II. PLANE WAVES PROPAGATION AND REFLECTION

A. Transverse Electric wave reflected from boundary

Consider a plane wave with its electric field polarized parallel to the surface of an interface between two media, (transverse electric, or TE, wave),

$$E_{in} = \hat{y}E_+ e^{-j\mathbf{k}^{(1)}\cdot\mathbf{r}} \tag{II.1}$$

The tangential E and E must be continuous at $z = 0$. This implies ,

$$k_x^{(1)} = k_x^{(2)} = k_x, \text{phase matching.} \tag{II.2}$$

The consequence is Snell's law,

$$\sqrt{\mu_1\epsilon_1}\sin\theta_1 = \sqrt{\mu_2\epsilon_2}\sin\theta_2. \tag{II.3}$$

At $z < 0$, the superposition of the incident and reflected waves is,

$$E_y = [E_+^{(1)}e^{-jk_z^{(1)}z} + E_-^{(1)}e^{+jk_z^{(1)}z}]e^{-jk_x x}, \tag{II.4}$$

from Faraday's law,

$$H_x = -\frac{k_z^{(1)}}{\omega\mu_1}[E_+^{(1)}e^{-jk_z^{(1)}z} - E_-^{(1)}e^{+jk_z^{(1)}z}]e^{-jk_x x}, \tag{II.5}$$

where

$$\frac{k_z^{(1)}}{\omega\mu_1} = \sqrt{\frac{\epsilon_1}{\mu_1}}\cos\theta_1 \equiv Y_0^{(1)}, \tag{II.6}$$

is the *characteristic admittance* by medium 1 to a TE wave at inclination θ_1 with respect at the z direction. The inverse of $Y_0^{(1)}$ is the *characteristic impedance* $Z_0^{(1)}$.
At $z > 0$, the transmitted waves is,

$$E_y = E_+^{(2)}e^{-jk_z^{(2)}z}e^{-jk_x x}, \tag{II.7}$$

with the x component of the H field,

$$H_x = -\frac{k_z^{(2)}}{\omega\mu_2}E_+^{(2)}e^{-jk_z^{(2)}z}e^{-jk_x x}, \tag{II.8}$$

with the characteristic admittance in medium 2,

$$\frac{k_z^{(2)}}{\omega\mu_2} = \sqrt{\frac{\epsilon_2}{\mu_2}}\cos\theta_2 \equiv Y_0^{(2)}. \tag{II.9}$$

Continuity of the tangential components of E and H requires the ratio

$$Z \equiv -\frac{E_y}{H_x} \tag{II.10}$$

to be continuous. Z is the *wave impedance* at the interface.
At $z = 0$,

$$Z_0^{(1)}\frac{E_+^{(1)} + E_-^{(1)}}{E_+^{(1)} - E_-^{(1)}} = Z_0^{(2)}. \tag{II.11}$$

The quantity,

$$\Gamma \equiv \frac{E_-}{E_+}, \tag{II.12}$$

is the *reflection coefficient.*
For $E_-^{(1)}/E_+^{(1)}$,

$$\Gamma^{(1)} = \frac{Z_0^{(2)} - Z_0^{(1)}}{Z_0^{(2)} + Z_0^{(1)}}, \tag{II.13}$$

using Snell's law,

$$\Gamma^{(1)} = \frac{\sqrt{1-\sin^2\theta_1} - \sqrt{1-\sin^2\theta_1 \frac{\epsilon_1\mu_1}{\epsilon_2\mu_2}}\sqrt{\frac{\epsilon_2\mu_1}{\epsilon_1\mu_2}}}{\sqrt{1-\sin^2\theta_1} + \sqrt{1-\sin^2\theta_1 \frac{\epsilon_1\mu_1}{\epsilon_2\mu_2}}\sqrt{\frac{\epsilon_2\mu_1}{\epsilon_1\mu_2}}}. \tag{II.14}$$

The density of power flow in the z direction is

$$\frac{1}{2}\mathrm{Re}[E \times H^*] \cdot \hat{z} = -\frac{1}{2}\mathrm{Re}[E_y H_x] = \frac{1}{2} Y_0^{(1)} |E_+^{(1)}|^2 (1 - |\Gamma^{(1)}|^2). \tag{II.15}$$

Thus $|\Gamma|^2$ is the ratio of reflected to incident power flow.

