# Summary of Wave Optics

- light propagates in form of waves
- wave equation in its simplest form is linear, which gives rise to superposition and separation of time and space dependence (interference, diffraction)
- waves are characterized by wavelength and frequency
- propagation through media is characterized by refractive index $n$, which describes the change in phase velocity
- media with refractive index $n$ alter velocity, wavelength and wavenumber but not frequency
- lenses alter the curvature of wavefronts

# Syllabus

1. Introduction to modern photonics (Feb. 26),
2. Ray optics (lens, mirrors, prisms, et al.) (Mar. 7, 12, 14, 19),
3. Wave optics (plane waves and interference) (Mar. 26, 28),
4. Beam optics (Gaussian beam and resonators) (Apr. 9, 11, 16),
5. Electromagnetic optics (reflection and refraction) (Apr. 18, 23, 25),
6. Fourier optics (diffraction and holography) (Apr. 30, May 2),
   Midterm (May 7-th),
7. Crystal optics (birefringence and LCDs) (May 9, 14),
8. Waveguide optics (waveguides and optical fibers) (May 16, 21),
9. Photon optics (light quanta and atoms) (May 23, 28),
10. Laser optics (spontaneous and stimulated emissions) (May 30, June 4),
11. Semiconductor optics (LEDs and LDs) (June 6),
12. Nonlinear optics (June 18),
13. Quantum optics (June 20),
    Final exam (June 27),
14. Semester oral report (July 4),

# Paraxial wave approximation

- paraxial wave = wavefronts normals are paraxial rays

$$U(r) = A(r)\text{exp}(-ikz),$$

- $A(r)$ slowly varying with at a distance of $\lambda$,
- paraxial Helmholtz equation

$$(\nabla^2 + k^2)U(r) = 0,$$
$$\rightarrow (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - 2ik\frac{\partial}{\partial z})A(r) = 0,$$

- solution of the paraxial Helmholtz equation is the $Gaussian$ beams,

國立清華大學
National Tsing Hua University

# Beam optics

- so far we have considered waves with infinite extension, optical beams have a limited spatial extension perpendicular to the direction of propagation,
- the solution of the paraxial Helmholtz equation describing the characteristics of an optical beam is a *"Gaussian function"*,

$$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - 2ik\frac{\partial}{\partial z})A(r) = 0,$$

- properties of Gaussian beams,
- other solutions of the wave equation,
- Experimental relaxation with Gaussian beams,

國立清華大學
National Tsing Hua University

# Fresnel approximation

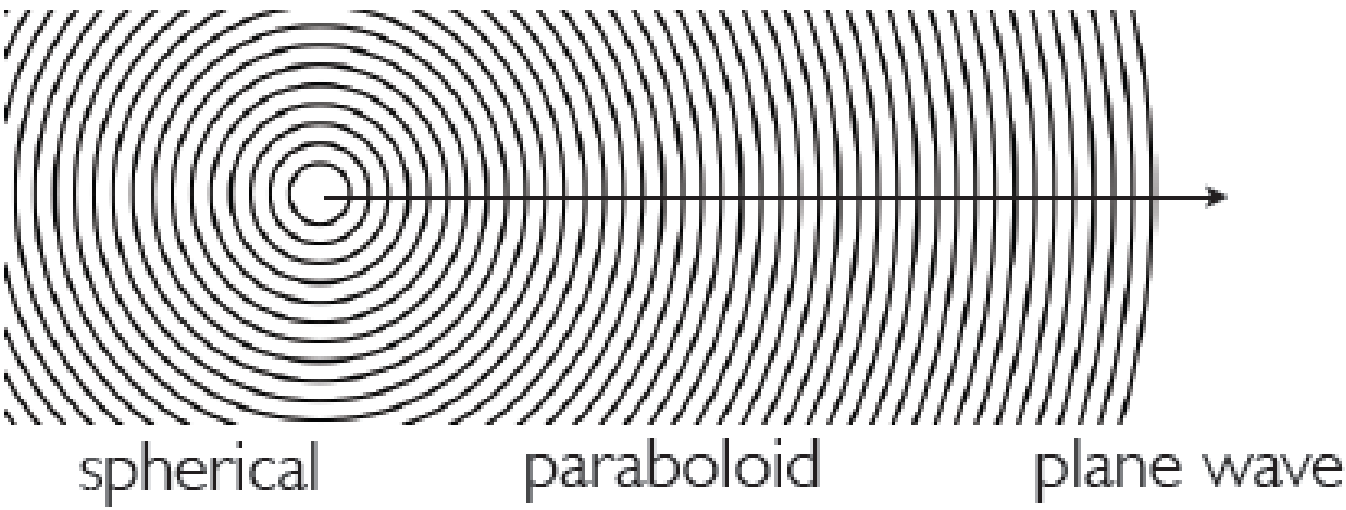
- For paraxial waves, $\sqrt{x^2+y^2} \ll z$,

$$r = \sqrt{x^2+y^2+z^2} \approx z + \frac{x^2+y^2}{2z},$$

- the spherical waves can be approximated by,

$$U(r) = \frac{A}{r}\text{exp}(-i\mathbf{k}\cdot r) \approx \frac{A}{z}\text{exp}(-ikz)\text{exp}(\frac{-ik(x^2+y^2)}{2z}),$$

- for the wavefront, constant phase plane, $\frac{x^2+y^2}{2z}$ is paraboloid,



# Gaussian Beams

- solution for $x, y \ll z$, is the paraboloidal wave, i.e. $U(r) = A(r)\text{exp}(-ikz)$,

$$A(r) = \frac{A_0}{z}\text{exp}[\frac{-ik(x^2+y^2)}{2z}] = \frac{A_0}{z}\text{exp}(\frac{-ik\rho^2}{2z}),$$

- shifted paraboloidal wave,

$$A(r) = \frac{A_1}{q(z)}\text{exp}(\frac{-ik\rho^2}{2q(z)}),$$

where

$$q(z) = z - z' - \zeta = z - z' + iz_0, \qquad z_0 \text{ is the Rayleigh range,}$$

- complex amplitude (general solution),

$$\frac{1}{q(z)} = \frac{1}{R(z)} - i\frac{\lambda}{\pi W^2(z)},$$

國立清華大學
National Tsing Hua University

# Complex amplitude of a Gaussian beam

$$U(r) = A_0 \frac{W_0}{W(z)} \text{exp}[-\frac{\rho^2}{W^2(z)}] \text{exp}[-ikz - ik\frac{\rho^2}{2R(z)} + i\zeta(z)],$$

