VII. COUPLED-MODE THEORY FOR RESONATORS AND COUPLERS

A. Single resonator

For a simple LC circuit,

$$v = L\frac{d\,i}{d\,t}, \tag{VII.1}$$

$$i = -C\frac{d\,v}{d\,t}, \tag{VII.2}$$

the two coupled first-order differential equations lead to the second-order differential equation for the voltage,

$$\frac{d^2\,v}{d\,t^2} + \omega_0^2 v = 0, \tag{VII.3}$$

where

$$\omega_0^2 = \frac{1}{LC}. \tag{VII.4}$$

Instead of the coupled first-order differential equations we may derive two uncoupled first-order differential equations, by defining the complex variables,

$$a_\pm = \sqrt{\frac{C}{2}}(v \pm j\sqrt{\frac{L}{C}}i), \tag{VII.5}$$

then,

$$\frac{d\,a_+}{d\,t} = j\omega_0 a_+, \tag{VII.6}$$

$$\frac{d\,a_-}{d\,t} = -j\omega_0 a_-. \tag{VII.7}$$

Therefore, a_+ is the *positive-frequency component* of the mode amplitude,

$$a_+ = \sqrt{\frac{C}{2}}Ve^{j\omega_0 t}, \tag{VII.8}$$

is normalized with the energy, W, in the circuit,

$$|a_+|^2 = \frac{C}{2}|V|^2 = W. \tag{VII.9}$$

If the circuit is lossy, the loss may e represented by a conductance G in parallel with L and C,

$$\frac{d\,a}{d\,t} = j\omega_0 a - \frac{1}{\tau_o}a, \tag{VII.10}$$

where $1/\tau_0$ is the decay rate due to the loss. The time-average loss is,

$$P_d = \frac{1}{2}G|V|^2 = \frac{G}{C}|a|^2, \tag{VII.11}$$

where the dimensionless quantity,

$$\frac{P_d}{\omega_0 W} = \frac{G}{\omega_0 C} = \frac{2}{\omega_0 \tau_0} = \frac{1}{Q_0}, \tag{VII.12}$$

is the inverse unloaded Q or quality factor.
If we had started from the equations of the circuit and had computed the complex frequency $s = -(1/\tau) + \omega$ exactly, we would have obtained,

$$s = -\frac{1}{\tau} + j\omega = -\frac{G}{2C} + j\sqrt{\frac{1}{LC} - \frac{G^2}{4C^2}}. \tag{VII.13}$$

Note that the decay rate has been properly evaluated by the perturbation, but a correction to the frequency $\omega_0 = 1/\sqrt{LC}$ that is of second order in G/C.

B. Single resonator with input wave

If the resonator is coupled to an external waveguide or to the outside space by a partially transmitting mirror,

$$\frac{d\,a}{d\,t} = j\omega_0 a - (\frac{1}{\tau_0} + \frac{1}{\tau_e})a, \tag{VII.14}$$

where $1\tau_e$ expresses the additional rate of decay due to escaping power, and the "external" Q of the resonator is,

$$\frac{P_e}{\omega_0 W} = \frac{2}{\omega_0 \tau_e} = \frac{1}{Q_e}. \tag{VII.15}$$

The waveguide may carry a wave traveling toward the resonator of amplitude s_+ due to a source,

$$\frac{d\,a}{d\,t} = j\omega_0 a - (\frac{1}{\tau_0} + \frac{1}{\tau_e})a + \kappa s_+, \tag{VII.16}$$

where κ is a coefficient expressing the degree of coupling between the resonator and the wave s_+. We normalized s_+ so that,

$$|s_+|^2 = \text{power carried by incident wave}, \tag{VII.17}$$

in contrast to $|a|^2$, which is normalized to the *energy.*
If the source is at the frequency ω, $s_+ \propto \exp(j\omega t)$, then the response is at the same frequency,

$$a = \frac{\kappa s_+}{j(\omega - \omega_0) + (1/\tau_0 + 1/\tau_e)}. \tag{VII.18}$$

