VI. OPTICAL WAVEGUIDES AND FIBERS

A. Wave equation in nonuniform dielectric

In a source-free, nonuniform, dielectric medium, the wave equation for the vector potential becomes,

$$\nabla^2 A - \mu_0 \epsilon \frac{\partial^2 A}{\partial t^2} = -\mu_0 (\nabla \epsilon) \frac{\partial}{\partial t} \Phi. \tag{VI.1}$$

The *nonuniform* dielectric couples the vector potential to the scalar potential, and vice versa. On the other hand, in a *uniform* charge-free medium, the wave equations for A and Φ are uncoupled,

$$\nabla^2 A - \mu_0 \epsilon \frac{\partial^2 A}{\partial t^2} = 0 \tag{VI.2}$$

$$\nabla^2 \Phi - \mu_0 \epsilon \frac{\partial^2 \Phi}{\partial t^2} = 0. \tag{VI.3}$$

The coupling is weak when the spatial variation of ϵ is small over distances of the order of a wavelength, then we can neglect this coupling entirely,

$$\nabla^2 A - \mu_0 \epsilon \frac{\partial^2 A}{\partial t^2} \approx 0, \tag{VI.4}$$

here $\epsilon(\rho)$ is not a constant, which represents an axially uniform medium, but with radial variation. We write

$$A = \hat{y} u(x, y) e^{-j\beta z}, \tag{VI.5}$$

where β is an unknown propagation constant. The wave equation for $u(x, y)$ is,

$$\nabla_T^2 u + [\omega^2 \mu_0 \epsilon(\rho) - \beta^2] u = 0, \tag{VI.6}$$

where $\rho = \hat{x} x + \hat{y} y$. Compared to the Schrödinger equation of a particle in a two-dimensional potential V, i.e.

$$\frac{-\hbar^2}{2m} \nabla_T^2 \Psi + (V - E) \Psi = 0, \tag{VI.7}$$

then the solutions are bounded, if and only if, there are local negative values of the function,

$$|V| - |E| > 0, \tag{VI.8}$$

with

$$V = -\omega^2 \mu_0 \epsilon(\rho), \tag{VI.9}$$

$$E = -\beta^2. \tag{VI.10}$$

Bounded solutions correspond to guided waves and are found only for specific values of the eigenvalues β^2.

B. Parabolic profile of dielectric constant

We assume a dielectric constant profile that depends quadratically on the distance from the axis:

$$\omega^2 \mu_0 \epsilon(\rho) = \omega^2 \mu \epsilon(0)(1 - \frac{x^2 + y^2}{h^2}). \tag{VI.11}$$

Denote by k_0 the propagation constant of an infinite parallel plane wave propagating in a medium of uniform dielectric constant $\epsilon(0)$,

$$k_0 \equiv \omega \sqrt{\mu_0 \epsilon(0)}. \tag{VI.12}$$

With the normalized variables,

$$\xi = \sqrt{\frac{k_0}{h}}x, \tag{VI.13}$$

$$\eta = \sqrt{\frac{k_0}{h}}y, \tag{VI.14}$$

then the wave equation for the u becomes,

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + [\lambda^2 - (\xi^2 + \eta^2)]u = 0, \tag{VI.15}$$

where

$$\lambda = \frac{k_0^2 - \beta^2}{k_0}h. \tag{VI.16}$$

The solutions to this equation are Hermite-Gaussian polynomials,

$$u_{mn}(\xi, \eta) = H_m(\xi)H_n(\eta)e^{-(\xi^2+\eta^2)/2}, \tag{VI.17}$$

with

$$\lambda = 2(m+n) + 2, \qquad m, n = 0, 1, 2, \ldots \tag{VI.18}$$

The fundamental mode, with $\lambda_{00} = 2$, has the pattern

$$u_{00}(x, y) = \sqrt{\frac{2}{\pi}}\frac{1}{w}exp(-\frac{x^2 + y^2}{w^2}), \tag{VI.19}$$

where

$$w^2 = \frac{2h}{k_0} = \frac{h\lambda}{\pi}\sqrt{\frac{\epsilon_0}{\epsilon(0)}}. \tag{VI.20}$$

