V. HERMITE-GAUSSIAN BEAMS

A. Gaussian beams

The exact solution of the scalar wave equation reduces to an exact solution of the paraxial wave equation,

$$\nabla_T^2 u - 2jk\frac{\partial u}{\partial z} = 0, \tag{V.1}$$

where

$$\nabla_T \equiv \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}. \tag{V.2}$$

This solution is proportional to the impulse response function (Fresnel kernel),

$$h(x,y,z) = \frac{j}{\lambda z}e^{-jk[(x^2+y^2)/2z]}, \tag{V.3}$$

i.e.

$$\nabla_T^2 h(x,y,z) - 2jk\frac{\partial h}{\partial z} = 0. \tag{V.4}$$

The paraxial wave equation is invariant with respect to a translation of the coordinate z, $z \to z - z_0$. Thus another solution of the equation is $h(x, y, z - z_0)$. To remove the singularity of the solution on the real z axis, the solution of the paraxial wave equation can be constructed by making z_0 imaginary, $z_0 = -jb$. Then the solution of the scalar paraxial wave equation becomes,

$$u_{00}(x,y,z) = c_1\frac{j}{\lambda(z+jb)}e^{-jk[\frac{x^2+y^2}{2(z+jb)}]}, \tag{V.5}$$

where c_1 is a normalization constant, which can be defined by the normalization condition,

$$\int_{-\infty}^{\infty} d\,x \int_{-\infty}^{\infty} d\,y |u_{00}(x,y,0)|^2 = 1. \tag{V.6}$$

When the integral is carried out, it is found that the required constant c_1 is $\sqrt{2b\lambda}$. Thus

$$u_{00}(x,y,z) = j\sqrt{\frac{kb}{\pi}}(\frac{1}{z+jb})e^{-jk[\frac{x^2+y^2}{2(z+jb)}]}. \tag{V.7}$$

Separation of the argument of the exponential into real and imaginary parts, put u_{00} into the form,

$$u_{00}(x,y,z) = \frac{\sqrt{2}}{\sqrt{\pi}w}exp(j\phi)exp(-\frac{x^2+y^2}{w^2})exp[-\frac{jk}{2R}(x^2+y^2], \tag{V.8}$$

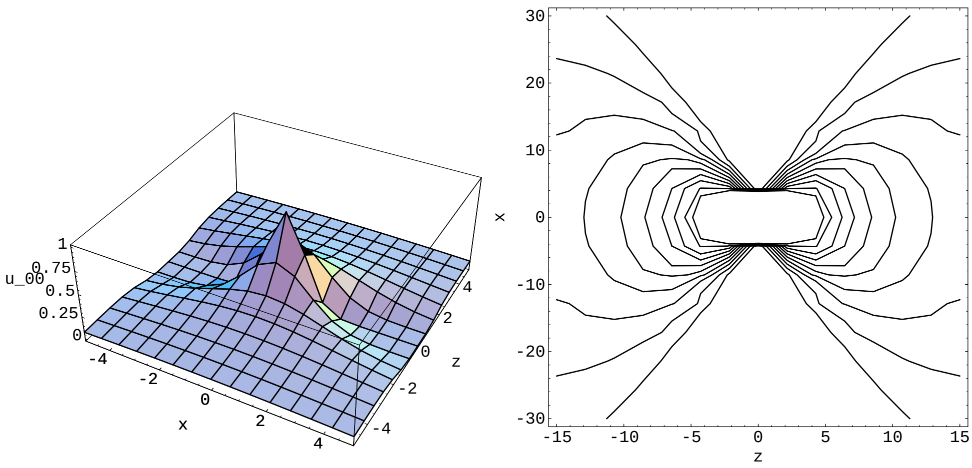
where

$$\text{beam width} \qquad w^2(z) = \frac{2b}{k}(1+\frac{z^2}{b^2} = w_0^2[1+(\frac{\lambda z}{\pi w_0^2})^2], \tag{V.9}$$

$$\text{radius of phase front} \qquad \frac{1}{R(z)} = \frac{z}{z^2+b^2} = \frac{z}{z^2+(\pi w_0^2/\lambda)^2}, \tag{V.10}$$

$$\text{phase delay} \qquad \tan\phi = \frac{z}{b} = \frac{z}{\pi w_0^2/\lambda}, \tag{V.11}$$

with the minimum beam radius $w_0 = \sqrt{2bk}$ and the confocal parameter $b = kw_0^2/2 = \pi w_0^2/\lambda$. The solution is a wave traveling in the $+z$ direction with a Gaussian amplitude profile and curved phase fronts of radius $R(z)$.