- beam parameters: $A_0 = \frac{A_1}{iz_0}$,
- Waist:

$$W(z) = W_0[1 + (\frac{z_0}{z})^2]^{1/2},$$

- wavefront curvature:

$$R(z) = z[1 + (\frac{z}{z_0})^2],$$

- phase retardation:

$$\zeta(z) = \tan^{-1}\frac{z}{z_0},$$

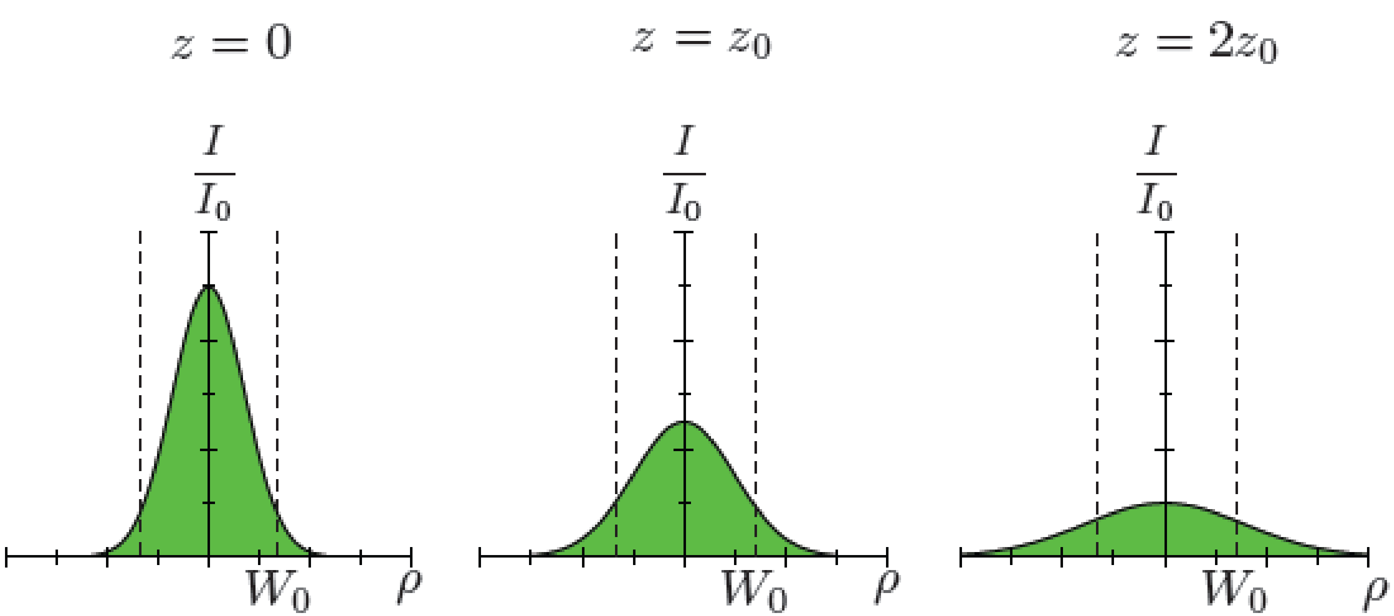
- Waist:

$$W_0 = (\frac{\lambda z_0}{\pi})^{1/2},$$

# Intensity of a Gaussian Beam

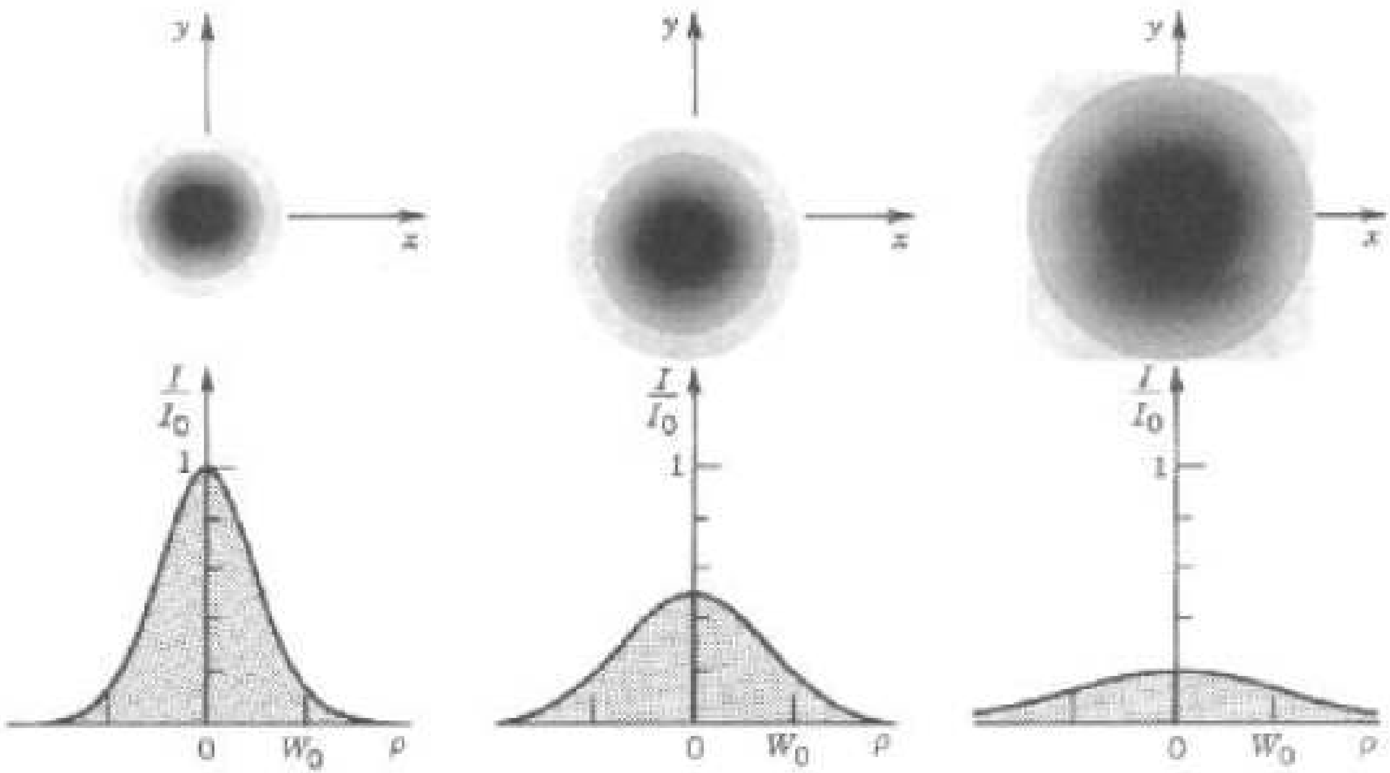
$$I(\rho, z) = I_0 [\frac{W_0}{W(z)}]^2 \text{exp}[-\frac{2\rho^2}{W^2(z)}],$$

- when $z$ is fixed, the intensity profile is perpendicular to $z$,
- when $W(z) =$ constant, $I = I(\rho) \propto \text{exp}(-\frac{2\rho^2}{W^2(z)})$, at each $z$, the intensity is a 2D Gaussian,
- width of the Gaussian, $W(z) = W_0[1 + (\frac{z}{z_0})^2]^{1/2}$,
- Gaussian gets wider with increasing $z$.