The propagation constant of the mn modes is,

$$\beta_{mn}^2 = k_0^2(1 - \frac{\lambda_{mn}}{k_0 h}). \tag{VI.21}$$

The propagation constant is smaller than k_0, thus indicating that the phase velocity of the mode is greater than the speed of an infinite parallel plane wave in a medium with a uniform dielectric constant $\epsilon = \epsilon(0)$. When $\lambda_{mn}/(k_0 h) \ll 1$, one may approximate the propagation constant by

$$\beta_{mn} \approx k_0(1 - \frac{m+n+1}{k_0 h}). \tag{VI.22}$$

The inverse group velocity is then

$$\frac{d\beta_{mn}}{d\omega} = \sqrt{\mu_0 \epsilon(0)}, \tag{VI.23}$$

independent of mode number m and n. The modes in a parabolic index profile are an approximate representation of modes in *multimode* fibers.

C. The dielectric slab waveguides

Transverse Electric modes:

$$E_y = A\cos(k_x x)e^{-j\beta z}, \qquad |x| < d, \tag{VI.24}$$

$$E_y = Be^{-j\beta z}e^{-\alpha_x x}, \qquad x > d, \tag{VI.25}$$

$$= Be^{-j\beta z}e^{\alpha_x x}, \qquad x < -d, \tag{VI.26}$$

and the magnetic field follows from Faraday's law,

$$H_z = -\frac{jk_x}{\omega\mu_0} A\sin(k_x x)e^{-j\beta z}, \qquad |x| < d, \tag{VI.27}$$

$$E_y = -\frac{j\alpha_x}{\omega\mu_0} Be^{-j\beta z}e^{-\alpha_x x}, \qquad x > d, \tag{VI.28}$$

$$= \frac{j\alpha_x}{\omega\mu_0} Be^{-j\beta z}e^{\alpha_x x}, \qquad x < -d. \tag{VI.29}$$

Continuity of E_y/H_z at $x = d$ gives,

$$\tan(k_x d) = \frac{\alpha_x}{k_x}. \tag{VI.30}$$

From the wave equation, we have

$$\beta^2 - \alpha_x^2 = \omega^2\mu_0\epsilon, \tag{VI.31}$$

$$\beta^2 + k_x^2 = \omega^2\mu_0\epsilon_i, \tag{VI.32}$$

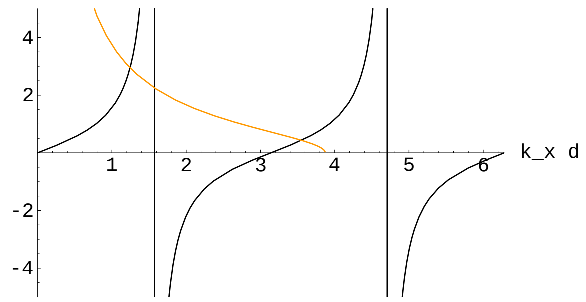
and combine these two equations,

$$\frac{\alpha_x}{k_x} = \sqrt{\frac{\omega^2\mu_0(\epsilon_i - \epsilon)}{k_x^2} - 1}, \tag{VI.33}$$

one can find the dispersion diagram, the dependence of the propagation constant β on frequency,

$$\tan(k_x d) = \sqrt{\frac{\omega^2\mu_0(\epsilon_i - \epsilon)}{k_x^2} - 1}, \tag{VI.34}$$

For decreasing ω, α_x/k_x moves toward the origin and intersections are lost, except for the first branch of the tan



function. This corresponds to the dominant mode, $m = 0$, with no *cutoff*. At low frequency, the fundamental mode acquires a small k_x, $\tan(k_x d) \approx k_x d$,