1. *The Electric and Magnetic fields of Gaussian beam*

The magnetic field H and the electric field E follow from the complex vector potential A,

$$\begin{aligned} \mu_0 H &= \nabla \times A; & \text{(V.12)} \\ E &= -j\omega A - \nabla \Phi, & \text{(V.13)} \end{aligned}$$

and according to the Lorentz gauge, the scalar potential is

$$\Phi = \frac{j}{\omega \mu_0 \epsilon_0} \nabla \cdot A. \tag{V.14}$$

Suppose that A is polarized along $\hat{x}$,

$$A = \hat{x} u_{00}(x, y, z) e^{-jkz}. \tag{V.15}$$

Thus

$$\mu_0 H = \nabla \times [\hat{x} u_{00}(x, y, z) e^{-jkz}] = -j[\hat{y} u_{00} - j\hat{z} \frac{\partial u_{00}}{k \partial y}] e^{-jkz}, \tag{V.16}$$

where we have ignored $\partial u_{00}/\partial z$ compared with ku_{00}. The electric field is,

$$\begin{aligned} E &= -j\omega[\hat{x} u_{00} - j\hat{z} \frac{\partial u_{00}}{k \partial x}] e^{-jkz} & \text{(V.17)} \\ &= \sqrt{\frac{kb}{\pi}} (\frac{\omega}{z + jb}) e^{-jk[\frac{x^2+y^2}{(z+jb)}]} exp(-jkz)[(\hat{x} - \hat{z}\frac{x}{R}) + j\hat{z}\frac{xb}{z^2 + b^2}]. & \text{(V.18)} \end{aligned}$$

B. Resonators with curved mirrors

Consider the Gaussian beam solution propagating in the $-z$ direction,

$$u_{00}(x, y, -z) = -j\sqrt{\frac{kb}{\pi}} (\frac{1}{z - jb}) e^{+jk[\frac{x^2+y^2}{2(z-jb)}]}, \tag{V.19}$$

and superposition of two gaussian beams propagating in the $+z$ and $-z$ directions produces standing waves. If mirror 1 is placed at position z_1, with the radius of curvature,

$$\frac{1}{R_1} = \frac{z_1}{z_1^2 + b^2}, \tag{V.20}$$

and mirror 2 at position z_2 with the radius of curvature,

$$\frac{1}{R_2} = \frac{z_2}{z_2^2 + b^2}. \tag{V.21}$$

Given a set of mirrors of radii of curvature R_1 and R_2 and their spacing d, $z_1 - Z_2 = d$, the resonant mode appropriate for this configuration is sought.

1. *One plane, one concave mirror*

When $R_1 = \infty$, $R_2 = R_0$, then $z_1 = 0$, $z_2 = d$, and

$$b = \frac{\pi w_0^2}{\lambda} = \sqrt{d(R_0 - d)}. \tag{V.22}$$

The minimum beam radius of the resonant mode occurs at the flat mirror when $R_0 > d$.

2. *The symmetric case*

When $R_1 = R_2 = R_0$, the distance d must be interpreted as $d/2$ and one has,

$$b = \frac{\pi w_0^2}{\lambda} = \sqrt{\frac{d}{2}(R_0 - \frac{d}{2})}. \tag{V.23}$$

3. *The general case*

When $R_1 \neq R_2$, one finds that

$$\frac{R_1}{2} + \frac{R_2}{2} \pm \sqrt{\frac{R_1^2}{4} - b^2} \pm \sqrt{\frac{R_2^2}{4} - b^2} = d, \tag{V.24}$$

and the equation for b^2,

$$4b^2 = (\frac{R_1^2 R_2^2}{4})\frac{1 - [(2d/R_1 R_2)(d - R_1 - R_2) + 1]^2}{[d - (R_1 + R_2)/2]^2}. \tag{V.25}$$

For real solution for b, one requires that

$$0 \leq (1 - \frac{d}{R_1})(1 - \frac{d}{R_2}) \leq 1. \tag{V.26}$$

Resonators for which no solutions of Gaussian modes exist that reproduce themselves are *unstable* resonators.

