2, Quantum theory of Radiation

1. Stimulated and Spontaneous Emission
2. Macroscopic theory of absorption
3. Microscopic theory of absorption
4. The Laser
5. Lamb shift
6. Quantum beats

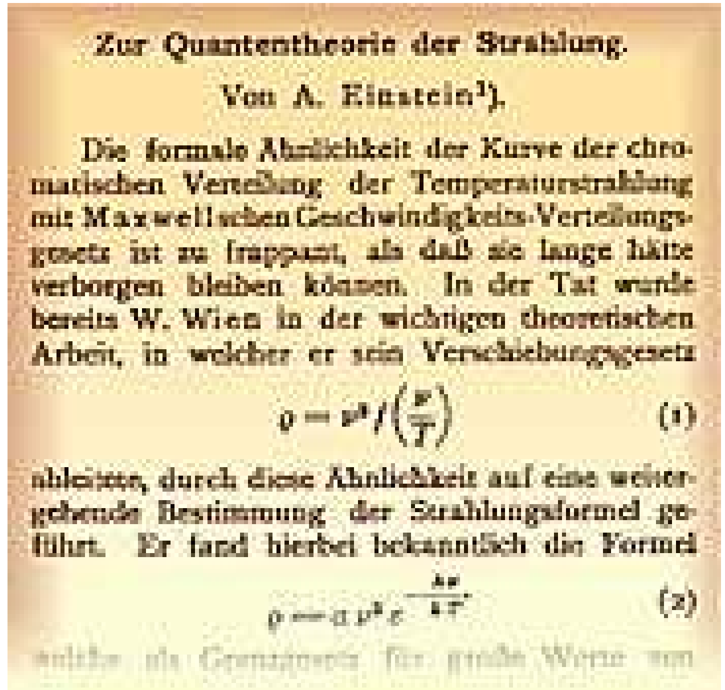
Ref:

Ch. 1 in *"Quantum Optics,"* by M. Scully and M. Zubairy.

Ch. 1 in *"The Quantum Theory of Light,"* by R. Loudon.

Ch. 2 in *"Modern Foundations of Quantum Optics,"* by V. Vedral.

Einstein on Radiation



Zur Quantentheorie der Strahlung.

Von A. Einstein[1]).

Die formale Ähnlichkeit der Kurve der chromatischen Verteilung der Temperaturstrahlung mit Maxwellschen Geschwindigkeits-Verteilungsgesetz ist zu frappant, als daß sie lange hätte verborgen bleiben können. In der Tat wurde bereits W. Wien in der wichtigen theoretischen Arbeit, in welcher er sein Verschiebungsgesetz

$$\rho = \nu^3 f\left(\frac{\nu}{T}\right) \quad (1)$$

ableitete, durch diese Ähnlichkeit auf eine weitergehende Bestimmung der Strahlungsformel geführt. Er fand hierbei bekanntlich die Formel

$$\rho = \alpha \nu^3 e^{-\frac{h\nu}{kT}} \quad (2)$$

welche als Grenzgesetz für große Werte von

"On the Quantum Theory of Radiation"

$$D(\omega) = \frac{A/B}{e^{\hbar\omega/k_B T} - 1}$$

$$\frac{A}{B} = \frac{\hbar\omega^3}{\pi^2 c^3}$$

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A. Einstein, *Phys. Z.* **18**, 121 (1917).

D. Kleppner, "Rereading Einstein on Radiation," *Physics Today* **58**, 30 (Feb. 2005).

Quantization of the Electromagnetic Field

- Like simple harmonic oscillator, $\hat{H} = \frac{p^2}{2m} + \frac{1}{2}kx^2$, where $[\hat{x}, \hat{p}] = i\hbar$,
- For EM field, $\hat{H} = \frac{1}{2}\sum_j [m_j \omega_m^2 q_j^2 + \frac{p_j^2}{m_j}]$, , where $[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}$,
- the Hamiltonian for EM fields becomes: $\hat{H} = \sum_j \hbar\omega_j (\hat{a}_j^\dagger \hat{a}_j + \frac{1}{2})$,
- the electric and magnetic fields become,

