IV. FRESNEL DIFFRACTION AND PARAXIAL WAVE EQUATION

A. Fresnel diffraction

Any physical optical beam is of finite transverse cross section. Beams of finite cross section may be described in terms of a superposition of plane waves, analogous to the representation of a time function of finite duration in terms of a Fourier superposition of sinusoids. The Fresnel diffraction integral expresses the amplitude distribution of a scalar wave at any cross section of constant, z, in terms of a given distribution at $z = 0$. A general plane-wave solution of the scalar wave equation in Cartesian coordinates is of the form,

$$e^{-jk_x x} e^{-jk_y y} e^{-jk_z z}, \tag{IV.1}$$

with

$$k_x^2 + k_y^2 + k_z^2 = k^2. \tag{IV.2}$$

If the propagation vector k is inclined by a small angle with respect to the z axis, then the wave vector is *paraxial*, and

$$k_z = \sqrt{k^2 - k_x^2 - k_y^2} \simeq k - \frac{k_x^2 + k_y^2}{2k}. \tag{IV.3}$$

This is the paraxial approximation for the z component of k. Let us build up an amplitude distribution $u(x, y, z)$ by superposition of plane waves,

$$u(x, y, z) = \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y U_0(k_x, k_y) e^{-j(k_x x + k_y y)} e^{[j(k_x^2 + k_y^2)/2k]z}, \tag{IV.4}$$

$U_0(k_x, k_y)$ is the amplitude of the plane wave solution with particular transverse components of k, k_x, and k_y. At $z = 0$, the amplitude distribution $u_0(x, y)$ is,

$$u_0(x, y) = \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y U_0(k_x, k_y) e^{-j(k_x x + k_y y)}. \tag{IV.5}$$

The wave amplitude function $U_0(k_x, k_y)$ is the Fourier transform of the amplitude distribution at $z = 0$, $u_0(x, y)$. Expressing $U_0(k_x, k_y)$ in terms of $u_0(x, y)$ by inverse Fourier transform,

$$U_0(k_x, k_y) = (\frac{1}{2\pi})^2 \int_{-\infty}^{\infty} dx_0 \int_{-\infty}^{\infty} dy_0 u_0(x_0, y_0) e^{j(k_x x_0 + k_y y_0)}. \tag{IV.6}$$

In the paraxial approximation, one is able to express the solution to the scalar wave equation $u(x, y, z)$ in terms of the know distribution $u_0(x, y)$, at $z = 0$,

$$u(x, y, z) = \int_{-\infty}^{\infty} dx_0 \int_{-\infty}^{\infty} dy_0 u_0(x_0, y_0) (\frac{1}{2\pi})^2 \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y e^{-j[k_x(x - x_0) + k_y(y - y_0)]} e^{[j(k_x^2 + k_y^2)/2k]z}. \tag{IV.7}$$

This expression is the convolution of $u_0(x, y)$ with the *Fresnel kernel*,

$$h(x, y, z) = (\frac{1}{2\pi})^2 \int_{\infty}^{\infty} dk_x \int_{\infty}^{\infty} dk_y e^{-j(k_x x + k_y y)} e^{[j(k_x^2 + k_y^2)/2k]z} \tag{IV.8}$$

$$= \frac{j}{\lambda z} e^{-jk[(x^2 + y^2)/2z]}. \tag{IV.9}$$

Then the *Fresnel diffraction integral* in the paraxial approximation,

$$u(x, y, z) = \frac{j}{\lambda z} \int_{-\infty}^{\infty} dx_0 \int_{-\infty}^{\infty} dy_0 u_0(x_0, y_0) e^{-j(k/2z)[(x - x_0)^2 + (y - y_0)^2)]} \tag{IV.10}$$

$$= h \otimes u_0. \tag{IV.11}$$

B. Amplitude distribution, vector potential, and E-field

In free space, the vector potential, A, is defined as

$$A(r,t) = \vec{n}\psi(x,y,z)e^{j\omega t}, \tag{IV.12}$$

which obeys the vector wave equation,

$$\nabla^2\psi + k^2\psi = 0. \tag{IV.13}$$

And the relation of scalar function to the amplitude distribution, $u(x,y,z)$, is