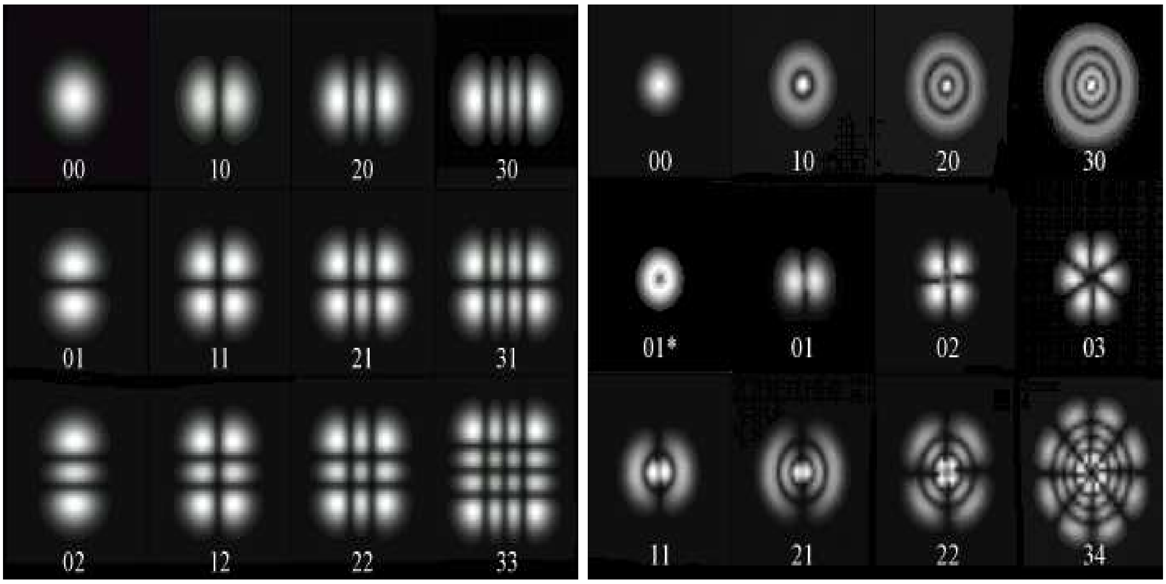
C. Higher-order modes

A Hermite-Gaussian of order m of the independent variable η is defined as the product of a Hermite polynomial of order m, $H_m(\zeta)$, and the Gaussian $exp(-\zeta^2/2)$. The lowest-order Hermite polynomials are,

$$H_0(\zeta) = 1; \tag{V.27}$$
$$H_1(\zeta) = 2\zeta; \tag{V.28}$$
$$H_2(\zeta) = 4\zeta^2 - 2. \tag{V.29}$$

Then the two-dimensional Hermite-Gaussian forward traveling wave of the modes of order m, n at $z = 0$ is

$$u_{mn}(x_0, y_0) = C_{mn}\psi_m(\frac{\sqrt{2}x_0}{w_0})\psi_n(\frac{\sqrt{2}y_0}{w_0}), \tag{V.30}$$



where $\psi_m(\zeta) \equiv H_m(\zeta)e^{-\zeta^2/2}$.

Assume a trial solution to the paraxial wave equation of the form,

$$u(x, y, z) = A(z)exp[-jk\frac{x^2 + y^2}{2q(z)}], \tag{V.31}$$

where $A(z)$ and $q(z)$ are initially unknown. If we substitute this trial solution into the paraxial wave equation, we obtain

$$[(\frac{k}{q})^2(\frac{d\,q}{d\,z} - 1)(x^2 + y^2) - \frac{2jk}{q}(\frac{q}{A}\frac{d\,A}{d\,z} + 1)]A(z) = 0. \tag{V.32}$$

Now, the only way in which this equation can be satisfied for all x and y is to set both of the differential expressions inside the large square brackets to zero, i.e.,

$$\frac{d\,q(z)}{d\,z} = 1 \qquad \text{and} \qquad \frac{d\,A(z)}{d\,z} = -\frac{A(z)}{q(z)}, \tag{V.33}$$

whose solutions are given by

$$q(z) = q_0 + z \equiv z + jb, \tag{V.34}$$

$$\frac{A(z)}{A_0} = \frac{q_0}{q(z)}. \tag{V.35}$$

The same trial solution approach can be extended to find high-order Hermite-Gaussian modes $u_{mn}(x, y, z)$ or Laguerre-Gaussian modes $u_{pm}(r, \theta, z)$ to the paraxial wave equation. In rectangular coordinates the elementary solutions can be separated into products of identical solutions in the x and y directions, i.e.,

$$u_{mn}(x, y, z) = u_n(x, z) \times u_m(y, z), \tag{V.36}$$

where $u_n(x, z)$ and $u_m(y, z)$ have the same mathematical form for the paraxial wave equation in one transverse coordinate,

$$\frac{\partial^2\, u_n(x, z)}{\partial x^2} - 2jk\frac{\partial u_n(x, z)}{\partial z} = 0. \tag{V.37}$$