$$\begin{aligned} \hat{E}_x(z,t) &= \sum_j (\frac{\hbar\omega_j}{\epsilon_0 V})^{1/2} [\hat{a}_j e^{-i\omega_j t} + \hat{a}_j^\dagger e^{i\omega_j t}] \sin(k_j z), \\ \hat{H}_y(z,t) &= -i\epsilon_0 c \sum_j (\frac{\hbar\omega_j}{\epsilon_0 V})^{1/2} [\hat{a}_j e^{-i\omega_j t} - \hat{a}_j^\dagger e^{i\omega_j t}] \cos(k_j z), \end{aligned}$$

- energy level for quantized field, $E_n = (n + \frac{1}{2})\hbar\omega$.

Planck's Law

- In the thermal equilibrium at temperature *T*, the probability P_n that the mode oscillator is thermally excited to the n-th excited state is given by the *Boltzman factor*,

$$P_n = \frac{\exp[-E_n/k_B T]}{\sum_n \exp[-E_n/k_B t]},$$

- the mean number $\bar{n}$ of photons is,

$$\bar{n} = \sum_n n P_n = \frac{U}{1-U} = \frac{1}{\exp(\hbar\omega/k_B T) - 1},$$

where $U \equiv \exp(-\hbar\omega/k_B T)$ and $\sum_{n=0}^{\infty} U^n = 1/(1-U)$.

- energy density of the radiation:

$$\begin{aligned} D(\omega)\mathrm{d}\omega &= \bar{n}\hbar\omega \mathrm{d}\omega = \bar{n}\hbar\omega\rho_\omega \mathrm{d}\omega, \\ &= \bar{n}\hbar\omega^3 \mathrm{d}\omega/\pi^2 c^3 = \frac{\hbar\omega^3}{\pi^2 c^3}\frac{\mathrm{d}\omega}{\exp[\hbar\omega/k_B T] - 1}. \end{aligned}$$

- total electromagnetic energy density: $\int_0^\infty D(\omega)\mathrm{d}\omega = 1/2V \int_{\text{cavity}} \epsilon_0 |\mathbf{E}(r,t)|^2 \mathrm{d}V$.

Fluctuations in Photon Number

- the ergodic theorem of statistical mechanics: time averages are equivalent to averages taken over a large number of exactly similar systems, each maintained in a fixed state (ensemble).
- the probability of finding $\bar{n}$ photons,

$$P_n = \frac{\text{exp}[-E_n/k_B T]}{\sum_n \text{exp}[-E_n/k_B t]} = (1-U)U^n = \frac{\bar{n}^n}{(1+\bar{n})^{1+n}},$$

  which is a thermal distribution or the geometric distribution (Poisson distribution).

- the root-mean-square deviation:

$$\Delta n^2 = \sum_n (n - \bar{n}^2) P_n = \bar{n}^2 + \bar{n} \approx \bar{n} + \frac{1}{2}, \quad \text{for} \quad \bar{n} \gg 1.$$