B. Transverse Magnetic wave reflected from boundary

Consider a plane wave with its magnetic field polarized parallel to the surface of an interface between two media, (transverse magnetic, or TM, wave), At $z < 0$, the superposition of the incident and reflected waves is,

$$H_y = [H_+^{(1)} e^{-jk_z^{(1)} z} + H_-^{(1)} e^{+jk_z^{(1)} z}] e^{-jk_x x}, \tag{II.16}$$

from Ampére's law,

$$E_x = \frac{k_z^{(1)}}{\omega\epsilon_1} [H_+^{(1)} e^{-jk_z^{(1)} z} - H_-^{(1)} e^{+jk_z^{(1)} z}] e^{-jk_x x}. \tag{II.17}$$

At $z > 0$, the transmitted waves is,

$$H_y = H_+^{(2)} e^{-jk_z^{(2)} z} e^{-jk_x x}, \tag{II.18}$$

with the x component of the E field,

$$E_x = \frac{k_z^{(2)}}{\omega\epsilon_2} H_+^{(2)} e^{-jk_z^{(2)} z} e^{-jk_x x}, \tag{II.19}$$

with the characteristic admittance of the traveling TM wave,

$$Y_0 = \frac{\omega\epsilon}{k_z} = \sqrt{\frac{\epsilon}{\mu}} \frac{1}{\cos\theta}. \tag{II.20}$$

Continuity of the tangential components of E and H requires the ratio

$$Z \equiv \frac{E_x}{H_y} \tag{II.21}$$

to be continuous. Z is the *wave impedance* at the interface.
At $z = 0$,

$$Z_0^{(1)} \frac{H_+^{(1)} - H_-^{(1)}}{H_+^{(1)} + H_-^{(1)}} = Z_0^{(2)}. \tag{II.22}$$

The quantity,

$$\Gamma \equiv -\frac{H_-}{H_+}, \tag{II.23}$$

is the *reflection coefficient*.
For $E_-^{(1)}/E_+^{(1)}$,

$$\Gamma^{(1)} = \frac{Z_0^{(2)} - Z_0^{(1)}}{Z_0^{(2)} + Z_0^{(1)}}, \tag{II.24}$$

using Snell's law,

$$\Gamma^{(1)} = -\frac{\sqrt{1-\sin^2\theta_1} - \sqrt{1-\sin^2\theta_1 \frac{\epsilon_1\mu_1}{\epsilon_2\mu_2}}\sqrt{\frac{\epsilon_1\mu_2}{\epsilon_2\mu_1}}}{\sqrt{1-\sin^2\theta_1} + \sqrt{1-\sin^2\theta_1 \frac{\epsilon_1\mu_1}{\epsilon_2\mu_2}}\sqrt{\frac{\epsilon_1\mu_2}{\epsilon_2\mu_1}}}. \tag{II.25}$$

TM waves can be transmitted reflection-free at a dielectric interface, when $\mu_1 = \mu_2 = \mu_0$, for the angle $\theta_1 = \theta_B$, the so-called *Brewster angle*,

$$\theta_B = \tan^{-1}\sqrt{\frac{\epsilon_2}{\epsilon_1}}. \tag{II.26}$$

Summary

	TE	TM
Reflection coefficient	$\Gamma(z) = \frac{E_-}{E_+}e^{+j2k_z z}$	$\Gamma(z) = -\frac{H_-}{H_+}e^{+j2k_z z}$
Wave impedance	$Z_0 = \sqrt{\frac{\mu}{\epsilon}}\frac{1}{\cos\theta}$	$Z_0 = \sqrt{\frac{\mu}{\epsilon}}\cos\theta$
	$Z(z) = -\frac{E_y}{H_x}$	$Z(z) = \frac{E_x}{H_y}$
	$\frac{Z(z)}{Z_0} = \frac{1+\Gamma(z)}{1-\Gamma(z)}$	
	$\Gamma(z) = \frac{Z(z)-Z_0}{Z(z)+Z_0}$	
Characteristic admittance	$Y_0 = \sqrt{\frac{\epsilon}{\mu}}\cos\theta$	$Y_0 = \sqrt{\frac{\epsilon}{\mu}}\frac{1}{\cos\theta}$
	$Y(z) = -\frac{H_x}{E_y}$	$Y(z) = \frac{H_y}{E_x}$
	$\frac{Y(z)}{Y_0} = \frac{1-\Gamma(z)}{1+\Gamma(z)}$	
	$\Gamma(z) = \frac{Y_0-Y(z)}{Y_0+Y(z)}$	