If there is no internal source, from the energy conservation,

$$\frac{d}{d\,t}|a|^2 = -\frac{2}{\tau_e}|a|^2 + |\kappa|^2 |s_+|^2 = 0, \tag{VII.19}$$

then,

$$|\kappa| = \sqrt{\frac{2}{\tau_e}}. \tag{VII.20}$$

The phase of κ can be disposed of by noting that the phase of a relative to s_+ can be defined arbitrarily. Thus,

$$\frac{d\,a}{d\,t} = j\omega_0 a - (\frac{1}{\tau_0} + \frac{1}{\tau_e})a + \sqrt{\frac{2}{\tau_e}} s_+. \tag{VII.21}$$

If the system is linear, so that s_- is the sum of a term proportional to s_+ and a term proportional to a,

$$s_- = c_s s_+ + c_a a, \tag{VII.22}$$

where

$$c_a = \sqrt{\frac{2}{\tau_e}}. \tag{VII.23}$$

The coefficient c_s can be evaluated by energy conservation,

$$|s_+|^2 - |s_-|^2 = \frac{d}{d\,t}|a|^2 + \frac{2}{\tau_0}|a|^2, \tag{VII.24}$$

and

$$\frac{d}{d\,t}|a|^2 = -2(\frac{1}{\tau_0} + \frac{1}{\tau_e})|a|^2 + \sqrt{\frac{2}{\tau_e}}(a^* s_+ + a s_+^*). \tag{VII.25}$$

Comparison of the coefficients, we obtain,

$$c_s = -1, \tag{VII.26}$$

thus,

$$s_- = -s_+ + \sqrt{\frac{2}{\tau_e}} a. \tag{VII.27}$$

The reflection coefficient of the resonator in the steady state is

$$\Gamma = \frac{s_-}{s_+} = \frac{1/\tau_e - 1/\tau_0 - j(\omega - \omega_0)}{1/\tau_e + 1/\tau_0 + j(\omega - \omega_0)}. \tag{VII.28}$$

If a resonator is connected to two guides, or power is coupled in and out at two mirrors as in the Fabry-Perot transmission resonator,

$$\frac{d\,a}{d\,t} = j\omega_0 a - (\frac{1}{\tau_0} + \frac{1}{\tau_{e1}} + \frac{1}{\tau_{e2}})a + \kappa_1 s_{+1} + \kappa_2 s_{+2}, \tag{VII.29}$$

where $1/\tau_{e1}$ and $1/\tau_{e2}$ express the contribution to the mode decay of the free running resonator due to the power escaping into each of the two guides, and the coupling coefficients,

$$\kappa_1 = \sqrt{\frac{2}{\tau_{e1}}}, \tag{VII.30}$$

$$\kappa_2 = \sqrt{\frac{2}{\tau_{e2}}}. \tag{VII.31}$$

The power transmitted to guide 2 from guide 1 is,

$$|s_{-2}|^2 = \frac{2|a|^2}{\tau_{e2}} = \frac{(4\tau^2/\tau_{e1}\tau_{e2})|s_{+1}|^2}{(\omega - \omega_0)^2\tau^2 + 1}, \tag{VII.32}$$

where $1/\tau = 1/\tau_{e1} + 1/\tau_{e2} + 1/\tau_0$. When there is no loss, $1/\tau = 1/\tau_{e1} + 1/\tau_{e2}$, and the transmitted power is at resonance, $\omega = \omega_o$,

$$|s_{-2}|^2 = \frac{(4/\tau_{e1}\tau_{e2})}{(1/\tau_{e1} + 1/\tau_{e2})^2}|s_{+1}|^2, \tag{VII.33}$$

which is the same result for the transmission of a Fabry-Perot interferometer at, and near, resonance of one of its modes, i.e.

$$|b_2|^2 = |S_{21}|^2|a_1|^2 = \frac{t_1^2 t_2^2 |a_1|^2}{(1 - r_1 r_2)^2 + 4r_1 r_2 \sin^2(\delta/2))}, \tag{VII.34}$$

or

$$|b_2|^2 = |a_1|^2 \frac{(1-R)^2}{(1-R)^2 + 4R\sin^2(\delta/2)}, \tag{VII.35}$$

when $t_1^2 = t_2^2 = 1 - r_1^2$, $r_1^2 = r_2^2 = r^2 \equiv R$, and $t_1^2 = t_2^2 = 1 - R$.