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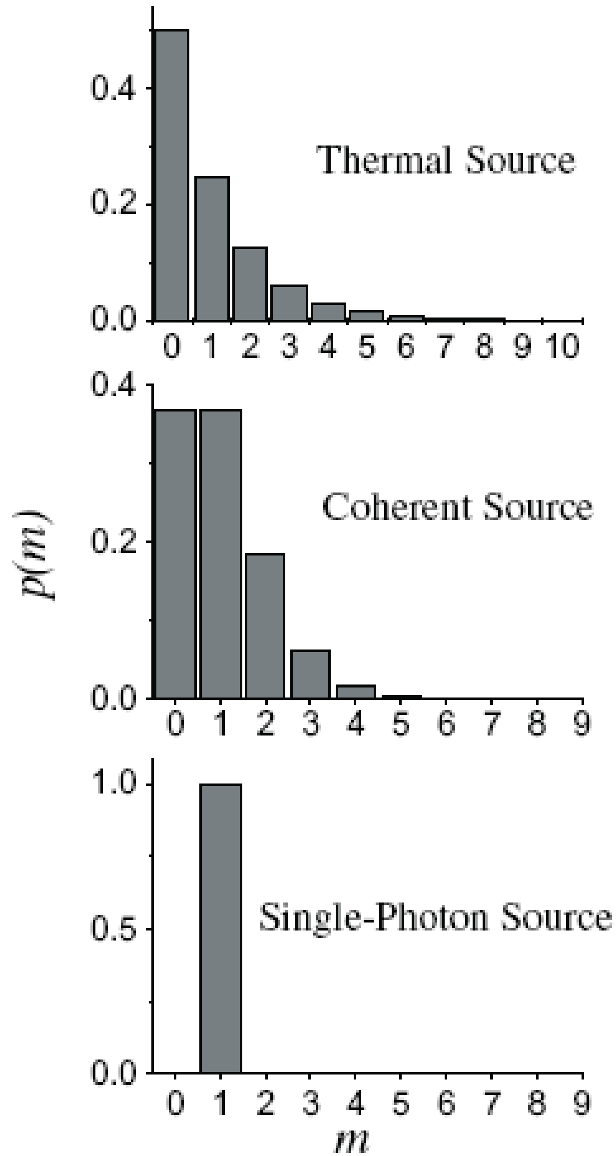
Probability distribution for $\bar{n} = 1$



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Einstein's A and B coefficients

- For a two-level atom, the rates of changes of N_1 and N_2 are,

$$\frac{\mathrm{d}N_1}{\mathrm{d}t} = -\frac{\mathrm{d}N_2}{\mathrm{d}t} = N_2 A_{21} - N_1 B_{12} D(\omega) + N_2 B_{21} D(\omega),$$

- A_{21} is the probability of photon in state 2 spontaneously fall into the lower state 1, i.e. spontaneous emission;
- B_{12} is the probability of photon absorption in state 1 into state 2, i.e. absorption;
- B_{21} is the probability of photon emission from state 2 into state 1, i.e. stimulated emission;
- in thermal equilibrium, $\frac{\mathrm{d}N_1}{\mathrm{d}t} = -\frac{\mathrm{d}N_2}{\mathrm{d}t} = 0$,

$$D(\omega) = \frac{A_{21}}{(N_1/N_2)B_{12} - B_{21}},$$

where the populations N_1 and N_2 are related by Boltzmann's law,

$$N_1/N_2 = (g_1/g_2)\exp[\hbar\omega/k_B T],$$

Einstein's A and B coefficients

- the density distribution of EM fields in a two-level atom,

$$D(\omega) = \frac{A_{21}}{(g_1/g_2)\text{exp}[\hbar\omega/k_B T]B_{12} - B21},$$

where g_1 and g_2 are the level degenerate parameters.

- compare it in free space,

$$D(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3}\frac{1}{\text{exp}[\hbar\omega/k_B T] - 1},$$

- at all temperatures T, we have

$$\begin{aligned}(g_1/g_2)B_{12} &= B_{21},\\ (\hbar\omega^3/\pi^2 c^3)B_{21} &= A_{21},\end{aligned}$$

- the consistency between the Einstein theory and Planck's law could not have been achieved without the introduction of the *stimulated emission* process.

