5, Nonlinear Equations and Nonlinear PDE

Nonlinear equation:

$$f(x) = x^3 + x^2 = 5,$$

- Iterative method
- Newton-Raphson method
- Secant Method

Nonlinear Schrödinger equation:

$$i\frac{\partial U}{\partial t} + \frac{\partial^2 U}{\partial x^2} + |U|^2 U = 0$$

- Crank-Nicholson method
- Slit-Step Fast Fourier Transform
- Pseudospectral method

Newton-Raphson method

- Newton-Raphson mouthed for root finding,

$$\text{Given} \quad F_i(x_1, x_2, \ldots, x_N) = 0 \quad i = 1, 2, \ldots, N$$
$$\mathbf{F}(\mathbf{x} + \delta\mathbf{x}) = \mathbf{F}(\mathbf{x}) + \mathbf{J} \cdot \delta\mathbf{x} + \mathbf{O}(\delta\mathbf{x}^2) = 0,$$

  where the Jacobian matrix **J**:

$$J_{ij} \equiv \frac{\partial F_i}{\partial x_j}.$$

  and the new solution $\mathbf{x}_{new} = \mathbf{x}_{old} + \delta\mathbf{x}$, with

$$\mathbf{J} \cdot \delta\mathbf{x} = -\mathbf{F}.$$

- For the boundary condition at x_2, we have a correction term for **V**

$$\mathbf{J} \cdot \delta\mathbf{V} = -\mathbf{F}.$$

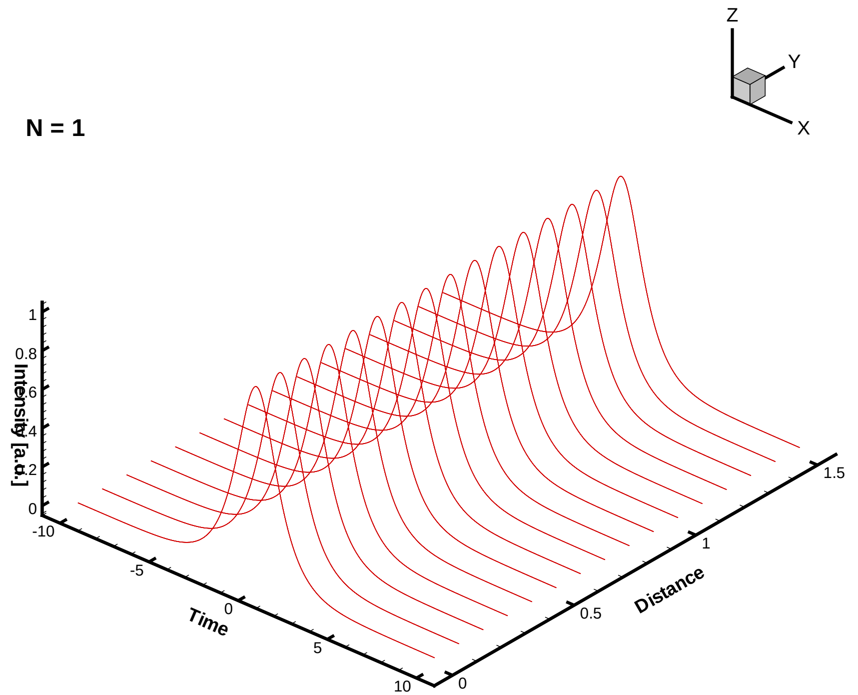
- Then adding the correction back until it converges,

$$\mathbf{V}^{new} = \mathbf{V}^{old} + \delta\mathbf{V}.$$

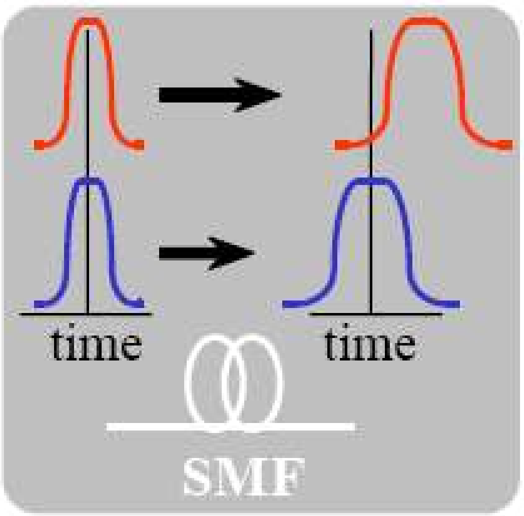
Solitons in optical fibers

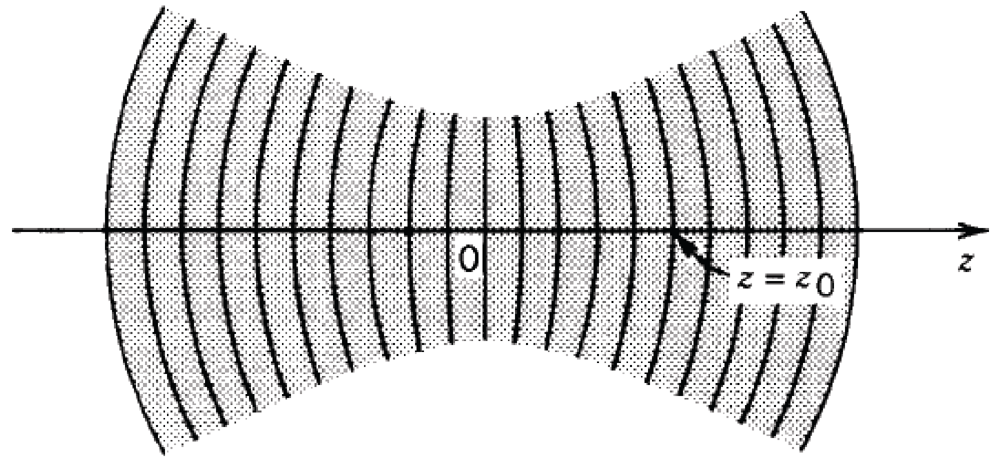
Nonlinear Schrödinger Equations: Hermitian System

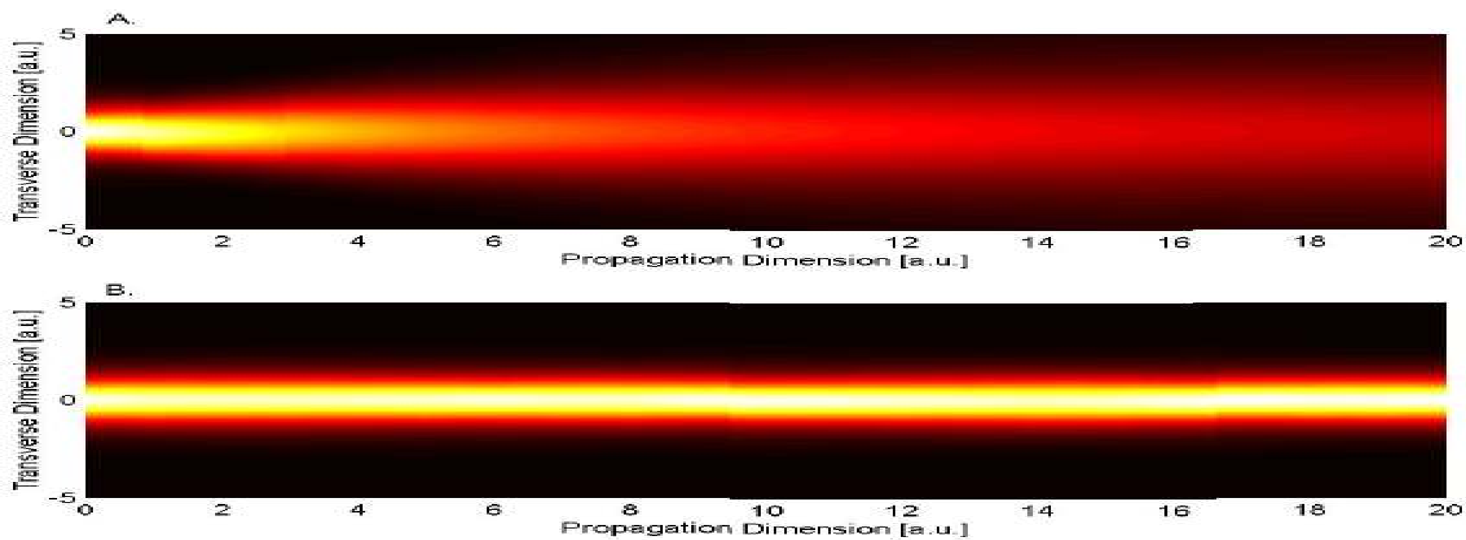
$$
\begin{aligned}
iU_z &= -\frac{D}{2}U_{tt} - |U|^2 U \qquad \text{, i.e.} \\
i\hbar\Psi_t &= -\frac{\hbar^2}{2m}\nabla^2\Psi + \mathcal{V}\Psi = \mathcal{H}\Psi
\end{aligned}
$$