C. Coupling of two resonator modes

Consider the equation of motion of the amplitudes a_1 and a_2 of the modes of two uncoupled lossless resonators of natural frequencies ω_1 and ω_2,

$$\frac{d\,a_1}{d\,t} = j\omega_1 a_1, \tag{VII.36}$$

$$\frac{d\,a_2}{d\,t} = j\omega_2 a_2. \tag{VII.37}$$

Suppose that the modes are coupled through some small perturbation of the system, such as the small connecting capacitor, or by a change of the totally reflecting mirror between the two resonators to a partially transmitting mirror. This change can be described by

$$\frac{d\,a_1}{d\,t} = j\omega_1 a_1 + \kappa_{12} a_2, \quad \text{(VII.38)}$$

$$\frac{d\,a_2}{d\,t} = j\omega_2 a_2 + \kappa_{21} a_1, \quad \text{(VII.39)}$$

where κ_{12} and κ_{21} are the coupling coefficients. Here *week coupling* means that $|\kappa_{12} \ll \omega_1$ and $|\kappa_{21} \ll \omega_2$. Energy conservation imposes a restriction on κ_{12} and κ_{21}. The time rate of change of energy, which must vanish, is derived as,

$$\frac{d}{d\,t}(|a_1|^2 + |a_2|^2) = a_1^* \kappa_{12} a_2 + a_1 \kappa_{12}^* a_2^* + a_2^* \kappa_{21} a_1 + a_2 \kappa_{21}^* a_1^* = 0. \quad \text{(VII.40)}$$

Because the initial amplitudes and phases of a_1 and a_2 can be set arbitrarily, the coupling coefficients must be related by,

$$\kappa_{12} + \kappa_{21}^* = 0. \quad \text{(VII.41)}$$

1. Transient response

We solve for the natural frequencies of the coupled systems,

$$\frac{d}{d\,t}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} j\omega_1 & \kappa_{12} \\ \kappa_{21} & j\omega_2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \quad \text{(VII.42)}$$

which have two homogeneous solutions,

$$\omega = \frac{\omega_1 + \omega_2}{2} \pm \sqrt{(\frac{\omega_1 - \omega_2}{2})^2 + |\kappa_{12}|^2}, \quad \text{(VII.43)}$$

$$\equiv \frac{\omega_1 + \omega_2}{2} \pm \Omega_0. \quad \text{(VII.44)}$$

The two frequencies of the coupled system are "forced apart" by the coupling. Suppose, initially, that at $t = 0$, $a_1(0)$ and $a_2(0)$ are specified, then the two solutions are,

$$a_1(t) = [a_1(0)(\cos\Omega_0 t - j\frac{\omega_2 - \omega_1}{2\Omega_0}\sin\Omega_0 t) + \frac{\kappa_{12}}{\Omega_0} a_2(0)\sin\Omega_0 t] e^{j[(\omega_1+\omega_2)/2]t}, \quad \text{(VII.45)}$$

$$a_2(t) = [\frac{\kappa_{21}}{\Omega_0} a_1(0)\sin\Omega_0 t + a_2(0)(\cos\Omega_0 t - j\frac{\omega_1 - \omega_2}{2\Omega_0}\sin\Omega_0 t)] e^{j[(\omega_1+\omega_2)/2]t}. \quad \text{(VII.46)}$$