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Einstein's A and B coefficients

- for nondegenerate two-level atom, $g_1 = g_2 = 1$ and $N_1 + N_2 = N$,

$$\frac{\mathrm{d}N_1}{\mathrm{d}t} = -\frac{\mathrm{d}N_2}{\mathrm{d}t} = N_2 A + (N_2 - N_1) B D(\omega),$$

- the solution for N_1 is,

$$N_1 = [N_1^0 - \frac{N(A + BD(\omega))}{A + 2BD(\omega)}]\mathrm{exp}[-(A + 2BD(\omega))t] + \frac{N[A + BD(\omega)]}{A + 2BD(\omega)}$$

where N_1^0 is the initial value of N_1 at $t = 0$,

- if $N_2^0 = 0$, all atoms are in the ground state at $t = 0$,

$$N_2 = \frac{NBD(\omega)}{A + 2BD(\omega)}[1 - \mathrm{exp}[-(A + 2BD(\omega))t],$$

- in the steady-state,

$$N_2 = \frac{NBD(\omega)}{A + 2BD(\omega)} \approx 0.5, \quad \text{if} \quad BD(\omega) \gg A,$$

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Macroscopic theory of Absorption

- for the excited state,

$$\frac{\mathrm{d}N_2}{\mathrm{d}t} = -N_2 A,$$

  with the solution $N_2 = N_2^0 \exp[-At]$, where $A \equiv 1/\tau_R$ the radiative lifetime of the excited states.

- in macroscopic, the polarization **P** by an applied electric field **E** is related with $\mathbf{P} = \epsilon_0 \chi \mathbf{E}$, where the susceptibility $\chi = \chi_1 + i\chi_2$,

- the relation between frequency and the wavevector, $kc/\omega = 1 + \chi = \eta + i\kappa$, where $\eta^2 - \kappa^2 = 1 + \chi_1$ and $2\eta\kappa = \chi_2$,

- the traveling-wave solution propagated in the $z-$direction becomes,

$$\exp[i(kz - \omega t)] = \exp[i\omega(\frac{\eta z}{c} - t) - \frac{\omega \kappa z}{c}],$$

- the averaged Poynting vector, $\bar{I} = \langle \mathbf{E} \times \mathbf{B}/\mu_0 \rangle = \frac{1}{2}\epsilon_0 c \eta |\mathbf{E}(r,t)|^2$, where

$$\bar{I}(z) = \bar{I}_0 \exp[-2\omega\kappa z/c],$$

  where $2\omega\kappa/c$ is called the absorption coefficient.

Microscopic theory of Absorption

- total electromagnetic energy density: $\int_0^\infty D(\omega)\mathrm{d}\omega = 1/2V \int_{\text{cavity}} \epsilon_0 |\mathbf{E}(r,t)|^2 \mathrm{d}V$.
- for a lossy dielectric medium, $\int_0^\infty D(\omega)\mathrm{d}\omega = 1/2V \int_{\text{cavity}} \epsilon_0 \eta^2 |\mathbf{E}(r,t)|^2 \mathrm{d}V$.
- in steady-state condition, $-\frac{\mathrm{d}N_2}{\mathrm{d}t} = N_2 A + (N_2 - N_1) B D(\omega)/\eta^2 = 0$, with an additional factor η^2 for the energy density,
- the attenuation energy within a small section of dz, cross-section A is,

$$\frac{\partial}{\partial t} D(\omega)\mathrm{d}\omega A\mathrm{d}z = -(N_1 - N_2) F(\omega)\mathrm{d}\omega B D(\omega)/\eta^2 \hbar\omega (A\mathrm{d}z/V),$$

- for the absorption, $-\frac{\partial}{\partial t} D(\omega)\mathrm{d}\omega A\mathrm{d}z = -\frac{\partial}{\partial z}\bar{I}\mathrm{d}\omega A\mathrm{d}z$, or $\frac{\partial}{\partial t} D(\omega) = \frac{\partial}{\partial z}\bar{I}$,
- for $\bar{I} = \frac{1}{2}\epsilon_0 c\eta |\mathbf{E}(r,t)|^2$, we have $cD(\omega) = \eta\bar{I}$, then,

$$\frac{\partial}{\partial z}\bar{I} = -(N_1 - N_2) F(\omega) (B\hbar\omega/Vc\eta)\hbar I,$$

where $F(\omega)$ is the distribution of atomic transition frequencies.