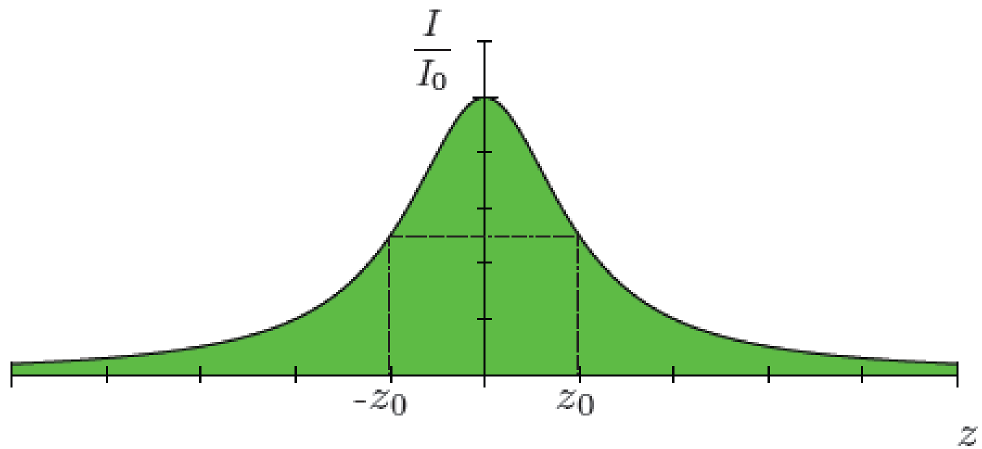

國立清華 National Tsing Hua U

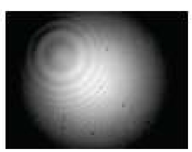
# Measured beam profile

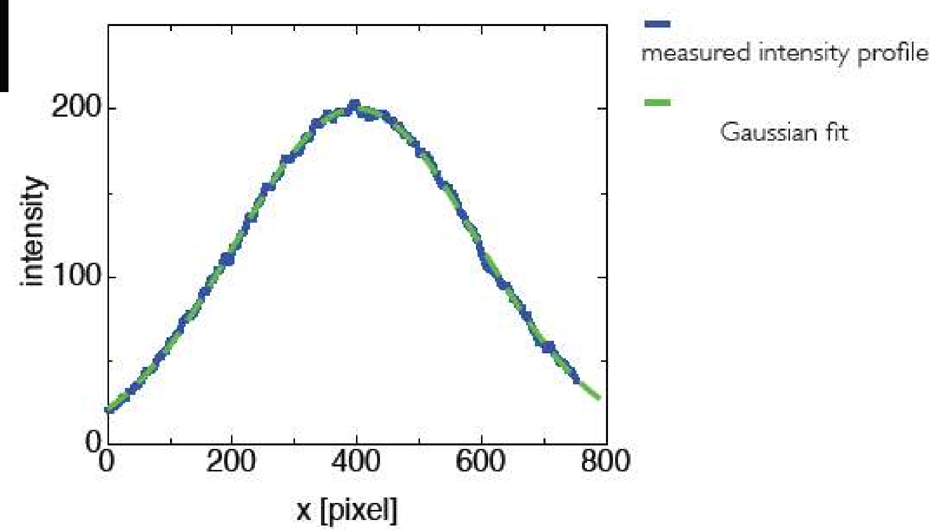


# Intensity at constant $\rho$

for a fixed $\rho$, along $z$, $I(0,z) = I_0[\frac{W_0}{W(z)}]^2 = \frac{I_0}{1+\frac{z}{z_0}^2}$,



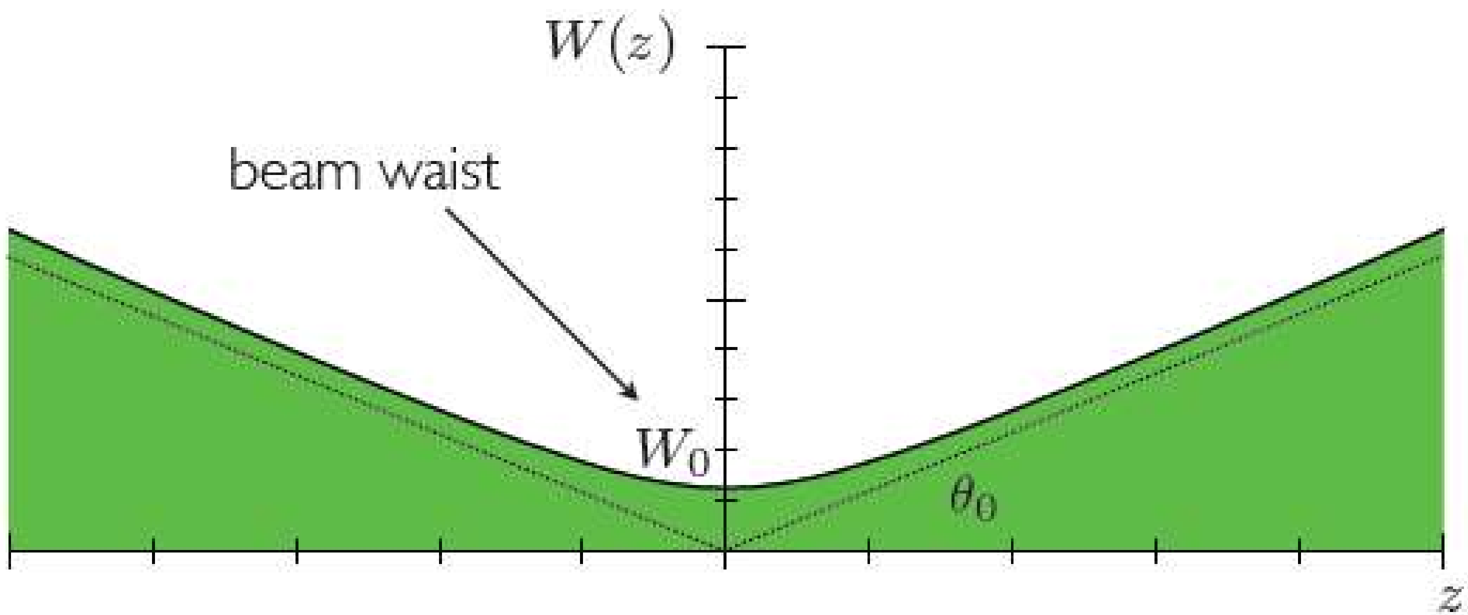


# Beam radius divergence

- Gaussian decayed to $1/e^2$ at $\rho = W(z)$,
- for large $z$, small $\theta_0$,

$$W(z) = W_0[1 + (\frac{z}{z_0})^2]^{1/2} \approx \frac{W_0}{z_0} z = \theta_0 z,$$

