III. MIRRORS AND INTERFEROMETERS

A. Reciprocity principle

For an isotropic medium, we can have two solutions of Maxwell's equations,

$$\nabla \times \mathbf{E}^{(a),(b)} = -j\omega\mu\mathbf{H}^{(a),(b)}, \tag{III.1}$$
$$\nabla \times \mathbf{H}^{(a),(b)} = j\omega\epsilon\mathbf{E}^{(a),(b)}, \tag{III.2}$$

then one may have the field-theoretical form of the *reciprocity theorem*,

$$\nabla \cdot (\mathbf{E}^{(a)} \times \mathbf{H}^{(b)} - \mathbf{E}^{(b)} \times \mathbf{H}^{(a)}) = 0, \tag{III.3}$$

or by using of Gauss's theorem

$$\oint_S (\mathbf{E}^{(a)} \times \mathbf{H}^{(b)}) \cdot d\,A = \oint_S (\mathbf{E}^{(b)} \times \mathbf{H}^{(a)}) \cdot d\,A. \tag{III.4}$$

It imposes a constraint on the solutions of Maxwell's equations in a medium described by a scalar dielectric constant and magnetic permeability.

B. Scattering Matrix and its properties

For plane waves passing through slabs of optical media, the power per unit area incident from the left is given by

$$\frac{1}{2}\mathbf{R}e[\mathbf{E} \times \mathbf{H}^* \cdot \vec{z}] = |a_1|^2 - |b_2|^2. \tag{III.5}$$

For TE-wave, we can write E_y as

$$E_y = \sqrt{2/Y_0^{(1)}}[a_1 + b_1]e^{-jk_x x}, \tag{III.6}$$

and

$$H_x = \sqrt{2Y_0^{(1)}}[a_1 - b_1]e^{-jk_x x}, \tag{III.7}$$

where

$$a_1 \equiv \sqrt{Y_0^{(1)}/2}E_+^{(1)}e^{-jk_z^{(1)}z}, \tag{III.8}$$
$$b_1 \equiv \sqrt{Y_0^{(1)}/2}E_-^{(1)}e^{+jk_z^{(1)}z}. \tag{III.9}$$

Take advantage of the fact that a_1 and a_2 can be chosen as independent variables, b_1 and b_2 are dependent variables, the latter being linear functions of the former. Then the system is describable by the *scattering matrix*

$$b_1 = S_{11}\,a_1 + S_{12}\,a_2, \tag{III.10}$$
$$b_2 = S_{21}\,a_1 + S_{22}\,a_2. \tag{III.11}$$

The reflection of a TE wave from an interface is an example of a particular excitation of a particular two-port. At $z = 0$, the elements of the scattering matrix are

$$S_{11} = \Gamma^{(1)} = \frac{E_-^{(1)}}{E_+^{(1)}} = \frac{Y_0^{(1)} - Y_0^{(2)}}{Y_0^{(1)} + Y_0^{(2)}}, \tag{III.12}$$

$$S_{12} = S_{21} = \frac{\sqrt{Y_0^{(2)}}E_+^{(2)}}{\sqrt{Y_0^{(1)}}E_+^{(1)}} = (1 + \Gamma^{(1)})\sqrt{\frac{Y_0^{(2)}}{Y_0^{(1)}}}, \tag{III.13}$$

$$S_{22} = -S_{11}, \tag{III.14}$$

the last equation comes from an interchange of indices 1 and 2.
It is convenient to write the scattering matrix into matrix form by defining the two column matrices

$$\mathbf{a} \equiv= \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \qquad \mathbf{b} \equiv= \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \tag{III.15}$$

and the matrix of second rank

$$\mathbf{S} \equiv= \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}. \tag{III.16}$$

With these definitions the compact expression for the scattering matrix is

$$\mathbf{b} = \mathbf{S}\mathbf{a}. \tag{III.17}$$

1. *Properties of scattering matrix*

1. Reciprocity condition: for a linear and isotropic medium,

$$\mathbf{S}_t = \mathbf{S}. \tag{III.18}$$

   The scattering matrix of a linear reciprocal system is *symmetric*, i.e. $S_{12} = S_{21}$.

2. Power conservation: for a lossless medium,

$$\mathbf{S}^\dagger \mathbf{S} = \mathbf{1}, \tag{III.19}$$

   or

$$\mathbf{S}^\dagger = \mathbf{S}^{-1}. \tag{III.20}$$

   The scattering matrix of a lossless system is *unitary*, i.e.

$$|S_{11}|^2 + |S_{21}|^2 = 1, \tag{III.21}$$
$$|S_{22}|^2 + |S_{12}|^2 = 1, \tag{III.22}$$
$$S_{11}^* S_{12} + S_{21}^* S_{22} = 0 \tag{III.23}$$
$$. \tag{III.24}$$

3. Time reversal: time reversibility leads to constraints on the scattering matrix coefficients,

$$\mathbf{S}^* = \mathbf{S}^{-1}. \tag{III.25}$$

   Note: When both ot time reversibility and power conservation are made,

$$\mathbf{S}^* = \mathbf{S}^\dagger, \tag{III.26}$$

   which imply reciprocity. A lossless reciprocal two-port is described in terms of *three* real parameters.