$$\psi(x,y,z) = u(x,y,z)exp(-jkz). \tag{IV.14}$$

In all our investigation of wave propagation in the z direction, the direction of the vector potential $\vec{n}$ will be picked in the $x-y$ plane. The electric field E will be polarized mainly in the $x-y$ plane, although small longitudinal components along z are predicted as required in order to satisfy the condition of $\nabla \cdot E = 0$.

$$E = -j\omega A - \nabla\Phi, \tag{IV.15}$$

and the scalar potential is

$$\Phi = \frac{j}{\omega\mu_0\epsilon_0}\nabla \cdot A. \tag{IV.16}$$

Thus

$$E = -j\omega(A + \frac{1}{\omega^2\mu_0\epsilon_0}\nabla\nabla \cdot A). \tag{IV.17}$$

If $A = \vec{n}u(x,y,z)exp(-jkz)$ is transverse to $\vec{z}$, the divergence involves derivatives with respect to the (much slower) transverse variation. The scalar potential contributes a longitudinal component of the electrical field that is much smaller than the transverse component and gives an even smaller contribution to the transverse component.

C. Near-field region

The wave of finite transverse extent retains its profile as it propagates in the "*near-field region*" defined by,

$$z \ll \frac{d_x^2}{\lambda}, \qquad \frac{d_y^2}{\lambda}. \tag{IV.18}$$

For a initial profile with an inclined phase front,

$$u_0(x_0,y_0) = f_0(x_0,y_0)e^{-jk_x x_0}, \tag{IV.19}$$

here $f_0(x_0,y_0)$ is assumed to vary with x_0 much less rapidly than $exp(-jk_x x_0$. Then the amplitude distribution at z is,

$$h \otimes u_0 = \frac{j}{\lambda z}\int_{-\infty}^{\infty} dx_0 \int_{-\infty}^{\infty} dy_0 f_0(x_0,y_0)e^{-jk_x x_0}e^{-j(k/2z)[(x-x_0)^2+(y-y_0)^2)]} \tag{IV.20}$$

$$\simeq \sqrt{\frac{j}{\lambda z}}\int_{-\infty}^{\infty} dx_0 f_0(x_0,y)e^{-j(k/2z)[x-(k_x/k)z-x_0)]^2}e^{-jk_x x}e^{j(k/2)(k_x/k)^2 z} \tag{IV.21}$$

$$\simeq f_0(x - \frac{k_x}{k}z, y)e^{-jk_x x}e^{j(k/2)(kx/k)^2 z}, \tag{IV.22}$$

the profile is undistorted but shifts in the transverse direction as it propagates along z.

D. Fraunhofer diffraction

Fraunhofer diffraction is the limit of Fresnel diffraction for large distances between the input plane at $z = 0$, at which $u_0(x_0, y_0)$ is specified, and the observation plane at z. In this limit, the *far-field limit*, one approximates the argument of the exponential,

$$\frac{k}{z}[(x-x_0)^2 + (y-y_0)^2] \simeq \frac{k}{z}[(x^2+y^2) - 2xx_0 - 2yy_0], \tag{IV.23}$$

and ignores the term $k(x_0^2 + y_0^2)/z$. Thus the Fraunhofer approximation is valid if the amplitude distribution in the input plane extends over a transverse dimension d such that

$$d \ll \sqrt{\frac{z}{k}}. \tag{IV.24}$$

In this limit

$$u(x,y,z) = \frac{j}{\lambda z} e^{-j[k(x^2+y^2)/2z]} \int_{-\infty}^{\infty} dx_0 \int_{-\infty}^{\infty} dy_0 u_0(x_0,y_0) exp[\frac{jk}{z}(xx_0 + yy_0)]. \tag{IV.25}$$