Microscopic theory of Absorption

- if $N_2^0 = 0$, all atoms are in the ground state at $t = 0$,

$$N_2 = \frac{NBD(\omega)}{A + 2BD(\omega)/\eta^2}[1 - \text{exp}[-(A + 2BD(\omega))t] \approx \frac{NBD(\omega)}{A + 2BD(\omega)/\eta^2},$$

and we have,

$$N_1 - N_2 = \frac{NA}{A + 2BD(\omega)/\eta^2} = \frac{NA}{A + 2B\bar{I}/c\eta},$$

- the equation for the average beam intensity becomes,

$$\frac{1}{\bar{I}}(1 + \frac{2B\bar{I}}{Ac\eta})\frac{\partial}{\partial z}\bar{I} = -\frac{NB\hbar\omega F(\omega)}{Vc\eta}$$

- for all ordinary light beams, $\frac{2B\bar{I}}{Ac\eta} \ll 1$, then we have,

$$\bar{I}(z) = \bar{I}_0\text{exp}[-NB\hbar\omega F(\omega)z/Vc\eta],$$
$$= \bar{I}_0\text{exp}[-Kz],$$

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where the absorption coefficient, $K = 2\omega\kappa/c$.

Microscopic theory of Absorption

- A dielectric with one single resonance may be modeled as a distribution of "$+$" and "$-$" charges, the $+$ charges immobile and the $-$ charges tied to the $+$ charges by a spring constant k,

$$m(\frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2)\mathbf{d} = -\frac{e}{m}\mathbf{E},$$

- for the incident field $\mathbf{E} = E_0 \text{exp}[-i(\omega t - kz)]$ and the dipole $\mathbf{d} = a\text{exp}[-i(\omega t - kz)$, we have

$$a = \frac{-(e/m)E_0}{\omega^2 - \omega_0^2 + 2i\beta\omega},$$

- the polarization

$$\mathbf{P} = Np = N\sum_j e\mathbf{d}_j = N\alpha(\omega)E_0 e^{-i(\omega t - kz)},$$

where $\alpha(\omega) = \frac{-e^2/m}{\omega^2 - \omega_0^2 + 2i\beta\omega}$.

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Microscopic theory of Absorption

- the dispersion relation,

$$k^2 = \frac{\omega^2}{c^2}[1 + \frac{N\alpha(\omega)}{\epsilon_0}] = \frac{\omega^2}{c^2} n^2(\omega^2),$$

- the real index of refraction,

$$n_R(\omega) = 1 + \frac{Ne^2}{m\epsilon_0} \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2},$$

- the absorption coefficient or extinction coefficient,

$$a(\omega) = 2n_I(\omega)\omega/c = \frac{2Ne^2}{m\epsilon_0 c} \frac{\beta\omega^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2},$$

which has the lineshape of the *Lorentzian* function,

$$a(\omega) = \frac{Ne^2}{2m\epsilon_0 c} \frac{\delta\omega_0}{(\omega_0 - \omega)^2 + \delta\omega_0^2},$$

where $\delta\omega = \beta$

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Population Inversion: the Laser

- for a three level atom, $N_1 + N_2 + N_3 = N$, the rate equations are:

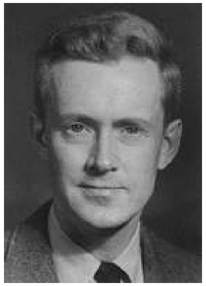
$$\begin{aligned}
\frac{\mathrm{d}N_2}{\mathrm{d}t} &= -N_2 A_{21} - N_2 A_{23} + D_p B_{23}(N_3 - N_2) - D(\omega) B_{21}(N_2 - N_1), \\
\frac{\mathrm{d}N_1}{\mathrm{d}t} &= N_2 A_{21} - N_1 A_{13} + D(\omega) B_{21}(N_2 - N_1), \\
\frac{\mathrm{d}N_3}{\mathrm{d}t} &= -N_2 A_{23} + N_1 A_{13} - D_p B_{23}(N_3 - N_2),
\end{aligned}$$