C. Partially transmitting mirror

We now apply the scattering matrix formalism to a lossless reflecting and transmitting system - the general case of a lossless partially transmitting mirror. We may choose the position of reference plane 1 so that the reflected wave is in anti-phase with the incident wave,

$$E_-^{(1)} = -r_1 E_+^{(1)}, \tag{III.27}$$

or

$$b_1 = -r_1\, a_1, \tag{III.28}$$

where r_1 is real and positive. The same may be done with reference plane 2, thus,

$$\begin{aligned} S_{11} &= -r_1, && \text{(III.29)} \\ S_{22} &= -r_2. && \text{(III.30)} \end{aligned}$$

Reciprocity requires a symmetric **S** matrix,

$$\begin{aligned} |S_{21}|^2 &= 1 - |S_{11}|^2 = 1 - r_1^2, && \text{(III.31)} \\ |S_{12}|^2 &= 1 - |S_{22}|^2 = 1 - r_2^2 = |S_{21}|^2. && \text{(III.32)} \end{aligned}$$

Thus the matrix elements $S_{11} = -r_1$ and $S_{22} = -r_2$ are equal, $r_1 = r_2 = r$. And the other elements of the scattering matrix are

$$S_{12} = -S_{21}^* \frac{S_{22}}{S_{11}^*}. \tag{III.33}$$

For a special choice of the reference planes, S_{22}/S_{11}^* is equal to unity, thus S_{12} is a pure imaginary quantity. We choose

$$S_{12} = j\,t, \tag{III.34}$$

where t is the *"transmissivity"*

$$t = \sqrt{1 - r^2}. \tag{III.35}$$

The transmissivity t need not be positive; both signs of the square root are permissible. Through a particular choice of reference planes the **S** matrix of a lossless systems has been cast into the form

$$\mathbf{S} = \begin{bmatrix} -r & j\,t \\ j\,t & -r \end{bmatrix}, \tag{III.36}$$

where

$$t = \sqrt{1 - r^2}. \tag{III.37}$$

The present general derivation of the mirror **S** matrix is independent of the angle and polarization of the incident waves, it can be TE, TM , or a combination of the two for particular reference planes.

D. Fabry-Perot interferometer

Two parallel partially transmitting mirrors separated by a distance l form a Fabry-Perot interferometer. The total wave leaving mirror 1 and traveling to the right, at the right-hand reference plane of mirror 1 is

$$a = \sum_{m=0}^{\infty} (r_1 r_2 e^{-j\delta})^m j\,t_1 a_1 = \frac{j\,t_1}{1 - r_1 r_2 e^{-j\delta}}\, a_1. \tag{III.38}$$

where $\delta/2 = (\omega n/c)\cos\theta l$ is the phase shift between two mirrors. From the scattering matrix of mirror 1,

$$\begin{aligned} b_1 &= -r_1 a_1 + j\,t_1 b, && \text{(III.39)} \\ b_1 &= j\,t_1 a_1 - r_1 b, && \text{(III.40)} \end{aligned}$$

and the solution for the reflected wave b_1 is

$$b_1 = -\frac{r_1 - r_2 e^{-j\delta}}{1 - r_1 r_2 e^{-j\delta}}\, a_1. \tag{III.41}$$

By the same method, the transmitted wave b_2 is

$$b_2 = -\frac{t_1 t_2 e^{-j\delta/2}}{1 - r_1 r_2 e^{-j\delta}}\, a_1. \tag{III.42}$$

By reciprocity, $S_{21} = S_{12}$, and S_{22} is obtained from S_{11} by interchange of the subscripts 1 and 2. Thus the complete scattering matrix of the Fabry-Perot interferometer is

$$\mathbf{S} = \frac{1}{1 - r_1 r_2 e^{-j\delta}} \begin{bmatrix} -(r_1 - r_2 e^{-j\delta}) & -t_1 t_2 e^{-j\delta/2} \\ -t_1 t_2 e^{-j\delta/2} & -(r_2 - r_1 e^{-j\delta}) \end{bmatrix}. \tag{III.43}$$

The important property of a Fabry-Perot interferometer is its frequency-dependent transmission of power. The transmitted power (per unit area) is

$$|b_2|^2 = |S_{21}|^2 |a_1|^2 = \frac{t_1^2 t_2^2 |a_1|^2}{(1 - r_1 r_2)^2 + 4 r_1 r_2 \sin^2(\delta/2))}. \tag{III.44}$$

In this form, the expression for the transmitted power is equally valid for mirrors with *loss*; then r_1, r_2, t_1, and t_2 may be complex. If we define $t_1^2 = t_2^2 = 1 - r_1^2$, $r_1^2 = r_2^2 = r^2 \equiv R$, and $t_1^2 = t_2^2 = 1 - R$, then

$$|b_2|^2 = |a_1|^2 \frac{(1 - R)^2}{(1 - R)^2 + 4R \sin^2(\delta/2)}. \tag{III.45}$$