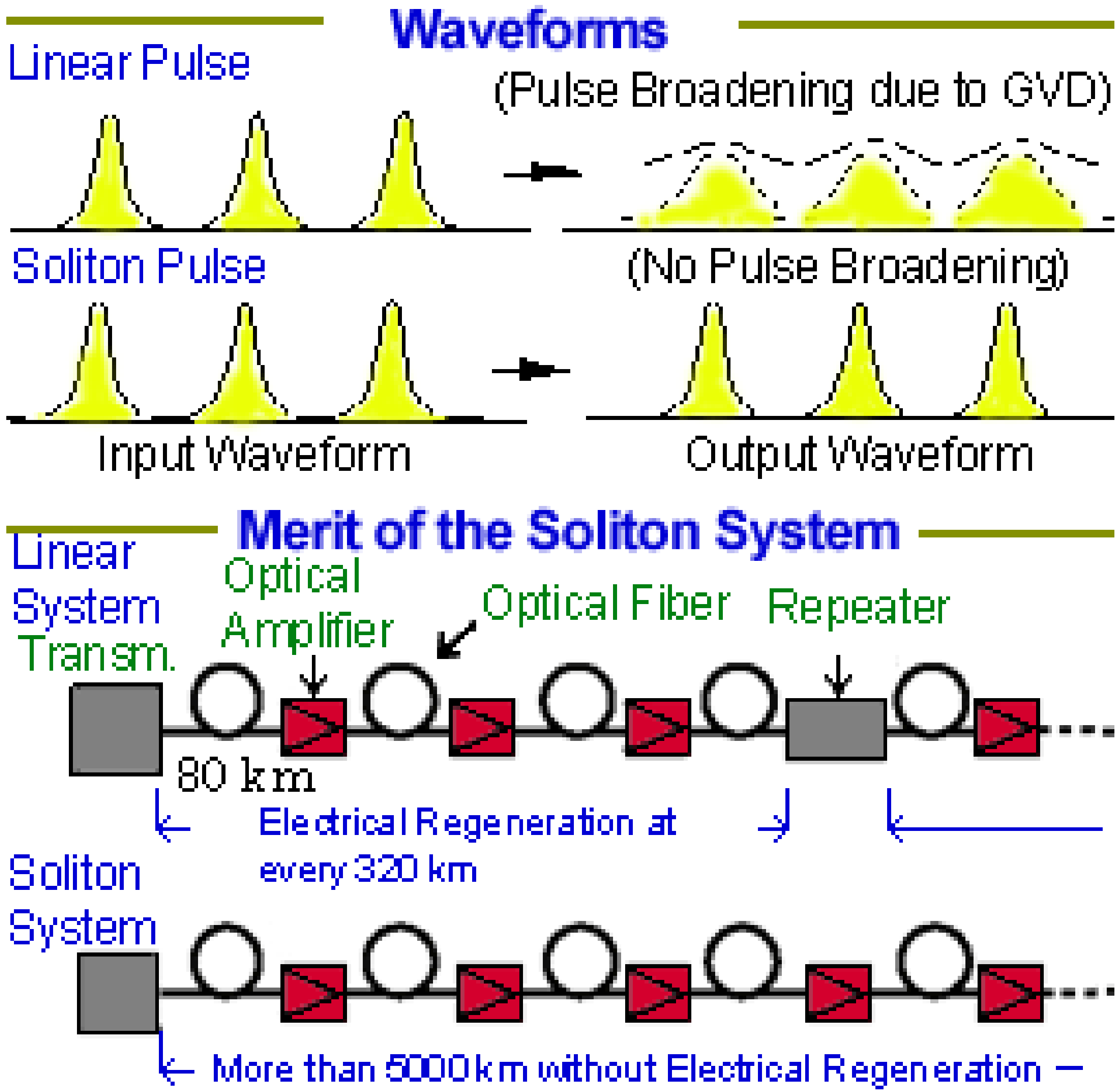
Dispersion/Diffraction effect



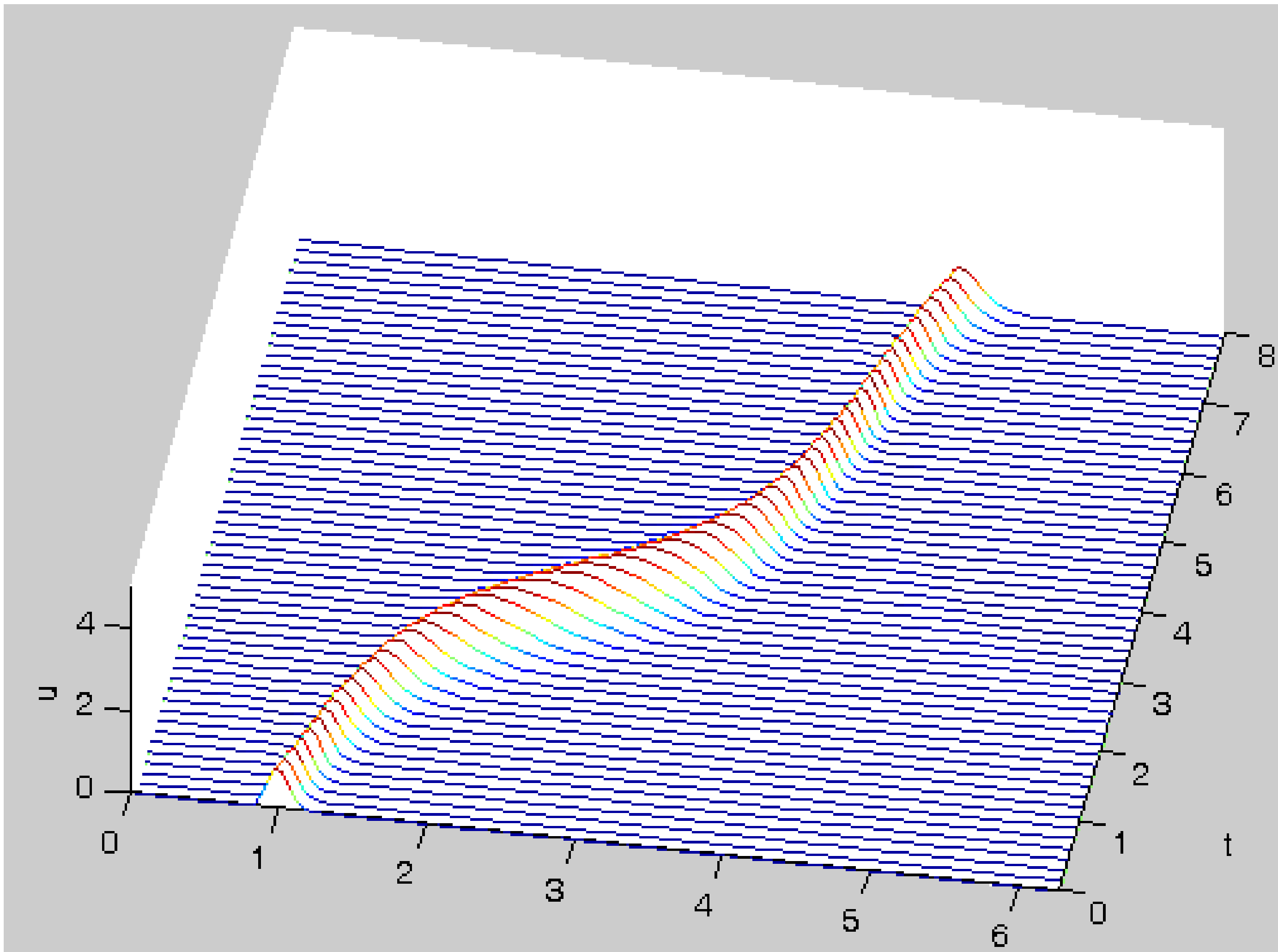




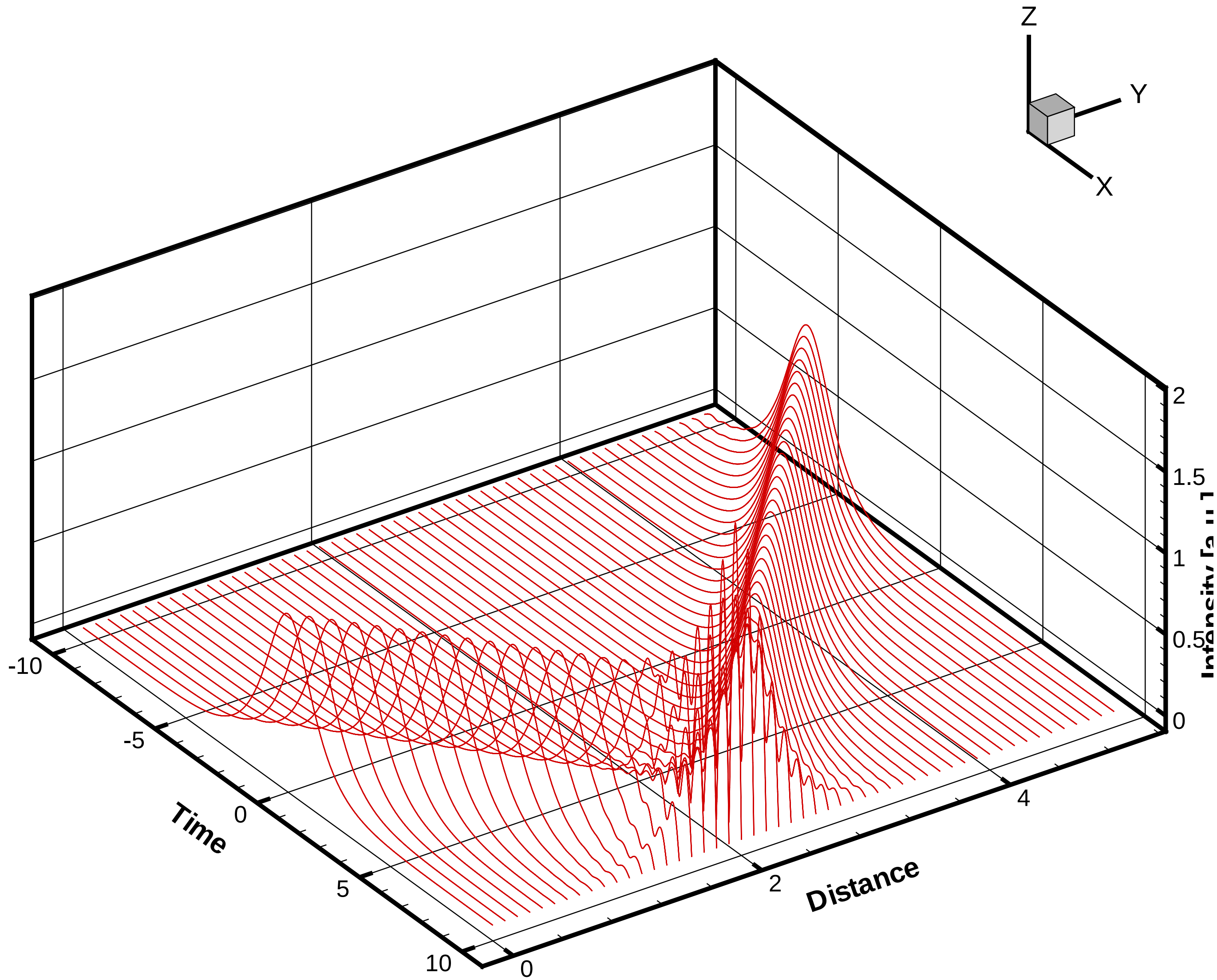
Soliton communication system



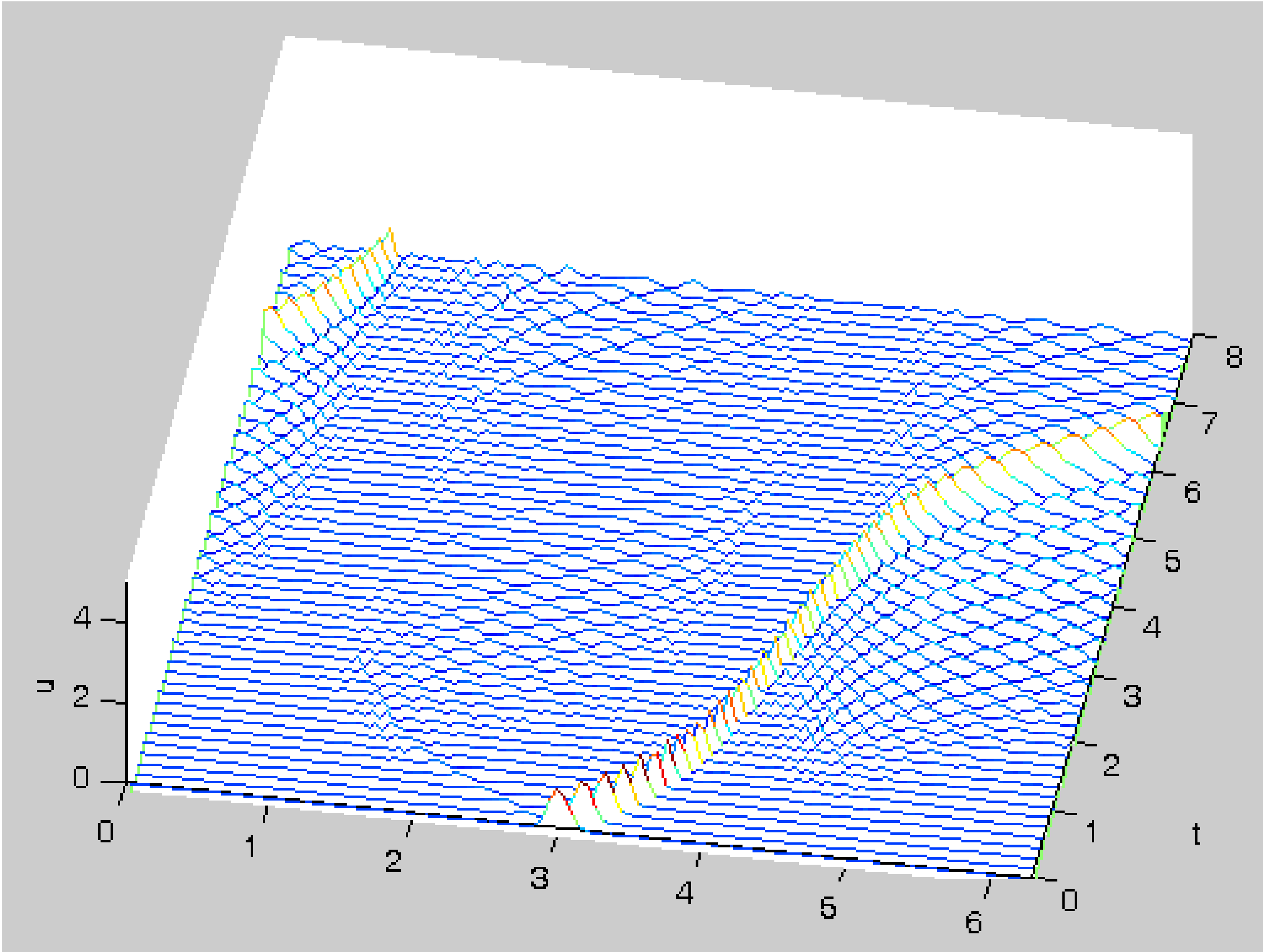
Wave propagation



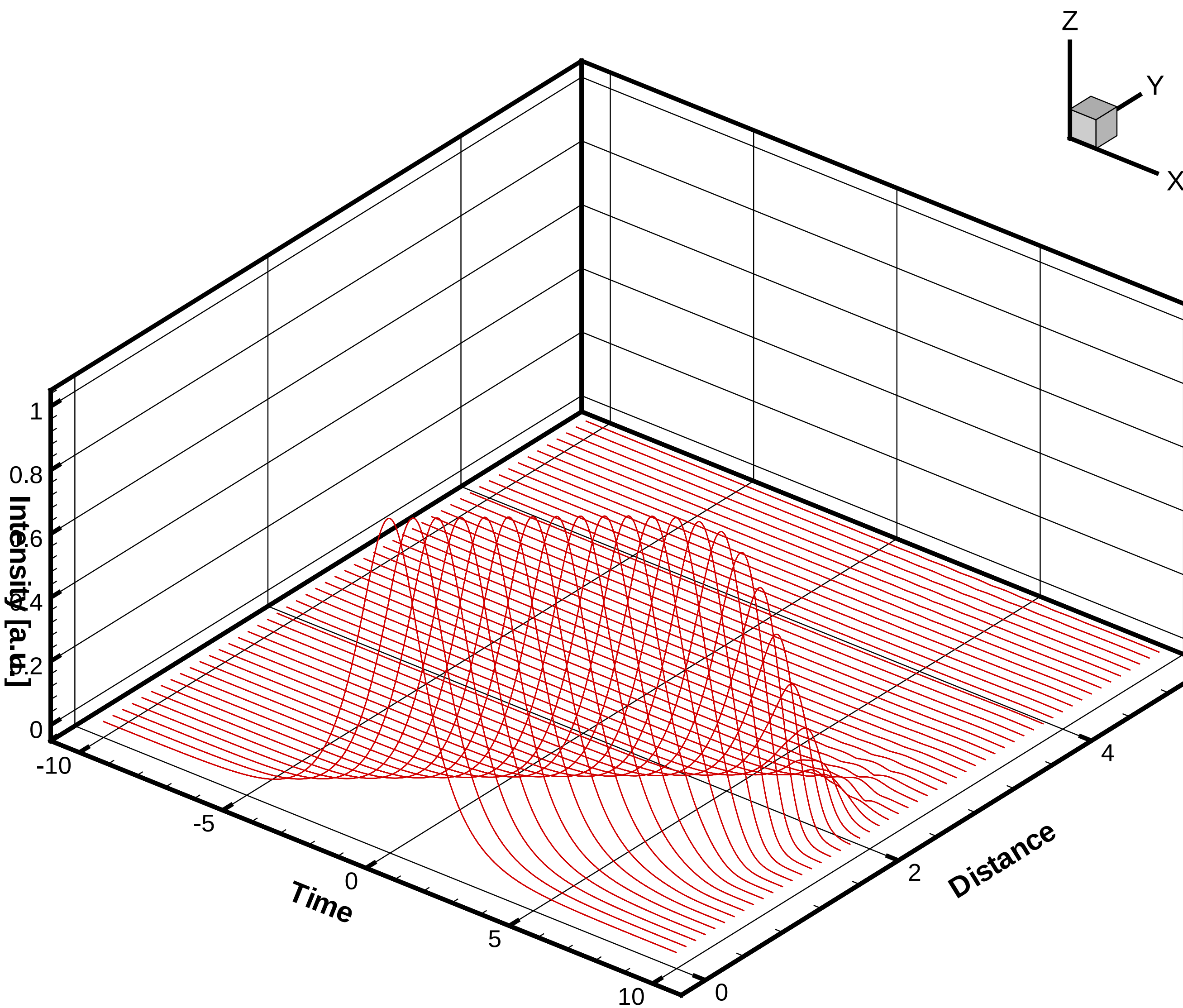
without Periodic Boundary Condition



with Periodic Boundary Condition



with Absorption Boundary Condition



Perfect Matching Layers

For the linear Schrödinger equation

$$i\frac{\partial}{\partial t}\Psi(x,t) = -\frac{1}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t)$$

which can be written as

$$i\frac{\partial}{\partial t}\Psi(x,t) = -\frac{1}{2n}\frac{\partial}{\partial x}\frac{1}{n}\frac{\partial}{\partial x}\Psi(x,t)$$

where m, the mass, has been split into two spatially dependent functions n.

$$\Psi = \int_0^\infty A(\omega)\text{exp}(\pm i\int k\text{d}x - i\omega t)\text{d}\omega,$$