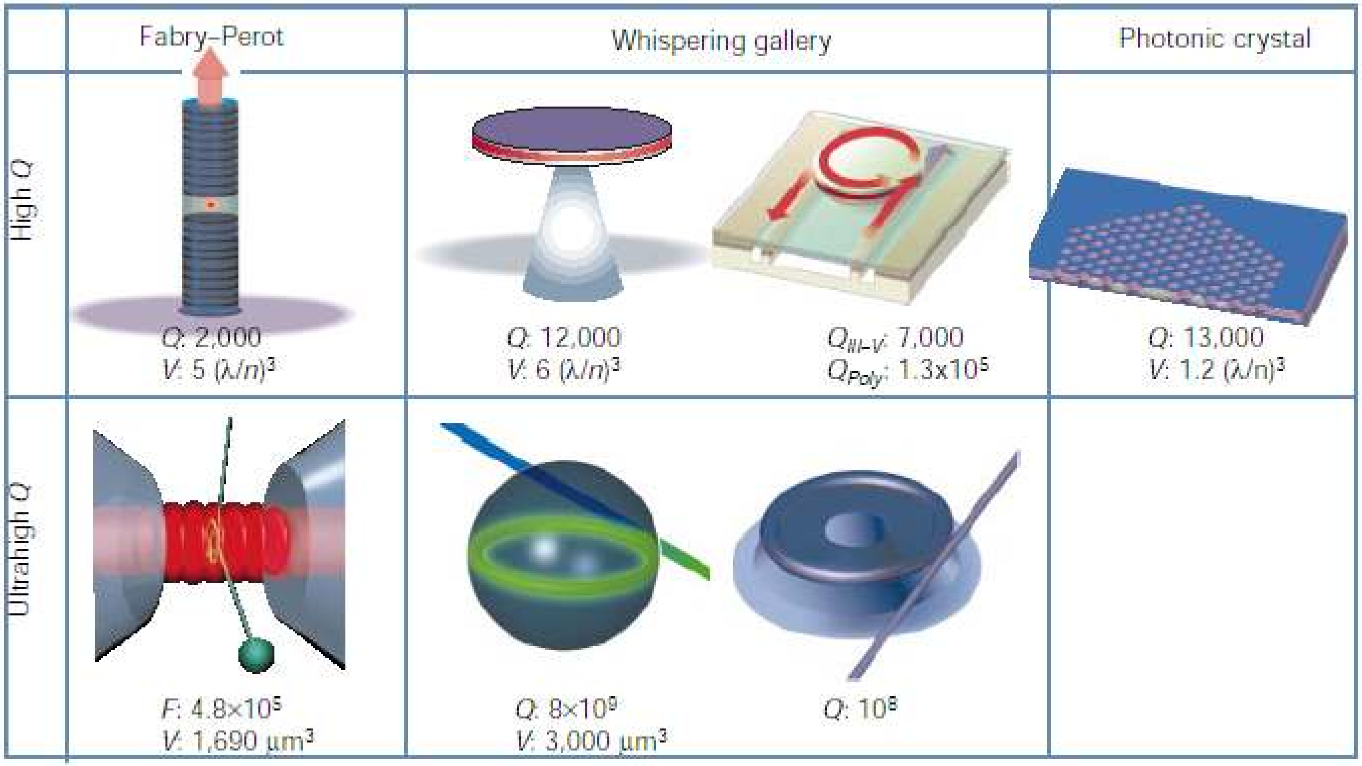
- the pumping rate $\gamma = D_p B_{23}(N_3 - N_2)/N$,
- in steady-state,

$$\begin{aligned}
N_2[A_{21} + B_{21} D(\omega)] &= N_1[A_{12} + B_{21} D(\omega)], \\
N_2 A_{23} + N_1 A_{13} &= N\gamma,
\end{aligned}$$

- for $A_{21} < A_{13}$, we have $N_2 > N_1$.