The integral over x_0 and y_0 produces the Fourier transform of $u_0(x_0, y_0)$, denoted by U_0,

$$u(x,y,z) = j\frac{(2\pi)^2}{\lambda z} e^{-j[k(x^2+y^2)/2z]} U_0(\frac{kx}{z}, \frac{ky}{z}). \tag{IV.26}$$

Note that there is a phase factor multiplying the amplitude distribution: $\psi(x,y,z) = u(x,y,z)exp(-jkz)$, where

$$\text{phase factor} = exp\{-j[\frac{k(x^2+y^2}{2z} + kz]\}, \tag{IV.27}$$

indicating that the phase front is curved, with the equation of constant phase,

$$\frac{k^2(x^2+y^2)}{2\phi} + kz = \phi. \tag{IV.28}$$

This is the equation of a paraboloid, which has the radius of curvature R at x =0,

$$\frac{1}{R} = \frac{-d^2z/dx^2}{\sqrt{1+(dz/dx)^2}^3} = \frac{k}{\phi} = \frac{1}{z}. \tag{IV.29}$$

E. Single rectangular slit

Consider as an example the uniform illumination of a slit at $z = 0$,

$$u_0(x_0,y_0) = \begin{cases} 1, & |x_0| < d_x/2; \quad 0 < |y_0| < d_y/2 \\ 0, & d_x/2 \le |x_0|; \quad\quad d_y/2 \le |y_0| \end{cases} \tag{IV.30}$$

Then

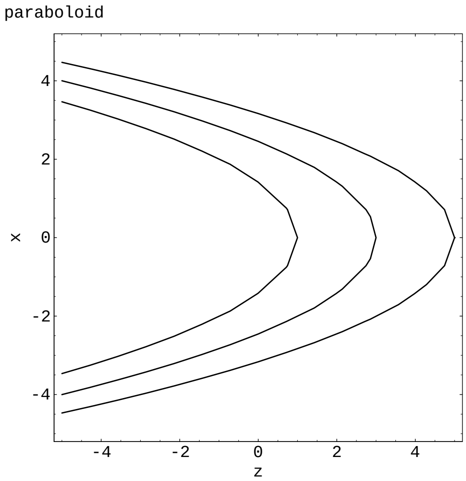
$$u(x,y,z) = \frac{j}{\lambda z} exp[-\frac{jk(x^2+y^2)}{2z}] d_x d_y \frac{\sin(kd_x x/2z)\sin(kd_y y/2z)}{(kd_x x/2z)(kd_x x/2z)}. \tag{IV.31}$$

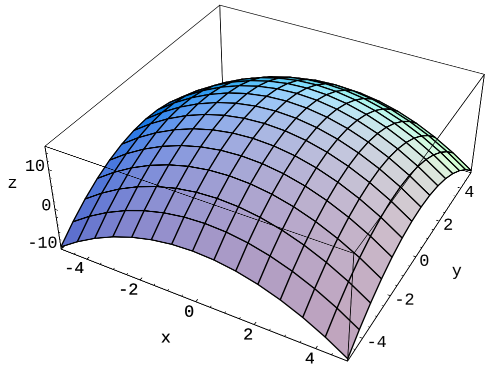
The widths of the diffraction patterns in the x and y directions are characterized by the first null at

$$x = \frac{2\pi z}{kd_x} \quad \text{and} \quad y = \frac{2\pi z}{kd_y}. \tag{IV.32}$$

the angles subtended from the center of the slit to the first nodal lines of the diffraction pattern are the diffraction angles,

$$\theta_x = \frac{2\pi}{kd_x} = \frac{\lambda}{d_x}, \qquad \theta_y = \frac{2\pi}{kd_y} = \frac{\lambda}{d_y}. \tag{IV.33}$$





F. Array of rectangular slits

Next, we consider an array of slits, N in number. They are excited uniformly and in phase. This array of slits can be the model of a grating. The Fraunhofer diffraction pattern is a summation over the spatially shifted amplitude distributions of the $n = 0$ slit.