C. Total internal reflection

If medium 1 has a larger value of $\sqrt{\mu\epsilon}$, *optical denser*, than medium 2, Snell's law fails to yield a real angle θ_2 for a certain range of angle of incidence.
For $\mu_1 = \mu_2 = \mu_0$,

$$\sin\theta_c = \sqrt{\frac{\epsilon_2}{\epsilon_1}} = \frac{n_2}{n_1}. \tag{II.27}$$

When no real solution of θ_2 are found, the propagation constant must be allowed to become negative imaginary,

$$k_x^{(2)} = k_x^{(1)}, \tag{II.28}$$
$$k_z^{(2)} = -j\alpha_z^{(2)}. \tag{II.29}$$

In this case,

$$[k_x^{(2)}]^2 + [k_z^{(2)}]^2 = [k_x^{(2)}]^2 - [\alpha_z^{(2)}]^2 = \omega^2\mu_0\epsilon_2, \tag{II.30}$$

and

$$k_x^{(2)} = \sqrt{\omega^2\mu_0\epsilon_2 + [\alpha_z^{(2)}]^2}. \tag{II.31}$$

In the case of a TE wave, the transmitted fields become,

$$E_y = E_+^{(2)} e^{-\alpha_z^{(2)} z} e^{-jk_x x}, \tag{II.32}$$
$$H_x = \frac{j\alpha_z^{(2)}}{\omega\mu_0} E_+^{(2)} e^{-\alpha_z^{(2)} z} e^{-jk_x x}. \tag{II.33}$$

The wave impedance, $-E_y/H_x$,

$$\frac{\omega\mu_0}{k_z^{(1)}} \frac{E_+^{(1)} + E_-^{(1)}}{E_+^{(1)} - E_-^{(1)}} = \frac{j\omega\mu_0}{\alpha_z^{(2)}} = Z_0^{(2)}. \tag{II.34}$$

The characteristic impedance of medium 2 is now *imaginary*, $Z_0^{(2))} = jX_0^{(2)}$, with $X_0^{(2)}$ real. Then the reflection coefficient, $\Gamma = E_-^{(1)}/E_+^{(1)}$,

$$\Gamma^{(1)} = \frac{E_-^{(1)}}{E_+^{(1)}} = \frac{jX_0^{(2)} - Z_0^{(1)}}{jX_0^{(2)} + Z_0^{(1)}}, \tag{II.35}$$

which shows that $|\Gamma^{(1)}| = 1$, and the magnitude of the reflected wave, $E_-^{(1)}$, equals to the magnitude of the incident wave, $E_+^{(1)}$.
At $z < 0$,

$$E_y = E_+^{(1)} [e^{-jk_z^{(1)} z} + \Gamma^{(1)} e^{+jk_z z}] e^{-jk_x x}, \tag{II.36}$$
$$= 2e^{-j\phi} E_+^{(1)} \cos(k_z^{(1)} z - \phi) e^{-jk_x x}, \tag{II.37}$$

where $\phi = -\frac{1}{2}\arg(\Gamma^{(1)})$, is the Goos-Hänchen shift.

D. Impedance and reflection coefficient

The wave impedance of a TE wave in medium 1 at any position z is,

$$Z(z) = -\frac{E_y}{H_x} = Z_0 \frac{1 + \Gamma e^{2jk_z z}}{1 - \Gamma e^{2jk_z z}}. \tag{II.38}$$

At $z = 0$,

$$Z(0) = Z_0 \frac{1+\Gamma}{1-\Gamma}, \tag{II.39}$$

then,

$$Z(z) = Z_0 \frac{Z(0) - jZ_0 \tan(k_z z)}{Z_0 - jZ(0)\tan(k_z z)}. \tag{II.40}$$