Again, we now write a more general trial solution for the wave amplitude $u_n(x, z)$ in the form

$$u_n(x, z) = A(z) \times H_n[\frac{x}{p(z)}] \times exp[-jk\frac{x^2 + y^2}{2q(z)}], \tag{V.38}$$

where $p(z)$ is a distance-dependent scaling factor. Substituting this form into the paraxial wave equation, the paraxial wave equation then is converted into a differential relation for $H_n(x/p)$,

$$H_n^{''} - 2jk[\frac{p}{q} - p^{'}]xH_n^{'} - \frac{jkp^2}{q}[1 + \frac{2q}{A}\frac{dA}{dq}]H_n = 0, \tag{V.39}$$

where $H_n^{'}$ and $H_n^{''}$ mean the first and second derivatives of H_n with respect to its total argument. If we can find solutions for $p(z)$ and $A(z)$ which satisfy simultaneously the two conditions,

$$2jk[\frac{p}{q} - p^{'}] = \frac{2}{p} \qquad \text{or} \qquad \frac{dp}{dz} = \frac{p}{q} + \frac{j}{kp}, \tag{V.40}$$

$$-\frac{jkp^2}{q}[1 + \frac{2q}{A}\frac{dA}{dq}] = 2n \qquad \text{or} \qquad \frac{2q}{A}\frac{dA}{dq} = \frac{2jnkp^2}{q} - 1. \tag{V.41}$$

The the differential relation for H_n becomes the Hermite polynomials function,

$$H_n^{''} - 2(x/p)H_n^{'} + 2nH_n = 0, \tag{V.42}$$

with the solutions $H_n(x/p)$. The standard approach to Hermite polynomial solutions is obtained by assuming that the scalar factor $p(z)$ is purely real,

$$\frac{1}{p(z)} = \frac{\sqrt{2}}{w(z)} = \sqrt{\frac{jk}{2q(z)}}. \tag{V.43}$$

The the higher-order modes are

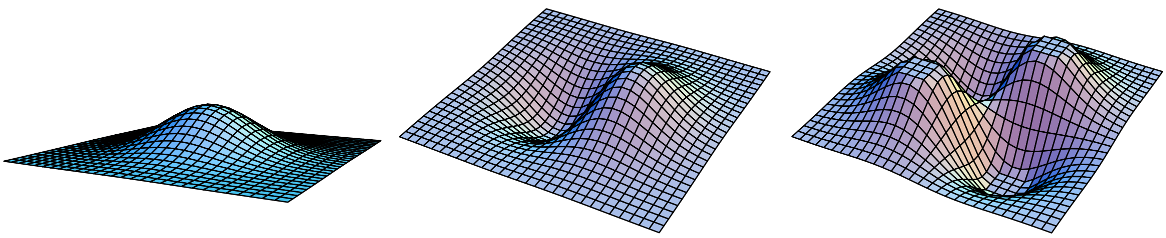
$$u_n(x,z) = H_n[\frac{\sqrt{2}z}{w(z)}] \times exp[\frac{-jkx^2}{2R(z)} - \frac{x^2}{w^2(z)}], \tag{V.44}$$

where the q-parameter become

$$\frac{1}{q(z)} = \frac{1}{R(z)} - j\frac{\lambda}{\pi w^2(z)}. \tag{V.45}$$

The complete normalized higher-order Hermite-Gaussian mode for a beam propagating in free-space is

$$u(x,z) = (\frac{2}{\pi})^{1/4}(\frac{1}{2^n n!\, w_0})^{1/2}(\frac{q_0}{q(z)})^{1/2}[\frac{q_0 q^*(z)}{q_0^* q(z)}]^{n/2} \times H_n[\frac{\sqrt{2}x}{w(z)}] \times exp[-j\frac{kx^2}{2q(z)}]. \tag{V.46}$$



D. Extended studies

1. Orthogonality property of Hermite-Gaussian modes
2. Laguerre-Gaussian modes
3. The q parameter and $ABCD$ matrix in ray optics
4. Methods of testing optical systems
5. Aberrations
6. The Twyman-Green interferometer