$$u(x,y,z) = \frac{j}{\lambda z} e^{-[jk(x^2+y^2)/2z]} \{ \int_{\infty}^{\infty} dx_0^{'} \int_{\infty}^{\infty} dy_0^{'} u_0(x_0^{'}, y_0^{'}) exp[\frac{jk}{z}(xx_0^{'} + yy_0^{'})]\} \sum_{n=-[(N-1)/2]}^{(N-1)/2} exp(nj\frac{kx}{z}\Lambda). \quad (IV.34)$$

The diffraction pattern is that of a single slit multiplied by the array factor

$$A(\theta,\phi) = \sum_{n=-[(N-1)/2]}^{(N-1)/2} exp(jnk\Lambda \sin\theta\cos\phi), \quad (IV.35)$$

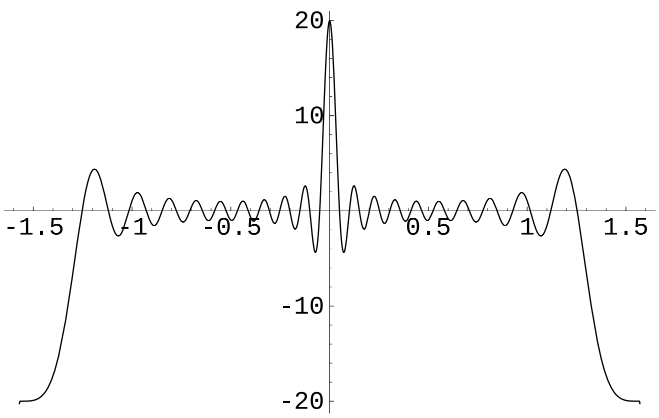
where we have introduced spherical coordinates and set

$$\frac{x}{r} \approx \frac{x}{z} = \sin\theta\cos\phi. \quad (IV.36)$$

The sum can be carried out in closed form:

$$A(\theta,\phi) = \frac{\sin(\frac{Nk\Lambda\sin\theta\cos\phi}{2})}{\sin(\frac{k\Lambda\sin\theta\cos\phi}{2})}. \quad (IV.37)$$

Array factor for array of slits, N=20, kL = 2p.



1. *Spatial coherence*

Fraunhofer diffraction of two slits provides a means for measuring spatial coherence. For two slits separated by a distance Λ and illuminated by wave amplitudes $a(0)$ and $a(\Lambda)$ gives, in the paraxial limit,

$$u = u_0(x, y, z)[a(0) + a(\Lambda)exp(-jk\Lambda\theta)], \tag{IV.38}$$

where $u_0(x, y, z)$ is the diffraction pattern of the slit centered at $x_0 = y_0 = 0$. The time-averaged intensity pattern is then

$$< |u|^2 >= |u_0|[< |a(0)|^2 > + < |a(\Lambda)|^2 + < a(0)a(\Lambda)^* > exp(jk\Lambda\theta) + < a(0)^* a(\Lambda) > exp(-jk\Lambda\theta)]. \tag{IV.39}$$

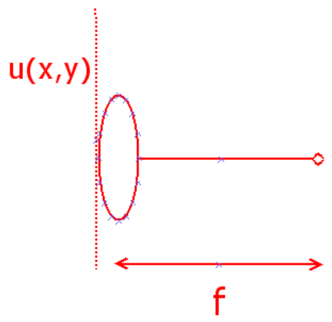
The observation of the fringe pattern of a pair of slits as a function of slit separation yields information about the coherence length.

