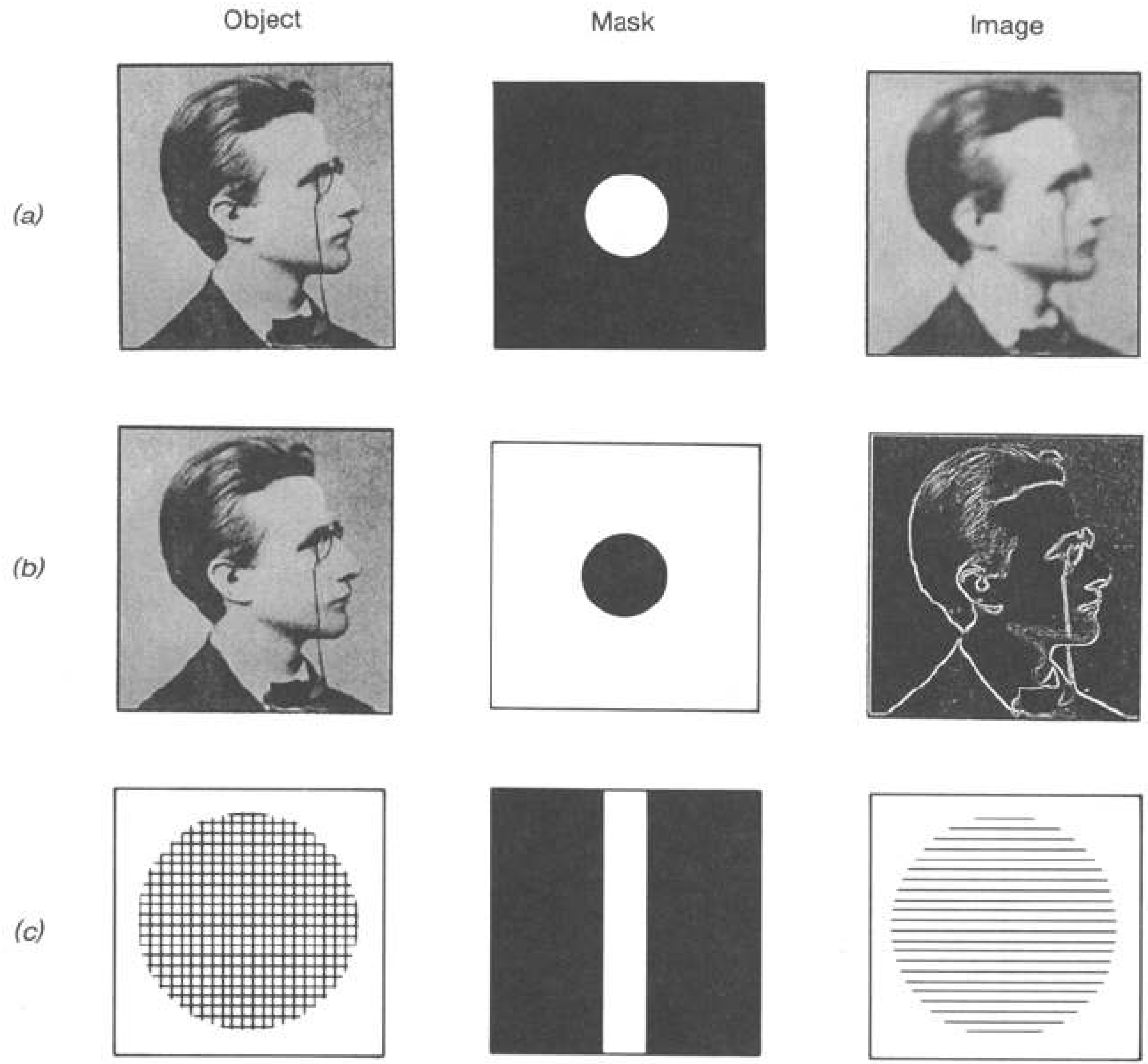
g(x, y)
Image plane
Fourier plane
Object plane
f
f
f
f
x

Spatial filters



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