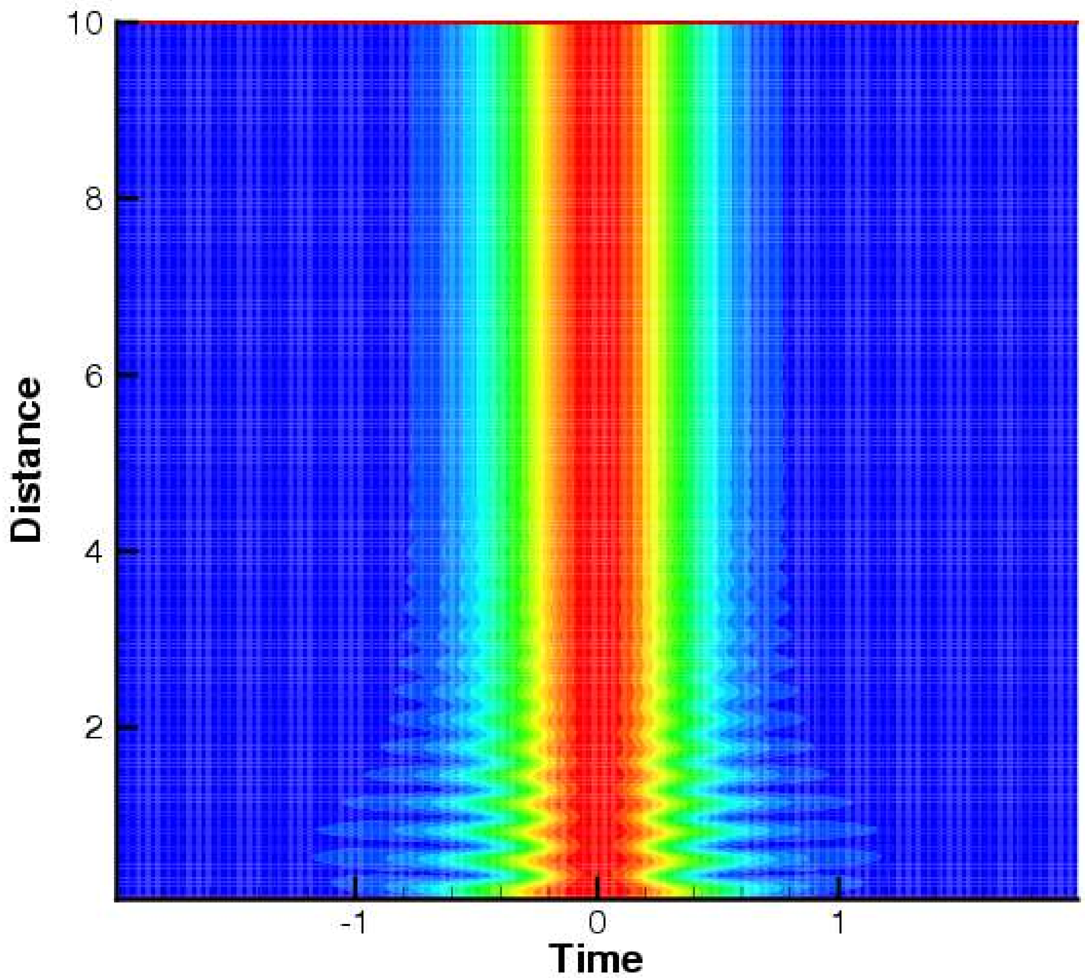
where $k = \pm n\sqrt{2\omega}$ with ther term inside the exponential is positive for waves moving to the left and negative for waves moving to the right. We can choose n to be, for example,

$$n = \text{exp}[\pm i\frac{\pi}{4}(1 - \tanh\frac{x - x_0}{a})]$$

where x_0 is the position where the PML starts and a is a parameter which determines the sharpness of the transition between 1 and i.

E.NTHU 國立清華大學 電機工程學系及研究所

Radiation wave



Fourier method

For the equation

$$\frac{\partial U}{\partial z} = (\hat{D} + \hat{N})U,$$

where $\hat{D}$ is a differential operator and $\hat{N}$ is a nonlinear operator,i.e.

$$\begin{aligned} \hat{D} &= i\frac{D}{2}\frac{\partial^2}{\partial t^2}, \\ \hat{N} &= i|U|^2 U. \end{aligned}$$

The split-step Fourier method approximat the differential and nonlinear operators independently,

$$U(z+h,t) \approx \text{exp}(h\hat{D})\text{exp}(h\hat{N})U(z,t).$$

The execution of the exponential operator exp$(h\hat{D})$ is carried out in the Fourier domain,

$$\begin{aligned} \text{exp}(h\hat{D})A(z,t) &= \{\mathbf{F}^{-1}\text{exp}[h\hat{D}(i\omega)]\mathbf{F}\}A(z,t), \\ &= \{\mathbf{F}^{-1}\text{exp}[-i\frac{D}{2}\omega^2 h)]\mathbf{F}\}A(z,t), \end{aligned}$$

where **F** denotes the Fourier-transform operation.

We replace the differential operator $\partial/\partial t$ by $i\omega$.

Slipt-Step Fourier method

The accuracy of the split-step Fourier method can be improved by adopting a differenet procedure to propagate the optical pulse over one segment from z to $z+h$,

$$U(z+h,t) \approx \text{exp}(\frac{h}{2}\hat{D})\text{exp}[\int_z^{z+h}\hat{N}(z')\text{d}z']\text{exp}(\frac{h}{2}\hat{D})U(z,t),$$

1. Fourier transform for z to $z+\frac{h}{2}$:

$$U_1(t) = \{\mathbf{F}^{-1}\text{exp}[-i\frac{D}{2}\omega^2\frac{h}{2})]\mathbf{F}\}U(z,t),$$

2. nonlinear integration:

$$U_2(t) = \frac{h}{2}[\hat{N}(z)+\hat{N}(z+h)]U_1(t),$$

3. Fourier transform for $z+\frac{h}{2}$ to $z+h$:

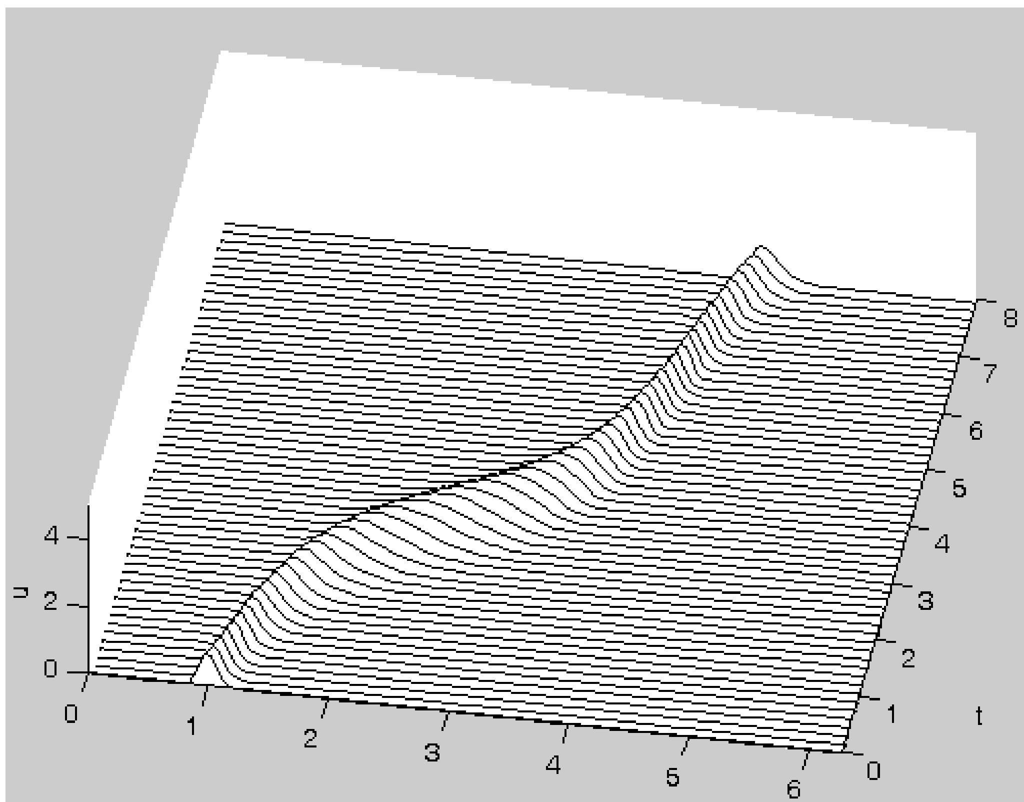
$$U(z+h,t) = \{\mathbf{F}^{-1}\text{exp}[-i\frac{D}{2}\omega^2\frac{h}{2})]\mathbf{F}\}U_2(t),$$

This is known as the symmetrized split-step Fourier method, with the accuracy of $\mathbf{O}(h^2)$.