Here, R is the reflectivity of the mirrors expressing the reflected power per unit incident power. The transmission peaks occur when $\delta/2 = (\omega n/c) \cos\theta l = m\pi$, with m an integer, when the frequency $f = \omega/2\pi$ is equal to one of a set of characteristic value

$$f_m = \frac{mc}{2nl\cos\theta}. \tag{III.46}$$

The frequency separation Δf of the peaks is given by

$$f_{m+1} - f_m = \Delta f = \frac{c}{2nl\cos\theta}. \tag{III.47}$$

The *free spectral range* of a Fabry-Perot interferometer is defined as the frequency separation of transmission peaks, expressed in (free-space) wavelength, $\Delta\lambda$,

$$|\Delta\lambda| = (\frac{|\Delta f|}{f})\lambda = \frac{\lambda^2}{2nl} \frac{1}{\cos\theta}. \tag{III.48}$$

The *full width at half-maximum* (FWHM) of the transmission peaks is approximately, when $1 - R \ll 1$,

$$\delta f_{1/2} = \frac{(1 - R)c}{2\pi\sqrt{R} nl \cos\theta}. \tag{III.49}$$

The quantity

$$\frac{\Delta f}{\delta f_{1/2}} \equiv F = \frac{\pi\sqrt{R}}{1 - R} \tag{III.50}$$

is the *finesse* of the interferometer.

E. Michelson interferometer

The scanning Fabry-Perot interferometer is used to measure the spectrum of an incident optical wave. The Michelson interferometer on the other hand can measure the autocorrelation function. Whereas the Fabry-Perot interferometer is ideally suited to the analysis of narrow band processes, the Michelson interferometer is well adapted to the measurement of broad band optical waveforms.
In the Michelson interferometer, an incident wave a_1 is split by the "half-silvered" mirror ($r = t = 1/sqrt2$) into two components. The other one at the "output" side of the mirror b_2 is

$$\begin{aligned} b_2 &= \frac{j}{2} exp(-j2kl_a) a_1 + \frac{j}{2} exp(-j2kl_b) a_1 && \text{(III.51)} \\ &= jexp[-jk(l_a + l_b)] \cos k(l_a - l_b) a_1 = S_{21} a_1. && \text{(III.52)} \end{aligned}$$

And the reflected wave amplitude at the input reference plane is

$$\begin{aligned} b_1 &= \frac{1}{2}exp(-j2kl_a)a_1 - \frac{1}{2}exp(-j2kl_b)a_1 && \text{(III.53)} \\ &= -jexp[-jk(l_a+l_b)]\sin k(l_a-l_b)a_1 = S_{11}a_1. && \text{(III.54)} \end{aligned}$$

The scattering matrix of the Michelson interferometer is

$$S = jexp[-jk(l_a+l_b)]\begin{bmatrix} -\sin k(l_a-l_b) & \cos k(l_a-l_b) \\ -\cos k(l_a-l_b) & \sin k(l_a-l_b) \end{bmatrix} \tag{III.55}$$

Thus, in contrast to the Fabry-Perot interferometer, the transmission as a function of frequency of the Michelson interferometer is not narrowly peaked around the transmission maxima.

F. Coherence

The Michelson interferometer superimposes two waves with interfere. If one treats a periodic time-dependent process,

$$a_1(t) = \sum_n a_1(n)e^{j\omega_n t}, \tag{III.56}$$

where $\omega_n = n\Delta\omega$, with $\Delta\omega = 2\pi/T$, the periodicity of the process, T. With $k = \omega/c$,

$$b_2(t) = \frac{j}{2}[a_1(t-\tau_a) + a_1(t-\tau_b)], \tag{III.57}$$

where

$$\tau_a = \frac{2l_a}{c}, \qquad \tau_b = \frac{2l_b}{c}. \tag{III.58}$$

The time-averaged power transmitted through the Michelson interferometer is

$$< |b_2(t)|^2 >= \frac{1}{4}2 < |a_1(t)|^2 > + < a_1(t-\tau_a)a_1^*(t-\tau_b) > + < a_1^*(t-\tau_a)a_1(t-\tau_b) >, \tag{III.59}$$

where we have taken into account that $< |a_1(t)|^2 >$ is time independent. For a stationary statistical process the quantity

$$< a_1(t-\tau_a)a_1^*(t-\tau_b) >=< a_1(t)a_1^*(t-\tau) >= \Gamma_{11}(\tau), \tag{III.60}$$

is the complex autocorrelation function $\Gamma_{11}(\tau)$ of a_1, where $\tau = \tau_a - \tau_b$. A detector that responds to time-averaged power mounted at the output of the interferometer detects

$$< |b_2(t)|^2 >= \frac{1}{4}2\Gamma_{11}(0) + \Gamma_{11}(\tau) + \Gamma_{11}^*(\tau). \tag{III.61}$$

The Michelson interferometer is well suited to the measurement of *temporal coherence*, while the spatial coherence is measured conveniently by diffraction.

G. Extended studies

- 1st-order correlation function;
- 2nd-order correlation function;
- Poisson distribution;
- Laser resonators with Fabry-Perot structures;
- Twyman-Green interferometer.