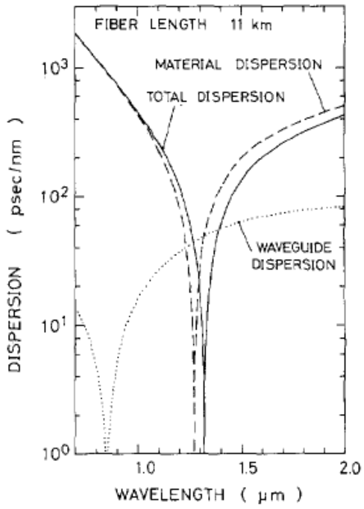
$$\omega^2\mu_0(\epsilon_i - \epsilon)d^2 - k_x^2 d^2 \approx k_x^4 d^4. \tag{VI.35}$$

Neglecting $k_x^4 d^4$ compared with $k_x^2 d^2$, we have $k_x^2 = \omega^2\mu_0(\epsilon_i - \epsilon)$ and $\beta \approx \omega\sqrt{\mu_0\epsilon}$. The wave propagates at the speed characteristic of the external region. In the other hand, when $\omega -> \infty$, $k_x d$ approaches $\pi/2$ and we find that $\beta \approx \omega\sqrt{\mu_0\epsilon_i}$.

D. Single-mode fiber

The propagation constant for a step index profile is bracketed between the limits,

$$\omega^2\mu_0\epsilon < \beta^2 < \omega^2\mu_0\epsilon_i. \tag{VI.36}$$



By making $(\epsilon_i - \epsilon)/\epsilon$ small enough, one can prevent solutions from occurring with greater curvatures at any given frequency and thus prevent the existence of any higher-order modes. In practice, the waveguide dispersion for a single mode fiber is overshadowed by the material dispersion, the frequency dependence of the dielectric constant, $\epsilon(\omega)$. suppose that we have solved the eigenvalue problem and have determined the propagation constant β as a function of frequency. The spatial dependence of the vector potential A_y is then,

$$A_y = a(z)u(x, y), \tag{VI.37}$$

where a_z is the amplitude with the z dependence $ex[-j\beta(\omega)z]$. If the spectrum $a(z, \omega)$ is narrow, and centered around ω_0 so that $\beta(\omega)$ can be expanded into,

$$\frac{\partial a(z,\omega)}{\partial z} = -j[\beta(\omega_0) + \frac{d\beta}{d\omega}(\omega - \omega_0) + \frac{1}{2}\frac{d^2\beta}{d\omega^2}(\omega - \omega)^2)]a. \tag{VI.38}$$

In terms of a function of $(\omega - \omega_0)$,

$$a(z, \omega) = A(z, \omega - \omega_0)exp[-j\beta(\omega_0)z], \tag{VI.39}$$

then the equation of motion for the *envelope* A is,

$$\frac{\partial}{\partial z}A(z, \omega - \omega_0) = -j[\frac{d\beta}{d\omega}(\omega - \omega_0) + \frac{1}{2}\frac{d^2\beta}{d\omega^2}(\omega - \omega)^2)]A(z, \omega - \omega_0), \tag{VI.40}$$

and its inverse Fourier transform,

$$(\frac{\partial}{\partial z} + \frac{1}{v_g}\frac{\partial}{\partial t})A(z, t) = \frac{j}{2}\frac{d^2\beta}{d\omega^2}\frac{\partial^2 A(z, t)}{\partial t^2}, \tag{VI.41}$$

where

$$\frac{1}{v_g} = \frac{d\beta}{d\omega}. \tag{VI.42}$$

The envelope $A(z, t)$ would propagate at the group velocity, but with *dispersion* effect.

E. Extended studies

1. Propagation of Gaussian pulses,
2. The solutions of Bessel functions for the single mode fibers with step index profile,
3. Index change in a fiber,
4. Orthogonality of guided waves,
5. A. W. Snyder and J. D. Love, *"Optical waveguide theory,"* Chapman & Hall (1995).