FFT method for wave equation

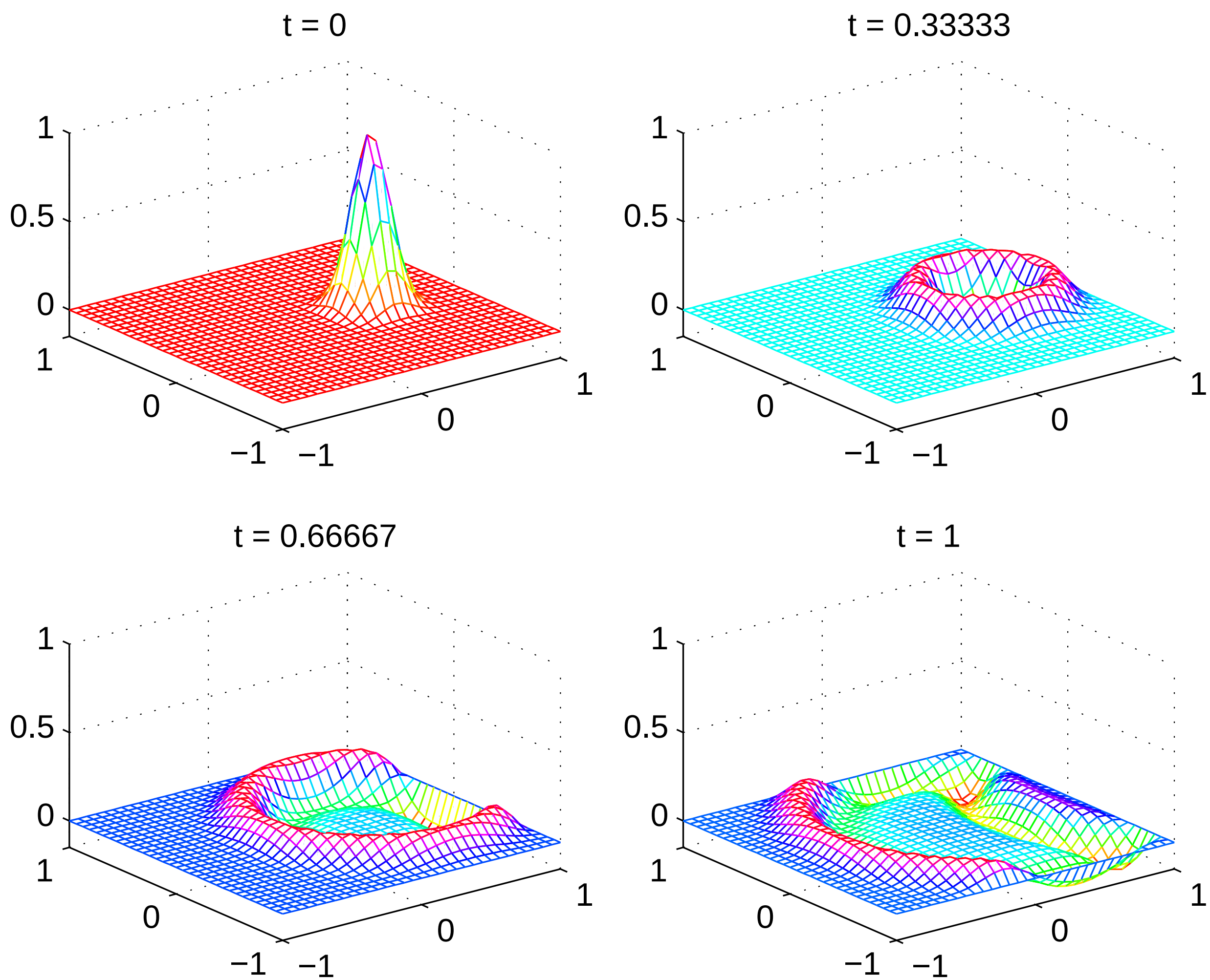
$$u_t + c(x)\, u_x = 0, \qquad c(x) = \frac{1}{5} + \sin^2(x-1),$$

- Given $u(x)$, compute $\tilde{U}(k)$,
- Define $\tilde{U}_k = (ik)^\mu \tilde{U}(k)$,
- Compute $\mathrm{D}_x u$ from $\tilde{U}_k$.



FFT method for wave equation

$$u_{tt} = u_{xx} + u_{yy}, \qquad -1 < x, y < 1, \quad t > 0, \quad u = 0 \quad \text{on the boundary}$$



Spectral method for Nonlinear Schrödinger equation

- Take the solutions of linear Schrödinger equation as basis, Hermite polynomials,

$$p_{N-1}(x) = \sum_{j=0}^{N-1} \frac{\text{exp}(-x^2/2)}{\text{exp}(-x_j^2/2)} f_j(x)\phi(x)_j,$$

where $f_j(x)$ are taken as

$$f_j(x) = \frac{H_{N-1}(x)}{H'_{N-1}(x_j)(x - x_j)},$$

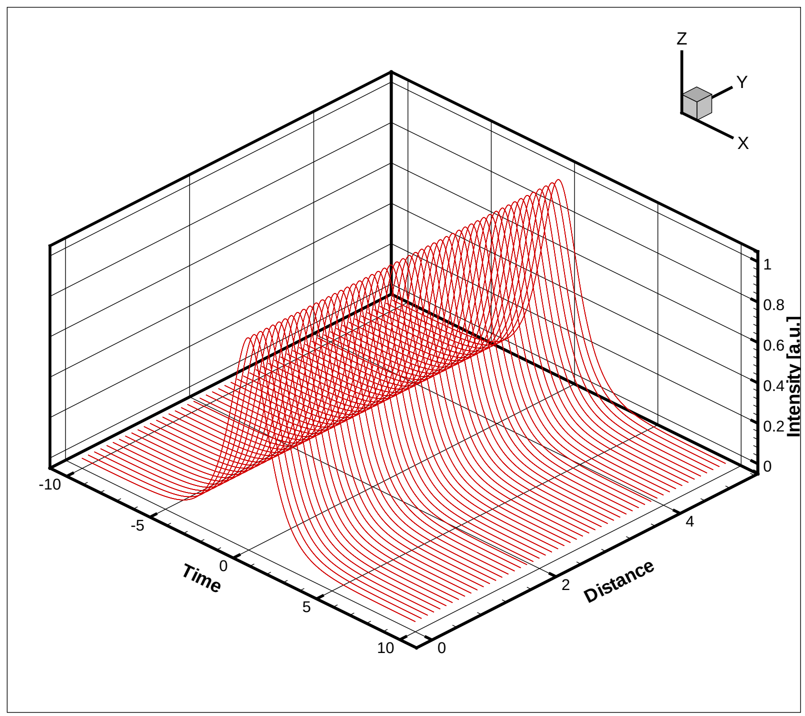
and the weight functions are taken as
$\alpha(x) = \text{exp}(-x^2/2)$.

Propagation of solitons

Nonlinear Schrödinger equation:

$$i\frac{\partial U}{\partial t} + \frac{\partial^2 U}{\partial x^2} + |U|^2 U = 0$$

supports soliton solutions, $U(t=0,x) = sech(x)$.