At $z = -[(2m+1)/2k_z]\pi$, with m an integer, where $\tan(k_z z) = \infty$,

$$Z(-\frac{2m+1}{2k_z}\pi) = \frac{Z_0^2}{Z(0)}. \tag{II.41}$$

E. Anti-reflection coating

Reflection of a plane wave from a dielectric interface may be eliminated by coating the interface with a layer of different dielectric constant. For example, the substrate is a medium described by μ_0, $\epsilon = \epsilon_0 n^2$. The wave impedance by a TE wave incident from the top is,

$$Z(0) = \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1}{n\cos\theta}. \tag{II.42}$$

If the thickness of the layer is a quarter wavelength,

$$d = \frac{\lambda^{(1)}}{4} \frac{1}{\cos\theta_1}, \tag{II.43}$$

we have at $z = -d$,

$$Z(-d) = \frac{Z_{01}^2}{Z(0)} = \sqrt{\mu_0/\epsilon_0} \frac{n\cos\theta}{n_1^2 \cos^2\theta_1}. \tag{II.44}$$

If the substrate is to be matched, we have

$$Z(-d) = Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1}{n\cos\theta}, \tag{II.45}$$

when

$$n_1^2 \cos^2\theta_1 = n\cos\theta\cos\theta_0. \tag{II.46}$$

A coating applied to match the substrate, and eliminate the reflected wave is an *anti-reflection coating.* In particular for $\theta_0 = 0$,

$$n_1^2 = n. \tag{II.47}$$

It may be difficult to find a dielectric material with an index n_1 for a given n. In the usual case, one would apply several quarter-wave layers. If m pairs of layers are applied to the substrate, the wave impedance seen at the "input plane" is,

$$Z(z_j) = \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1}{n\cos\theta} (\frac{n_1\cos\theta_1}{n_2\cos\theta_2})^{2m}. \tag{II.48}$$

The multiple layers act as an antireflection coating, if, $(n_1\cos\theta_1/n_2\cos\theta_2)$ is chosen so that $Z(z_j)$ is equal to the characteristic impedance of the wave in the "input" medium i.

F. Reflection gratings

A reflection grating is formed by a periodically "corrugated" reflecting surface. We idealize a grating surface as perfectly reflecting. The spatial period of the grating is Λ. Consider the incident wave is a TE wave, $\hat{y}E_i exp(-j\vec{k}\cdot\vec{r})$, then the superposition of incident and reflected wave will be,

$$E_i exp[-(jk_{ix}x + jk_{iz}z]|_{z=h(x)} + E_R(x,z)|_{z=h(x)} = 0. \tag{II.49}$$

The reflected field can only cancel the incident field at $z = h(x)$ of the form,

$$E_R(x,z) = exp(-jk_{ix}x)\sum_m R_m exp(jk_{Rz}^{(m)})exp(-j\frac{2\pi m}{\Lambda}x), \tag{II.50}$$

where the R_m's are constants and $k_{Rz}^{(m)}$'s obey the constraint

$$(k_{ix} + \frac{2\pi}{\Lambda}m)^2 + k_{Rz}^{(m)^2} = \omega^2\mu_0\epsilon_0. \tag{II.51}$$

The reflected electric field is composed of an infinite sum of plane waves propagating at different angles. The angle of reflection of mth order may be related to the angle of incidence,

$$\sin\theta_R^{(m)} = \sin\theta_i + \frac{m\lambda}{\Lambda}. \tag{II.52}$$

This is the grating reflection law.
Example 1: Sinusoidal corrugation,

$$h(x) = h_0\cos\frac{2\pi}{\Lambda}x, \qquad (\frac{h_0}{\Lambda} \ll 1). \tag{II.53}$$

In this case, only $R_{\pm 1}$ are non-zero,

$$R_{+1} = R_{-1} = jk_{iz}h_0E_i, \tag{II.54}$$

and all other R_m's are zero.
Example 2: Step grating with the step height is a multiple of half wavelengths.
In this case, we consider the condition of reflection of the -1 order into the direction of incidence,

$$-\sin\theta_R^{(-1)} = \sin\theta_i = \frac{\lambda}{\Lambda} - \sin\theta_i. \tag{II.55}$$

Here, the *blaze angle*, θ_B is defined as

$$\sin\theta_B = \frac{\lambda}{2\Lambda}. \tag{II.56}$$

G. Extended studies

- Grating spectrometers, (Optical Spectral Analyzer);
- Handbook of anti-reflection coating;
- Dispersion compensation by grating pairs;
- Left-handed materials.