G. The action of a thin lens

Consider a wave with the complex amplitude profile $u(x, y)$ incident upon a doubly convex *converging* lens of focal distance f. The portion of the wavefront near the axis of the lens travels through a thicker section of the optically dense material of the lens than the portion farther away from the axis. Tf the lens is *thin*, then the output profile $u(x, y)$ has the same amplitude as the input profile, the lens only introduces an (x, y)-dependent phase delay $\phi(x, y)$,

$$\phi(x, y) = \phi_0 - \frac{k}{2f}(x^2 + y^2), \tag{IV.40}$$

where ϕ_0 is the delay at the center, and f is the focal distance. Ignoring the constant phase, the action of the lens is



represented by the output(u')-input(u) relation:

$$u'(x, y) = u(x, y)exp[j\frac{k}{2f}(x^2 + y^2)]. \tag{IV.41}$$

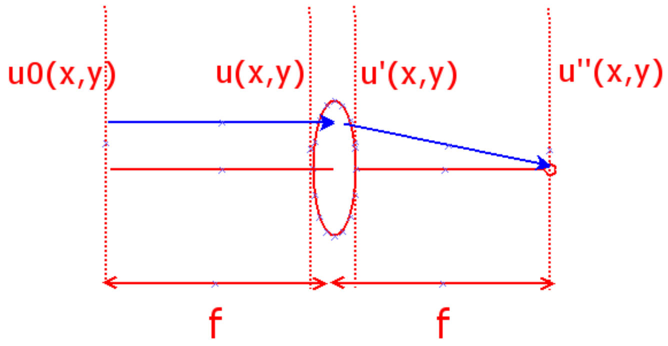
1. *Fourier transformation by a lens*

Consider a general illumination over the input plane, $u_0(x_0, y_0)$. If the lens is in the far-field, then the excitation at the front face reference plane of the lens is ,

$$u(x, y) = \frac{j(2\pi)^2}{\lambda f} e^{-[jk(x^2+y^2)/2f]} U_0(kx/f, ky/f). \tag{IV.42}$$

The transmission through the lens removes the exponential factor, then at the output plane,

$$u''(x, y) = \frac{j(2\pi)^2}{\lambda f} U_0(kx/f, ky/f), \tag{IV.43}$$



H. The paraxial wave equation

There are two equivalent descriptions of the response(output) of a linear system to an input excitation. One starts with the impulse response of the system and writes the output in terms of the convolution of the impulse response and excitation (*by Fresnel diffraction integral*). The other approach starts with the differential equation of the system and looks for particular and homogeneous solutions of these equations (*by paraxial wave equation*).
Suppose the $x-y$ dependence of the function $\psi(x,y,z)$ is much less rapid than its z dependence, then we can write the function $\psi(x,y,z)$ as a product of $exp(-jkz)$ times a factor $u(x,y,z)$,

$$\psi(x,y,z) = u(x,y,z)e^{-jkz}. \tag{IV.44}$$

By the wave equation of ψ, one obtains

$$\nabla_T^2 u + \frac{\partial^2}{\partial z^2}u - 2jk\frac{\partial u}{\partial z} = 0, \tag{IV.45}$$

where

$$\nabla_T \equiv \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}. \tag{IV.46}$$

Because $|(\partial/\partial z)u| \ll ku$ we may neglect $\partial^2 u/\partial z^2$ compared with $k(\partial u/\partial z)$ and obtain the *paraxial wave equation*

$$\nabla_T^2 u - 2jk\frac{\partial u}{\partial z} = 0. \tag{IV.47}$$

It is easy to confirm by differentiation that $h(x,y,z)$ of the Fresnel diffraction kernel is an exact solution of the paraxial wave equation, as an impulse response must be. The paraxial wave equation gives a simple picture of the incipient effect of diffraction, at small distances from the input plane positioned, say, at $z=0$,

$$\Delta u = -\frac{j}{2k}\Delta z \nabla_T^2 u. \tag{IV.48}$$

Suppose that we start with a plane wavefront, real u. Then the initial effect of diffraction is to add a quadrature component to u . The phase delay $\Delta\phi$ imparted to u is,

$$\Delta\phi = \frac{\Delta z}{2k}\frac{\nabla_T^2 u}{u}. \tag{IV.49}$$

I. Extended studies

1. Chromatic resolving power of grating spectrometers
2. Fresnel light
3. Non-paraxial wave equation
4. Lens design
5. Sub-wavelength images