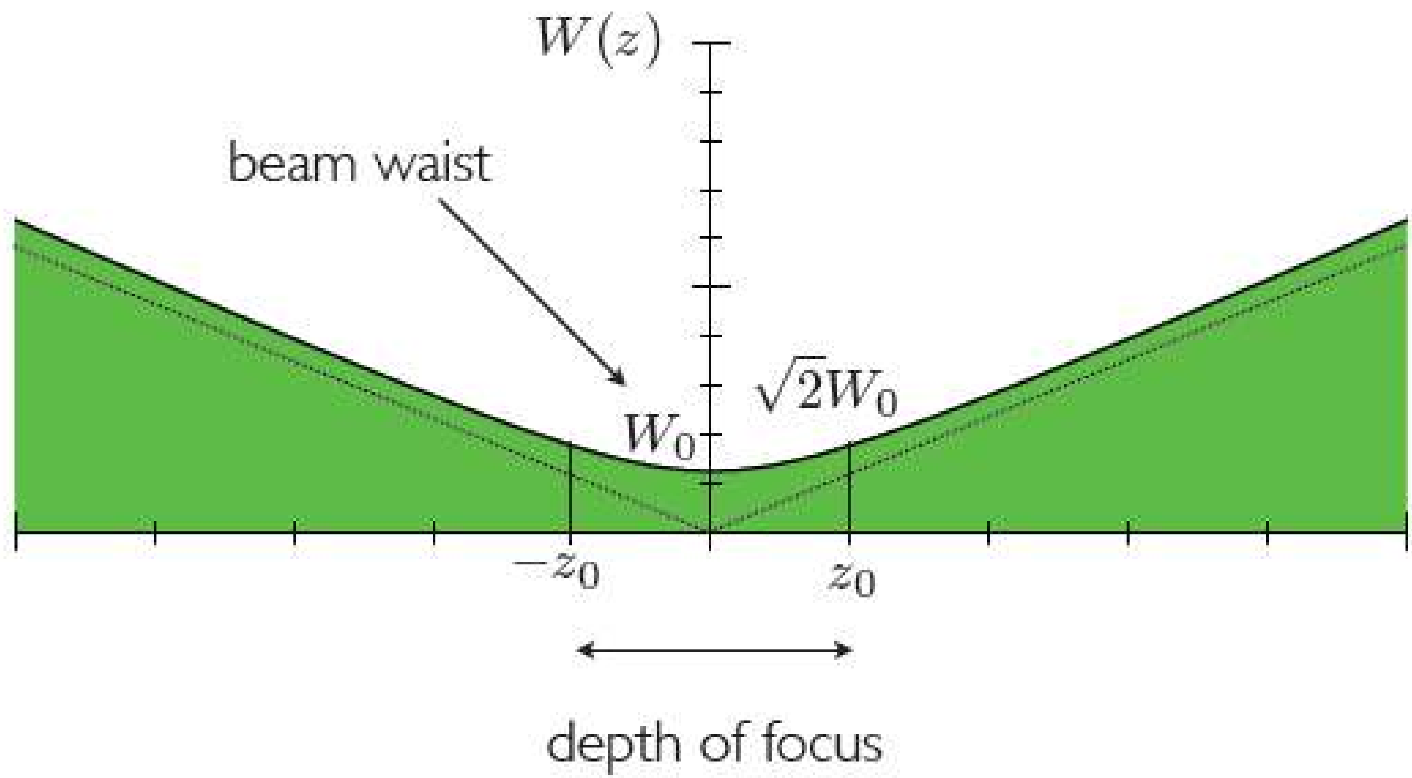
where $\theta_0 = \frac{\lambda}{\pi W_0}$,




國立清華大學
National Tsing Hua University

# Depth of focus

- depth of focus, *DOF* $= 2z_0 = \frac{2\pi W_0^2}{\lambda}$,
- example: for He-Ne Lasers, at $\lambda = 633nm$, *DOF* $= 1km$ for, $W_0 = 2cm$, and *DOF* $= 1mm$ for, $W_0 = 20\mu$,



# Power of a Gaussian beam

- power:

$$P(z) = \int_0^\infty I(\rho, z) 2\pi\rho \mathrm{d}\rho, \\ = \frac{1}{2} I_0(\pi W_0^2),$$

  which is independent of $z$,

- the power of a Gaussian beam equals to
  *maximum intensity* $\times$ *beam area* $\times\ 1/2$,

- write the intensity of a Gaussian beam in terms of its power,

$$I(\rho, z) = \frac{2P}{\pi W^2(z)} \exp[-\frac{2\rho^2}{W^2(z)}],$$

國立清華大學
National Tsing Hua University

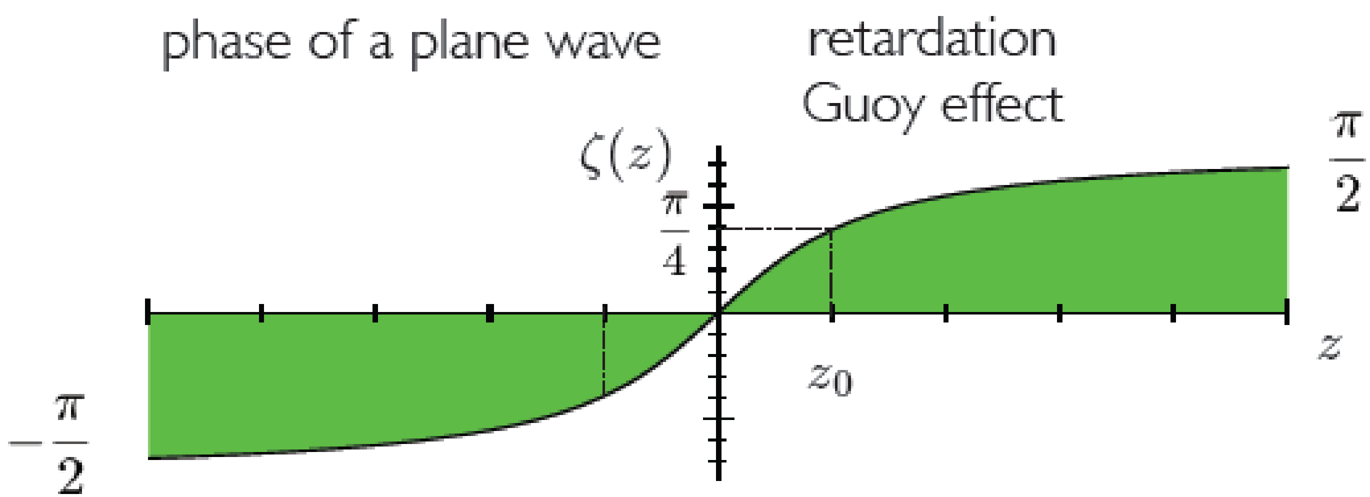
# Phase of a Gaussian beam

$$U(r) = A_0 \frac{W_0}{W(z)} \text{exp}[-\frac{\rho^2}{W^2(z)}] \text{exp}[-ikz - ik\frac{\rho^2}{2R(z)} + i\zeta(z)],$$

