X. SYSTEMS WITH THIRD-ORDER NONLINEARITY

A. Self-Focusing

When a beam of finite transverse dimensions propagates through a nonlinear medium, with an index that depends on the optical intensity in the medium,

$$n = n_0 + n_2 I, \tag{X.1}$$

the index within the beam is different from that outside the beam. The vector potential A obeys approximately the wave equation,

$$\nabla^2 A + \omega^2 \mu_0 \epsilon A = 0, \tag{X.2}$$

where

$$\epsilon = \epsilon_0 n^2 = \epsilon_0 (n_0 + n_2 I)^2. \tag{X.3}$$

We assume that the vector potential is polarized at y direction,

$$A \propto \hat{y}\, u(x, y, z) e^{-jk_0 z}, \tag{X.4}$$

where u varies slowly with z, and the propagation constant,

$$k_0 = \omega \sqrt{\mu_0 \epsilon_0}\, n_0. \tag{X.5}$$

Then the wave equation for the slowly varied envelope u becomes, in the paraxial limit,

$$\nabla_T^2 u - 2jk_0 \frac{\partial u}{\partial z} = -\omega^2 \mu_0 \epsilon_0 (n^2 - n_0^2) u. \tag{X.6}$$

In the one-dimension, the paraxial wave equation becomes,

$$-2jk_0 \frac{\partial u}{\partial z} + \frac{\partial^2}{\partial x^2} u + 2k_0^2 \frac{n_2}{n_0} |u|^2 u = 0. \tag{X.7}$$

If we introduce the variable,

$$q \equiv \frac{z}{2k_0}, \tag{X.8}$$

$$\kappa \equiv 2k_0^2 \frac{n_2}{n_0}, \tag{X.9}$$

the paraxial wave equation with nonlinearity is put into the standard form of the *nonlinear Schrödinger equation*,

$$\frac{\partial^2 u}{\partial x^2} - j \frac{\partial u}{\partial q} + \kappa |u|^2 u = 0. \tag{X.10}$$

For $\kappa > 0$ (i.e. $n_2 > 0$), self-focusing, the nonlinear Schrödinger equation has the solution,

$$u = \sqrt{\frac{2}{\kappa}} \eta\, \mathrm{sech}[\eta(x - x_0) + 2\eta\xi q] \exp[j(\xi^2 - \eta^2) q + j\xi x - j\phi], \tag{X.11}$$

with the arbitrary parameters, η, ξ, x_0, and ϕ. When $\xi = \phi = 0$, this solutions is simplified into,

$$u = \sqrt{\frac{2}{\kappa}} \eta\, \mathrm{sech}[\eta(x - x_0)] \exp[-j\eta^2 q]. \tag{X.12}$$

This is a beam with an x-dependent, but z-independent profile of width proportional to $1/\eta$. The area integral of the beam is independent of the beam parameters, i.e.

$$\sqrt{\frac{\kappa}{2}} \int_{-\infty}^{\infty} |u| dx = 2\pi. \tag{X.13}$$

B. Soliton propagation in fiber

For the guiding medium of an index that increases with intensity, such as dispersive nonlinear fibers, *soliton* wave can be formatted. A soliton is a pulse excitation of a nonlinear dispersive medium which propagates without distortion. The spreading of the pulse that would be caused by the dispersion acting alone is counteracted via the nonlinear phase modulation of the pulses by the nonlinearity of the medium. For the slowly varied envelope function of the pulse in optical fiber we have,

$$j\frac{\partial A}{\partial \zeta} + \frac{1}{2}\frac{d^2\beta}{d\omega^2}\frac{\partial^2 A}{\partial \tau^2} - \kappa |A|^2 A = 0, \tag{X.14}$$

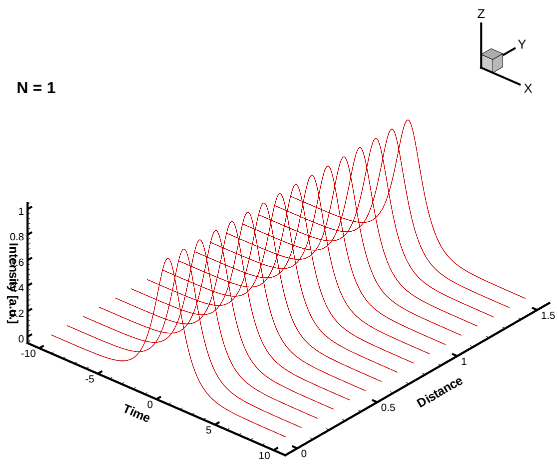
where

$$\kappa \equiv \frac{\omega_0^2 \mu_0 \epsilon_0}{\beta(\omega_0)} \frac{\int da\, n_0 n_2 |u|^4}{\int da\, |u|^2} \tag{X.15}$$

This equation is identical in form with the equation of self-focusing, nonlinear Schrödinger equation, and thus must have identical solutions, provided that

$$\frac{d^2\beta}{d\omega^2} < 0, \tag{X.16}$$

the fiber has anomalous dispersion.