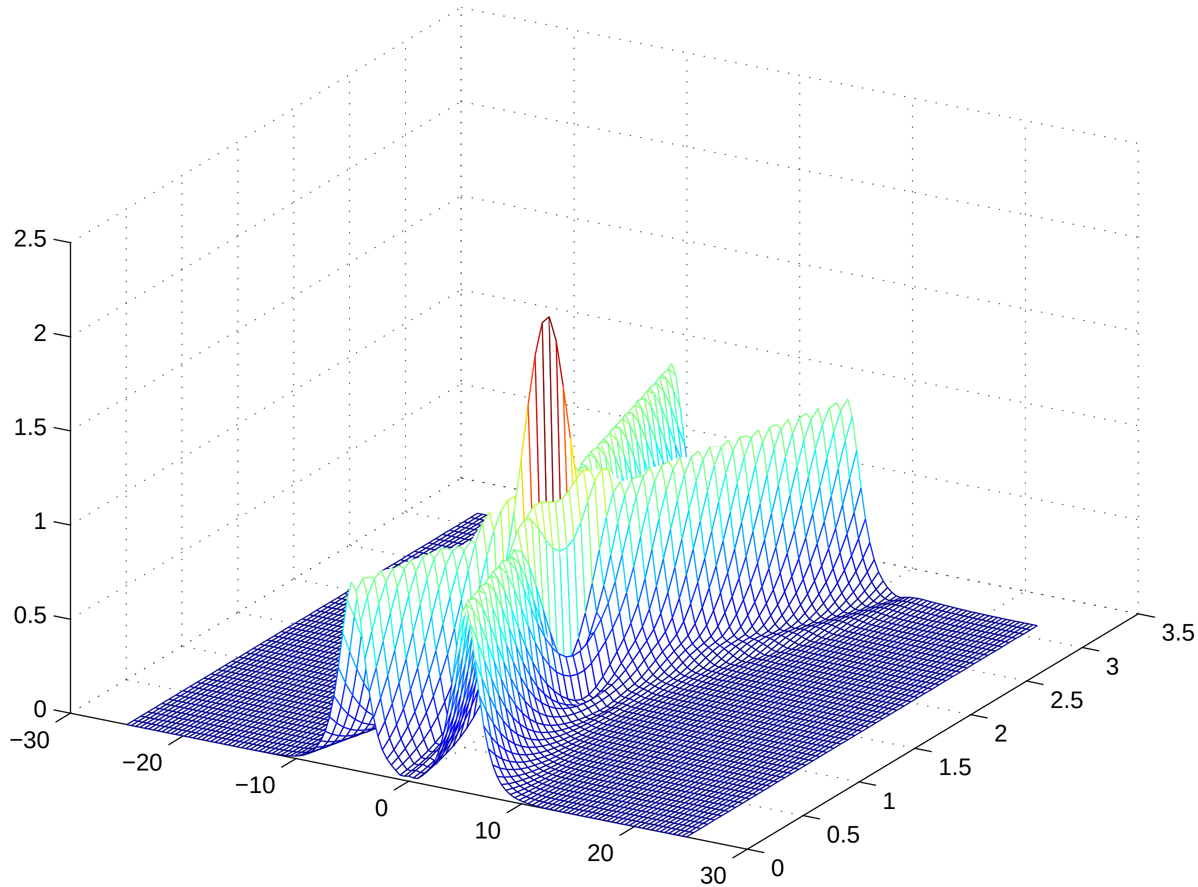


simulated by Fourier spectral + 4th-order explicit Runge-Kutta methods,

N_x = 128, N_t = 50, error = 10^{-6}, indep. of N_t; nlse.m.

Soliton collisions

$$U(t = 0, x) = sech(x + x_0) + sech(x - x_0)$$

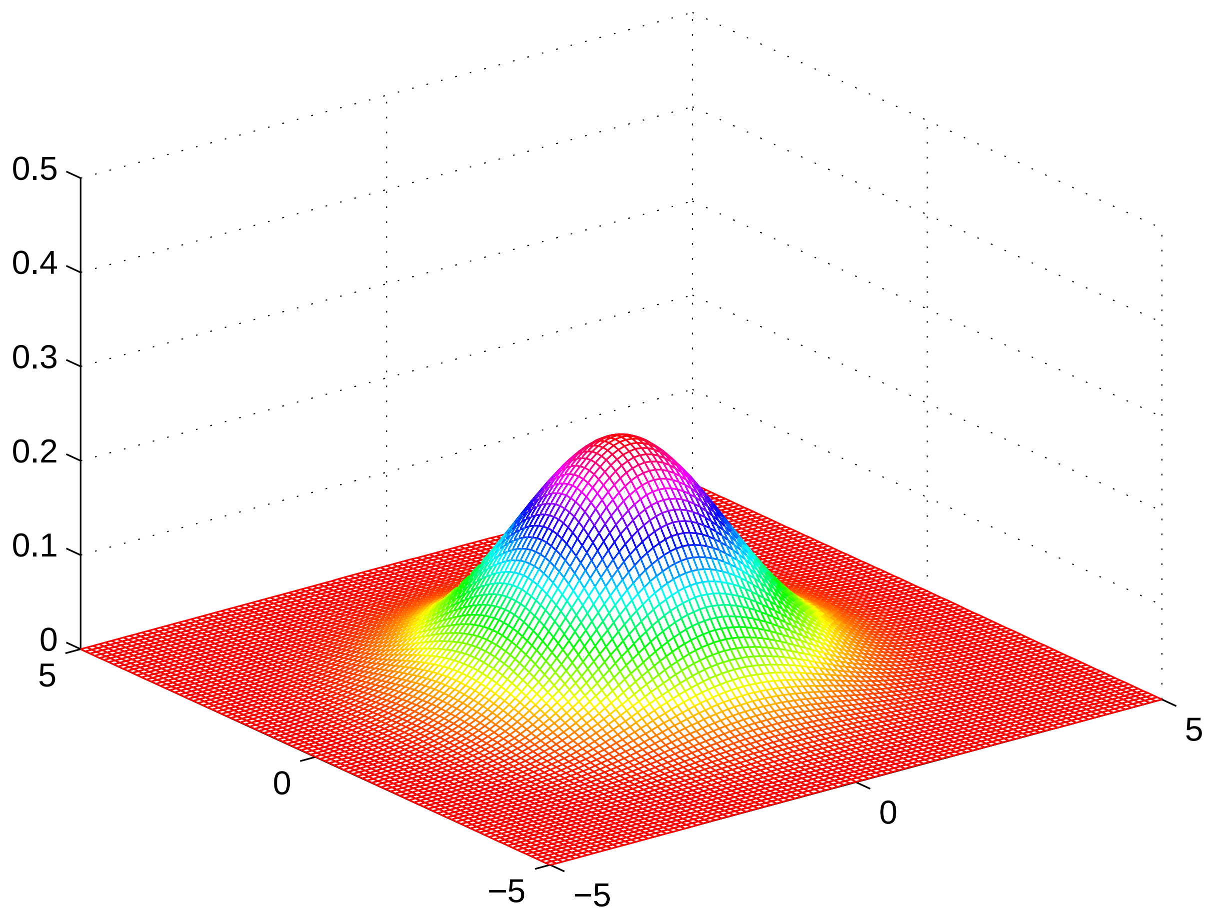


simulated by Fourier spectral + 4th-order explicit Runge-Kutta methods,

N_x = 128, N_t = 50.