- phase of a Gaussian beam,

$$\phi(\rho, z) = kz + k\frac{\rho^2}{2R(z)} - \zeta(z),$$

- at the axis, $\rho = 0$, $\phi(0, z) = kz - \zeta(z)$ and $\zeta(z) = \tan^{-1}\frac{z}{z_0}$,



# Wavefront of a Gaussian beam

- phase of a Gaussian beam,

$$\phi(\rho, z) = kz - \zeta(z) + k\frac{\rho^2}{2R(z)},$$

- the first and second terms, $kz + k\frac{\rho^2}{2R(z)}$ depend only on $z$,
- the third term depends on $x, y, z$ defines wavefront bending,
- constant phase plane,

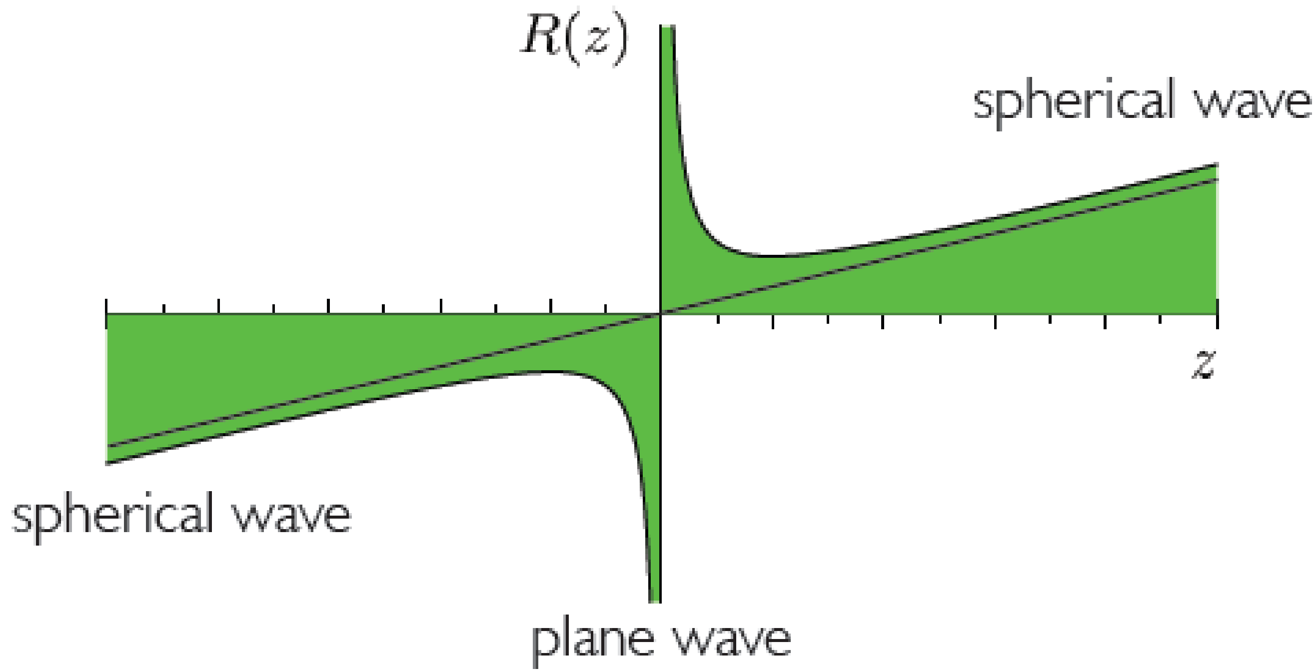
$$k(z + \frac{\rho^2}{2R(z)}) - \zeta(z) = 2m\pi, \qquad m \in N,$$