C. Optical bistability

Instead of pulses, the *steady state* response to an input of the system also exhibits nonlinear phenomena when it depends on the past history of the excitation, *hysteresis*. For a Fabry-Perot resonator containing a saturable-absorber medium, the equation for the amplitude a of the field in the resonator, excited by an incident wave s_+, is

$$\frac{d\,a}{d\,t} = (j\omega_0 - \frac{1}{\tau})\,a + \sqrt{\frac{2}{\tau_e}}\,s_+, \tag{X.17}$$

where

$$\frac{1}{\tau} = \frac{1}{\tau_0} + \frac{1}{\tau_e}. \tag{X.18}$$

If the resonator contains a saturable absorber, the $1/\tau_0$ depends on the energy $|a|^2$, and one replaces $1/\tau_0$ with,

$$\frac{1}{\tau_0} \rightarrow \frac{1}{\tau_0}\frac{1}{1+|a|^2/|a_0|^2}, \tag{X.19}$$

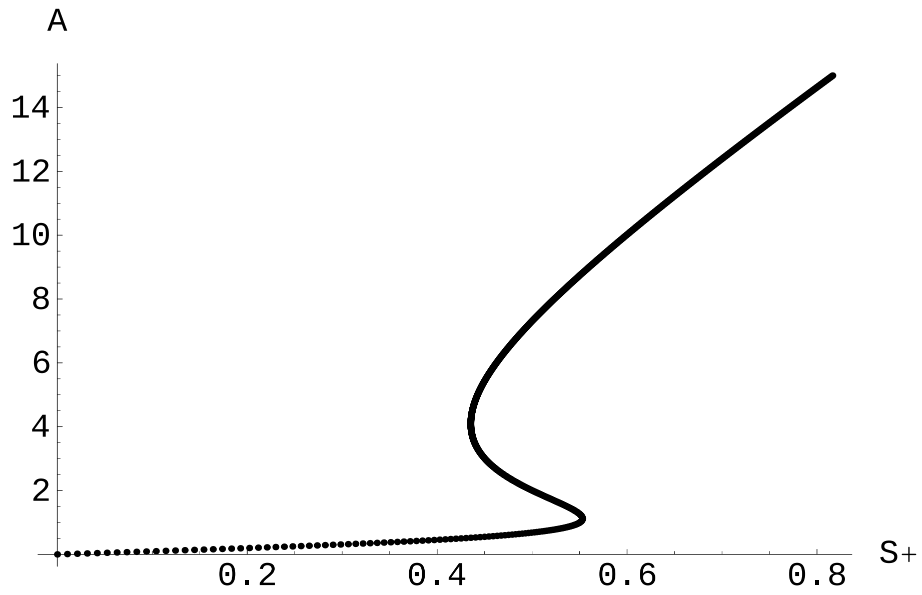
where a_0^2 is the energy in the resonator for which the loss decrease to one-half of its low-amplitude value. Then the mode equation for a becomes,

$$\frac{d\,a}{d\,t} = j\omega_0 a - (\frac{1}{\tau_0}\frac{1}{1+|a|^2/|a_0|^2} + \frac{1}{\tau_e})\,a + \sqrt{\frac{2}{\tau_e}}\,s_+, \tag{X.20}$$

Consider an excitation at the resonance frequency of the resonator, $s_+ = S_+\exp(j\omega_0 t)$ and $a = A\exp(j\omega_0 t)$, in the steady state,

$$\frac{A}{a_0}(\frac{1}{1+A^2/|a_0|^2} + \frac{\tau_0}{\tau_e}) = \sqrt{\frac{2}{\tau_e}}\tau_0\frac{S_+}{a_0}. \tag{X.21}$$

This equation shows bistability if we plot the left-hand side as the abscissa versus the ordinate A/a_0, i.e. plotted with $\sqrt{\frac{2}{\tau_e}}\tau_0\frac{S_+}{a_0}$ as x-axis and A/a_0 as y-axis.



D. Extended studies

1. Saturable absorber modelocking;
2. Self-Phase Modulation (SPM) and Cross-Phase Modulation (XPM);
3. Second-order nonlinearity: Second-Harmonic Generation (SHG), Optical Parametric Oscillation/Amplifier (OPO/OPA), Sum Frequency Generation (SFG), and Difference Frequency Generation (DFG);
4. Four-Wave Mixing (FWM);
5. Raman and Bouillon Scattering