Purcell effect: Cavity-QED (Quantum ElectroDynamics)





E. M. Purcell, *Phys. Rev.* **69** (1946).

Nobel laureate Edward Mills Purcell (shared the prize with Felix Bloch) in 1952,

for their contribution to nuclear magnetic precision measurements.

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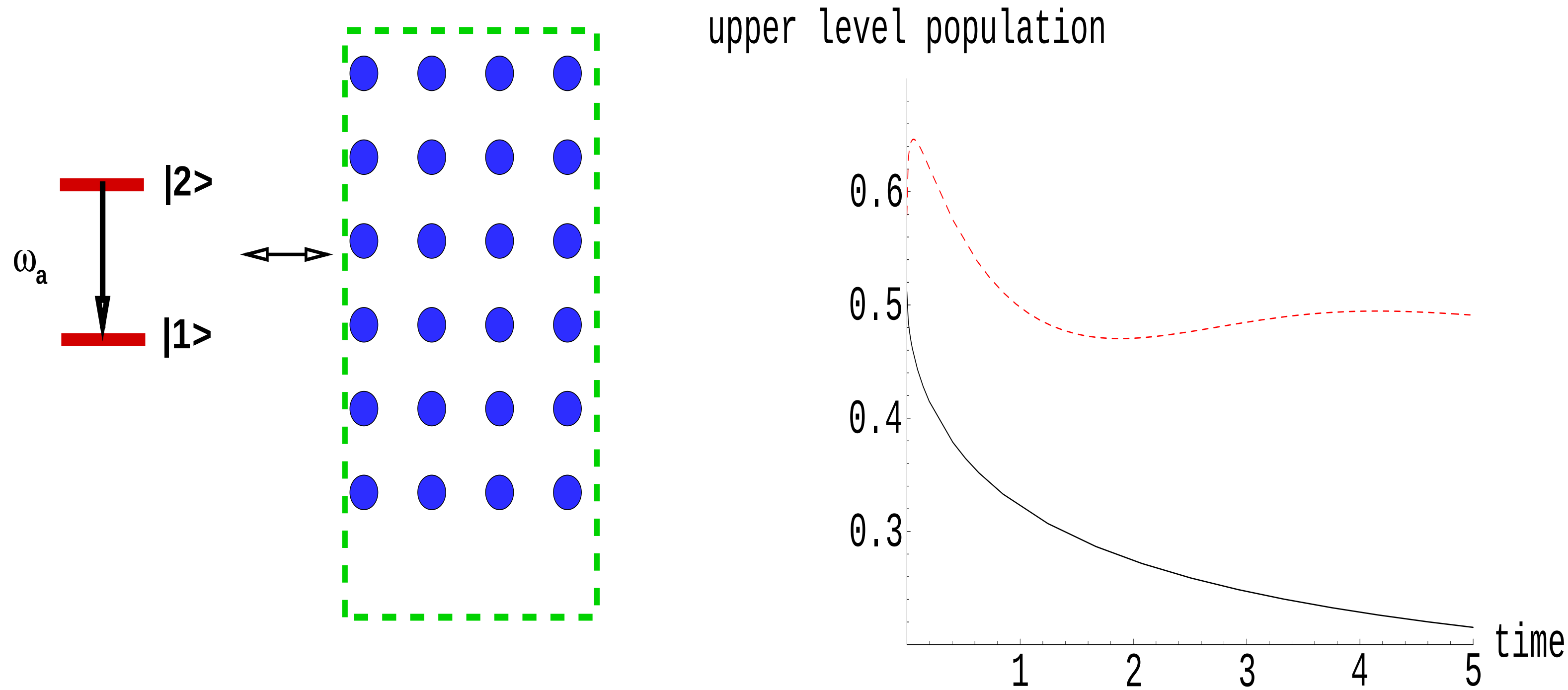
from: K. J. Vahala, *Nature* **424**, 839 (2003).

photon-atom bound state



S. John and J. Wang, *Phys. Rev. Lett.* **64**, 2418 (1990).