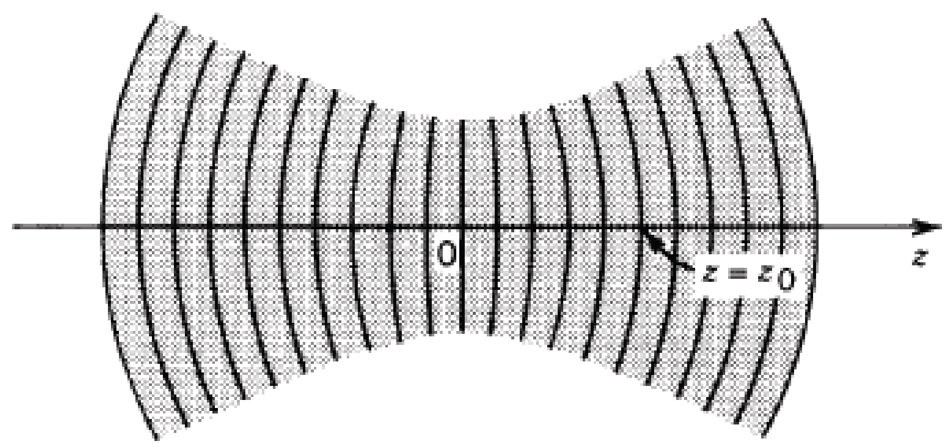
- *paraboloid* constant phase plane,

$$z = -\frac{\rho^2}{2R(z)} + m\lambda + \zeta(z)\frac{\lambda}{2\pi},$$

國立清華大學
National Tsing Hua University

# Wavefront of a Gaussian beam

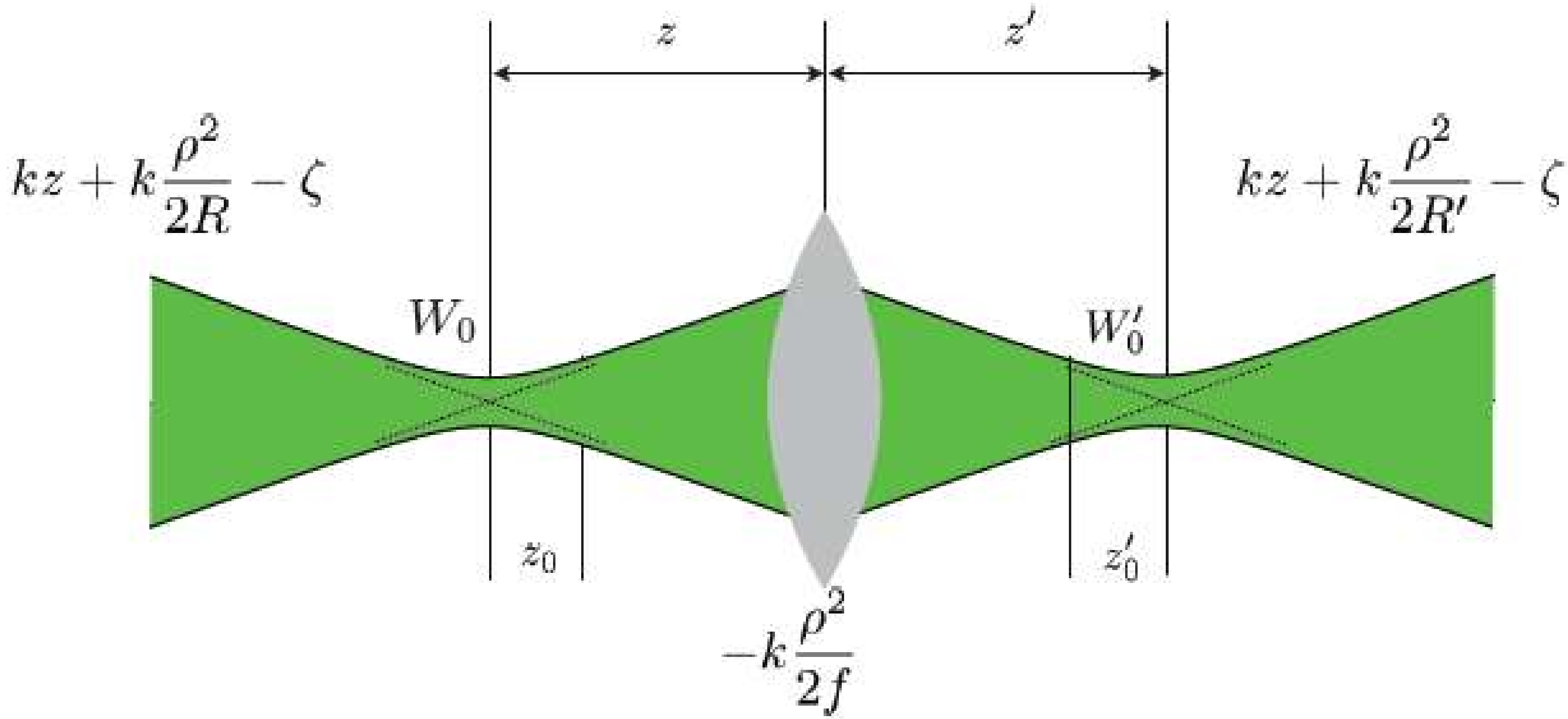



National Tsing Hua

# Transmission through a lens

$$\frac{1}{R'} = \frac{1}{R} - \frac{1}{f},$$



# Transmission through a lens

- waist radius:

$$W_0' = M W_0,$$

- waist location:

$$(z' - f) = M^2 (z - f),$$

- depth of focus:

$$2{z'}_0^2 = M^2 (2 z_0),$$

- divergence:

$$2{\theta'}_0 = \frac{2\theta_0}{M},$$

- magnification:

$$M = \frac{M_r}{\sqrt{1 + r^2}}, \qquad M_r = \left|\frac{f}{z - f}\right|, \qquad \text{and} \qquad r = \frac{z_0}{z - f},$$

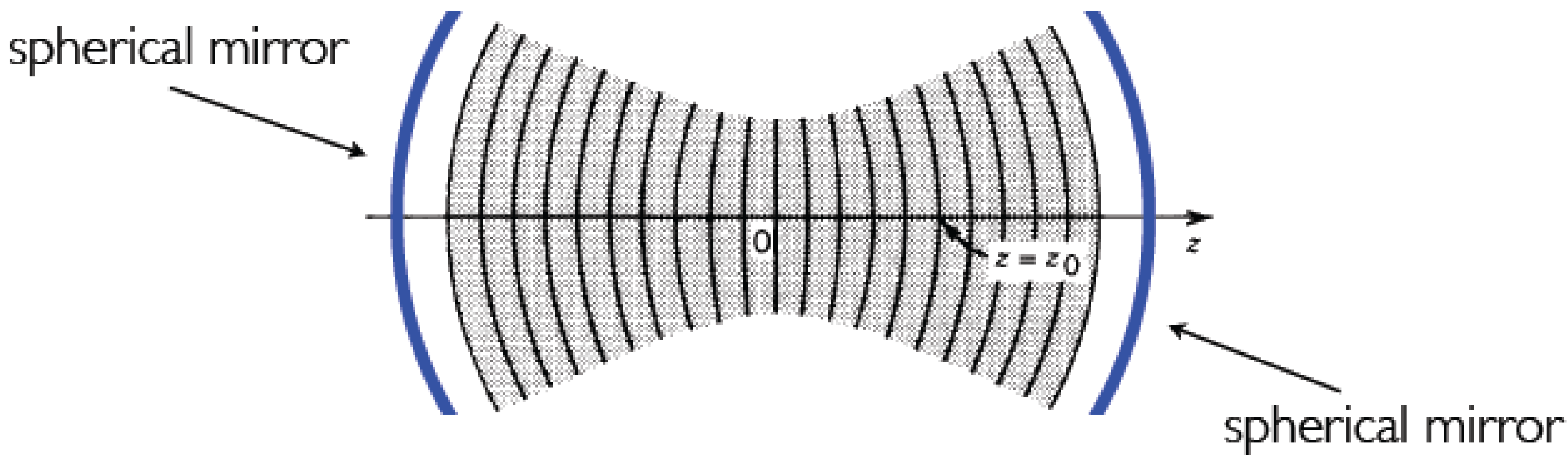
- limit of ray optics:

$$(z - f) \gg z_0, \qquad \frac{1}{z'} + \frac{1}{z} \approx \frac{1}{f},$$

國立清華大學
National Tsing Hua University

# Hermite Gaussian beams

- there are other solutions to the paraxial Helmholtz equation,
- main interest in solutions with paraboloidal wavefronts of special interest for resonators with spherical mirrors,




- paraboloidal wavefronts are unaltered by spherical mirror,

# Hermite Gaussian beams

- Hermite Gaussian beams,

$$U_{l,m}(x,y,z) = A_{l,m}[\frac{W_0}{W(z)}]G_l(\frac{\sqrt{2}x}{W(z)})G_m(\frac{\sqrt{2}y}{W(z)}) \times \text{exp}[-ikz - ik\frac{x^2+y^2}{2R(z)} + i(l+m+1)\zeta(z)],$$