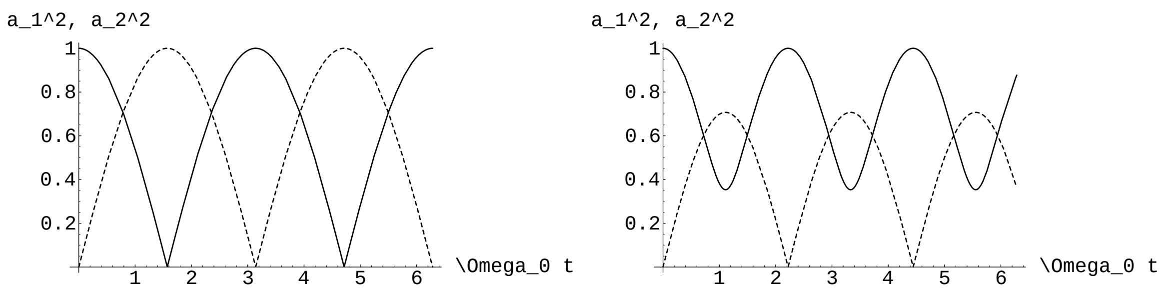

FIG. 1: Left: $\omega_1 - \omega_2 = 0$; Right: $\omega_1 - \omega_2 = 1$.

D. Coupling of modes in space

If two optical waveguides are coupled to each other via their fringing fields. A wave set up initially in one guide is transferred to the other guide. Consider two waves a_1 and a_2, of modes 1 and 2, which, in the absence of coupling , have propagation constants β_1 and β_2,

$$\frac{d\,a_1}{d\,z} = -j\beta_1 a_1, \tag{VII.47}$$

$$\frac{d\,a_2}{d\,z} = -j\beta_2 a_2. \tag{VII.48}$$

Suppose next that the two waves are weakly coupled by some means,

$$\frac{d\,a_1}{d\,z} = -j\beta_1 a_1 + \kappa_{12} a_2, \tag{VII.49}$$

$$\frac{d\,a_2}{d\,z} = -j\beta_2 a_2 + \kappa_{21} a_1. \tag{VII.50}$$

If power is to be conserved, there are restrictions imposed on κ_{12} and κ_{21}. Because the waves may carry power in opposite directions, we must distinguish the directions of power flow by a sign, $p_{1,2} = \pm 1$, depending upon whether the power flow is in the plus or minus z direction. The net power P is,

$$P = p_1|a_1|^2 + p_2|a_2|^2. \tag{VII.51}$$

Power conservation requires that the power be independent of distance z,

$$\frac{d\,P}{d\,z} = p_1\frac{d\,|a_1|^2}{d\,z} + p_2\frac{d\,|a_2|^2}{d\,z} = 0, \tag{VII.52}$$

from which if follows that,

$$p_1\kappa_{12} + p_2\kappa_{21}^* = 0. \tag{VII.53}$$

The determinantal equation for an assumed $exp(-j\beta z)$ dependence is,

$$\beta = \frac{\beta_1+\beta_2}{2} \pm \sqrt{(\frac{\beta_1-\beta_2}{2})^2 - \kappa_{12}\kappa_{21}}. \tag{VII.54}$$

For waves carrying power in the same direction, $p_1 p_2 = +1$, $\kappa_{12}\kappa_{21} = -|\kappa_{12}|^2$, and β is always real. But for $p_1p_2 = -1$ (i.e. waves carrying power in opposite directions), $\kappa_{12}\kappa_{21} = |\kappa_{12}|^2$ and β is complex for,

$$|\frac{\beta_1-\beta_2}{2}| < |\kappa_{12}|. \tag{VII.55}$$

Note the appreciable coupling can occur only if $|\beta_1 - \beta_2|$ is of order $|\kappa_{12}|$, which is small compared with $|\beta_1|$ and $|\beta_2|$ (weak-coupling assumption). Consider the case of codirectional, positive, group velocities, $p_1 = p_2 = +1$, with the initial waves $a_1(0)$ and $a_2(0)$, the solutions is analogous to the coupling of modes *in time* solutions,