- remember the Harmonic Oscillator in Quantum Mechanics,
- Hermite Gaussian Function,

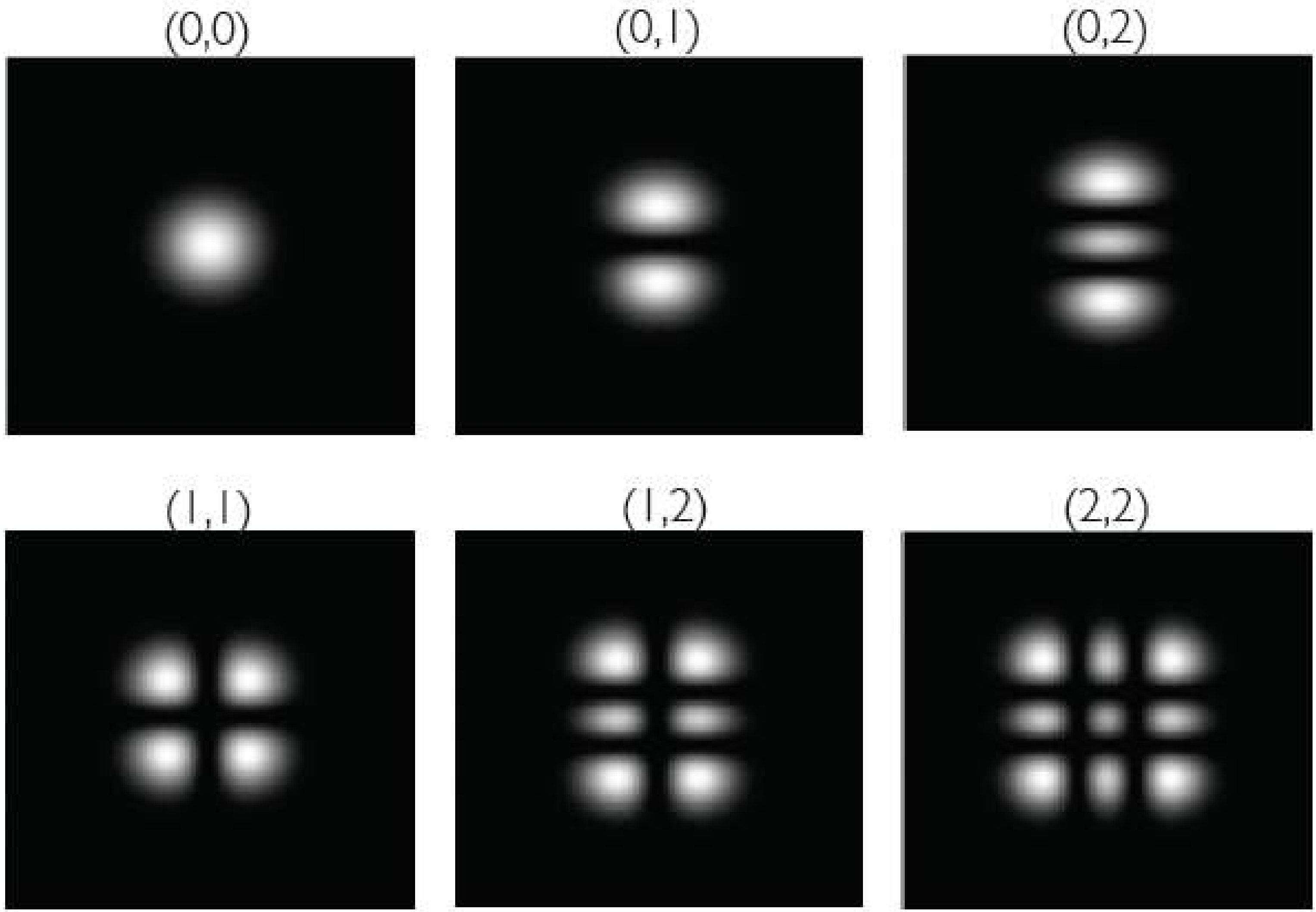
$$G_l(u) = H_l(u)\text{exp}(-\frac{-u^2}{2}), \qquad l = 0,1,2,\ldots,$$

- Hermite Polynomials,

$$H_l(u) = 2uH_l(u) - 2lH_{l-1}(u), \qquad l = 0,1,2,\ldots,$$

國立清華大學
National Tsing Hua University

# Hermite Gaussian beams - Intensity distributions

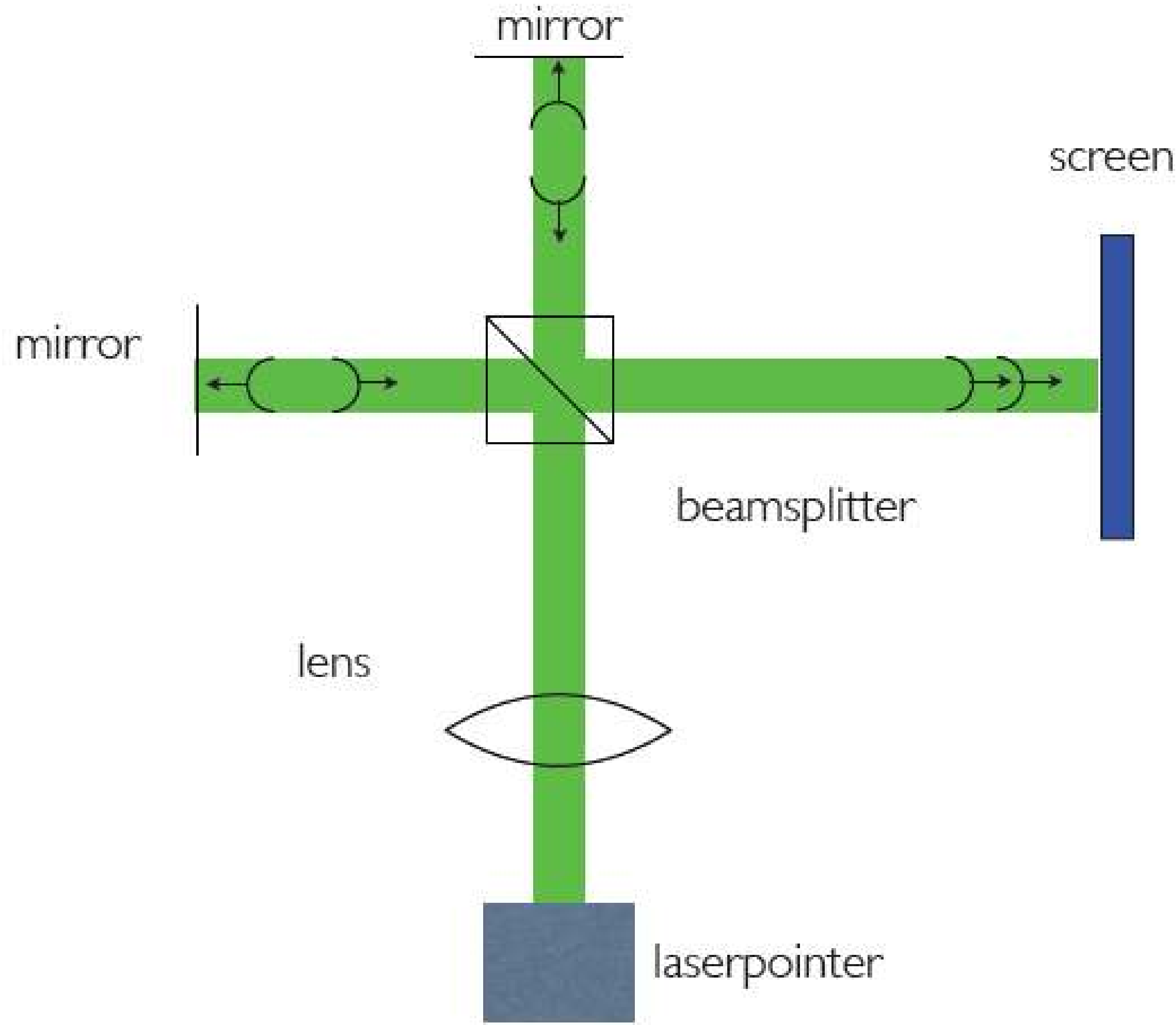



國立清華大學
National Tsing Hua University

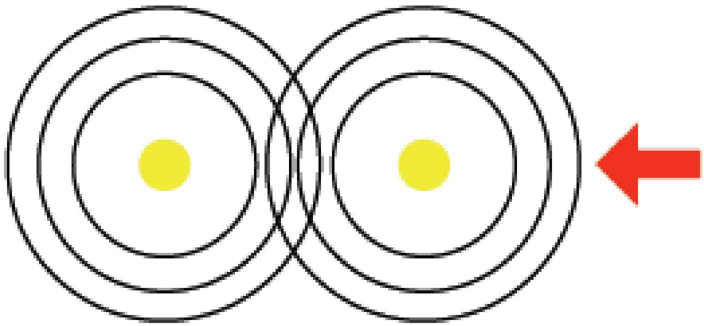
# Laguerre-Gaussian Beams

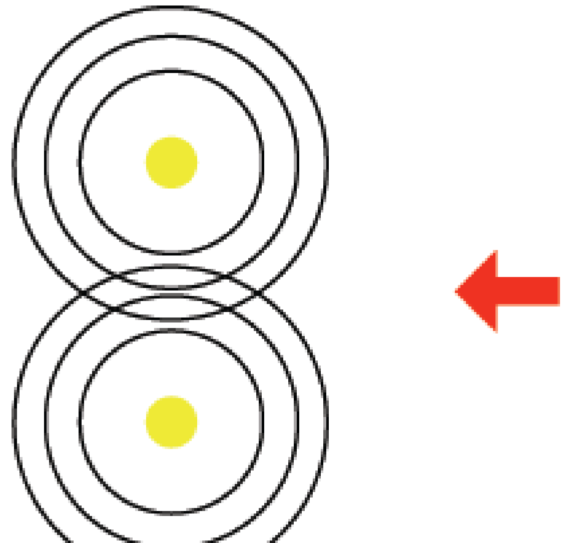
- for cylinder coordinates, the solutions of the paraxial Helmholtz equation are *Laguerre-Gaussian Beams*
- they have cylinder symmetry composed by Bessel functions,

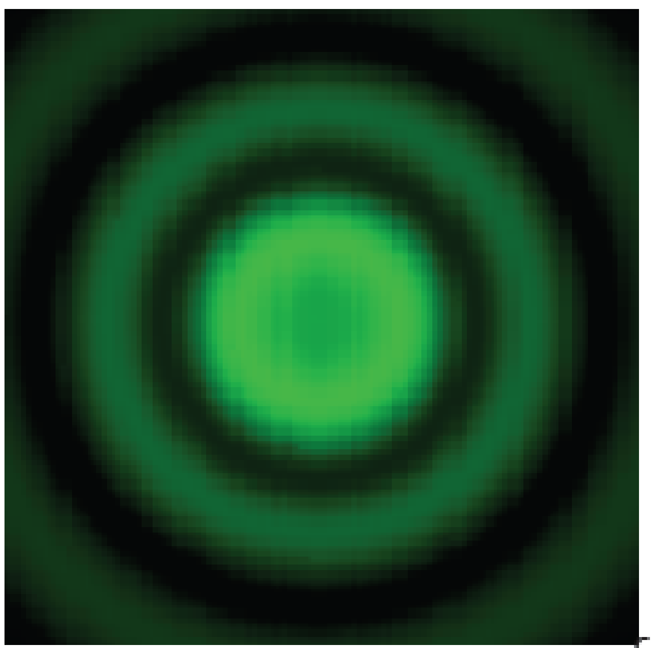
# Experimental relaxation with Gaussian beams

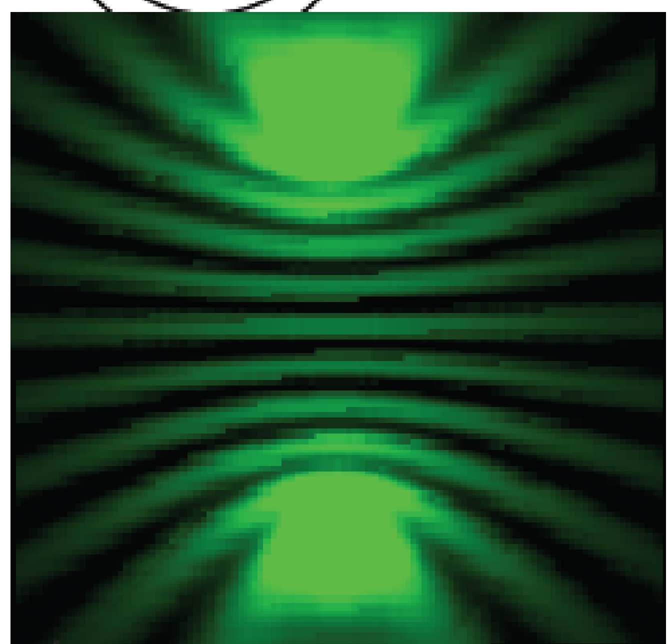


# Interference of two spherical waves









www.falstad.com

# Summary of Beam Optics

- Gaussian beams,
- waves with limited spatial extension perpendicular to propagation direction,
- Gaussian beam is solution of paraxial Helmholtz equation,
- Gaussian beam has parabolic wavefronts, (as seen in lab experiment),
- Gaussian beams characterized by focus waist and focus depth,