$$a_1(z) = [a_1(0)(\cos\beta_0 z + j\frac{\beta_2-\beta_1}{2\beta_0}\sin\beta_0 z) + \frac{\kappa_{12}}{\beta_0}a_2(0)\sin\beta_0 z]e^{-j[(\beta_1+\beta_2)/2]z}, \tag{VII.56}$$

$$a_2(z) = [\frac{\kappa_{21}}{\beta_0}a_1(0)\sin\beta_0 z + a_2(0)(\cos\beta_0 z + j\frac{\beta_1-\beta_2}{2\beta_0}\sin\beta_0 z)]e^{-j[(\beta_1+\beta_2)/2]z}, \tag{VII.57}$$

where

$$\beta_0 = \sqrt{(\frac{\beta_1-\beta_2}{2})^2 + |\kappa_{12}|^2} \tag{VII.58}$$

E. Quality factors, laser threshold, and output power

The advantage of the perturbation approach is that one may evaluate quite easily the different Q factor for a Fabry-Perot resonator for a given mirror transmissivity and internal loss. The energy W in the resonator is, as defined, $W = |a|^2$. In the limit of high reflectivity $|a|^2/2$ is the energy associated with each of the oppositely directed traveling waves. The powers in the two counter-traveling waves is approximately,

$$< P_\pm >= \frac{|a|^2}{2} \frac{v_g}{l}, \tag{VII.59}$$

where l is the length of the resonator, and v_g is the group velocity of the mode. The power P_e escaping through the partially transmitting mirror of transmissivity $t^2 = T$ is,

$$P_e = T < P_- >= T \frac{|a|^2}{2} \frac{v_g}{l}. \tag{VII.60}$$

The external Q is thus,

$$\frac{1}{Q_{ext}} = \frac{P_e}{\omega_0 W} = \frac{2}{\omega_0 \tau_e} = \frac{T v_g}{2\omega_0 l}. \tag{VII.61}$$

And the incident wave $|s_+|^2$ exciting the resonator can be obtained, with zero internal resonator loss,

$$|s_+|^2 = \frac{2}{\tau_e}|a|^2 = T < P_+ > . \tag{VII.62}$$

Suppose that the medium filling the resonator has a spatial decay rate α for the field, then the unloaded Q is,

$$\frac{1}{Q_0} = \frac{P_d}{\omega_0 W} = \frac{2}{\omega_0 \tau_0} = \frac{2\alpha v_g}{\omega_0}, \tag{VII.63}$$

where the power dissipated P_d is,

$$P_d = 4\alpha l < P_\pm >= 2\alpha |a|^2 v_g. \tag{VII.64}$$

Again, the gain is produced by some form of "pumping" over a length l_g of the resonator with the generated power P_g,

$$P_g = 4\alpha_g l_g < P_\pm > . \tag{VII.65}$$

The equation for the mode amplitude in the laser is now,

$$\frac{d\,a}{d\,t} = (j\omega_0 - \frac{1}{\tau_0} - \frac{1}{\tau_e} + \frac{1}{\tau_g})a + \sqrt{\frac{2}{\tau_e}} s_+. \tag{VII.66}$$

If the laser is to oscillate in the steady state with no drive, $s_+ = 0$, one must have,

$$\frac{1}{\tau_g} = \frac{1}{\tau_0} + \frac{1}{\tau_e}, \tag{VII.67}$$

or

$$\alpha_g = \frac{l}{l_g}\alpha + \frac{T}{4l_g}. \tag{VII.68}$$

This is the gain coefficient which must be achieved to reach *threshold*, the gain level for self-starting of the oscillator.

F. Extended studies

1. Propagation of Gaussian pulses,
2. Waveguide couplers: tunable filters, switch,
3. Injection locking of an oscillator.