Spectral method for Gross-Pitaevskii equation

$$[-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + \frac{x^2 + \kappa^2 y^2 + \nu^2 z^2}{4} + N|\phi(x,y,z,t)|^2 - i\frac{\partial}{\partial t}]\phi(x,y,z,t) = 0,$$



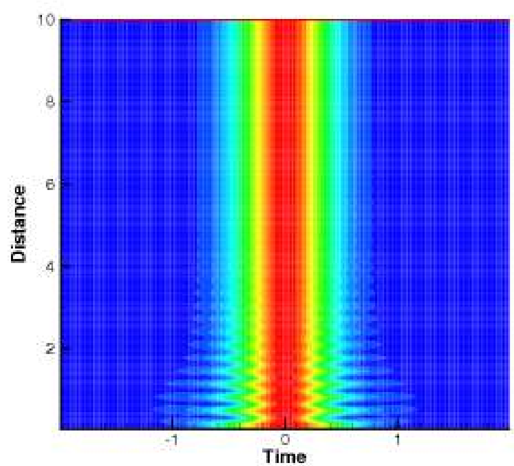
Bound solitons in CGLE

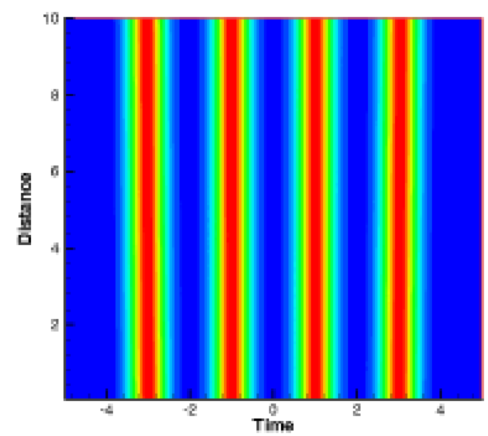
Complex Ginzburg-Landau Equation:

$$
\begin{aligned}
iU_z + \frac{D}{2}U_{tt} + |U|^2 U &= i\delta U + i\epsilon |U|^2 U + i\beta U_{tt} \\
&+ \; i\mu |U|^4 U - v|U|^4 U,
\end{aligned}
$$

seek for bound-state solutions by propagation method.

$$
U(z,t) = \sum^{N} U_0(z, t+\rho_j) e^{i\theta_j}
$$



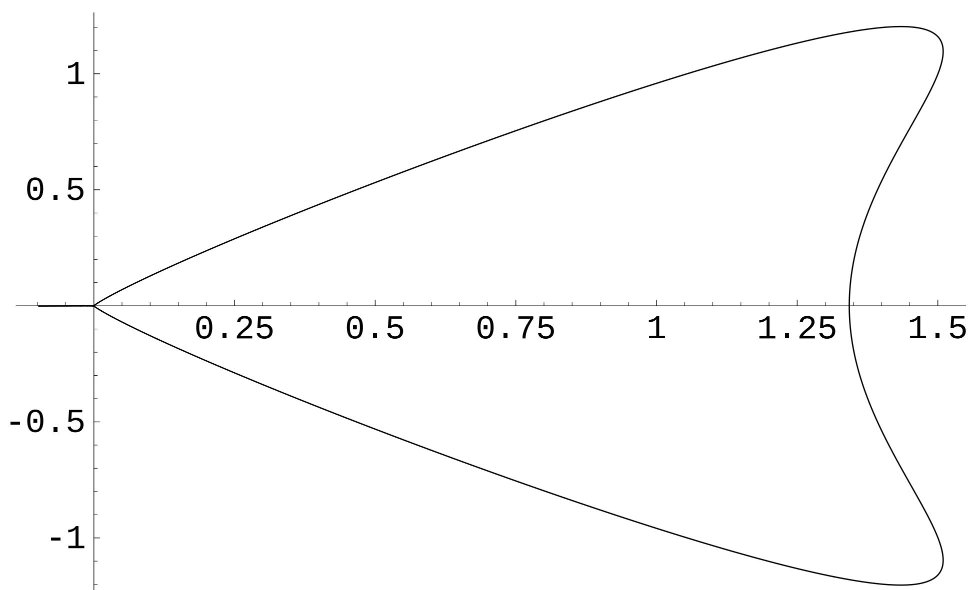


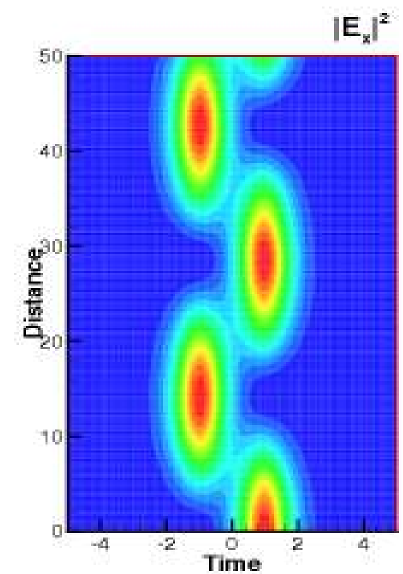
R.-K. Lee, Y. Lai, and B. A. Malomed, *Phys. Rev. A* **70**, 063817 (2004).

Coupled Nonlinear Schrödinger Equations

$$
\begin{aligned}
i\frac{\partial U}{\partial z} + \frac{1}{2}\frac{\partial^2 U}{\partial t^2} + A|U|^2U + B|V|^2U &= 0, \\
i\frac{\partial V}{\partial z} + \frac{1}{2}\frac{\partial^2 V}{\partial t^2} + A|V|^2V + B|U|^2V &= 0,
\end{aligned}
$$

seek for bound-state solutions by separatrix method.





R.-K. Lee, Y. Lai, and B. A. Malomed, *Phys. Rev. A* **71**, 013816 (2005).

Numerical methods for solving Nonlinear PDEs

- 1950s: finite difference methods
- 1960s: finite element methods
- 1970s: spectral methods

Newton iteration method

1. To solve
$$f(x) = 0,$$
2. Start with a guess $x^{(i)}$, then we can Taylor expand $f(x)$,
$$f(x) = f(x^{(i)}) + f_x(x^{(i)})[x - x^{(i)}] + O([x - x^{(i)}]^2).$$
3. If the guess is sufficiently good, we can ignore the quadratic terms and solve the *linear* equation for x,
$$x^{(i+1)} = x^{(i)} - \frac{f(x^{(i)})}{f_x(x^{(i)})}.$$
4. Iterate $x^{(i+1)}$ by *Step 3* until converge.

Newton-Kantorovich iteration method

1. Given a general nonlinear operator, $\mathcal{N}(u) = 0$.
2. We can expand it by a generalized Taylor expansion:

$$\mathcal{N}(u+\Delta) = \mathcal{N}(u) + \mathcal{N}_u(u)\Delta + O(\Delta^2),$$

   where $\mathcal{N}_u$ is called the Frechet differential,

$$\mathcal{N}_u\Delta \equiv \frac{\partial \mathcal{N}(u+\epsilon\Delta)}{\partial \epsilon}\Big|_{\epsilon=0},$$

   and is a *linear* operator.

3. Iterate u^{i+1} by $u^i + \Delta$ until $\Delta = 0$, where Δ satisfies

$$\mathcal{N}_u(u)\Delta = -\mathcal{N}(u).$$

NLSE

1. For time-independent NLSE, $U(t,x) = e^{-i\mu t}\Psi(x)$

$$\mu\Psi(x) + \frac{\partial^2\Psi}{\partial x^2} + |\Psi|^2\Psi = 0$$

2. Expand $\Psi^{i+1}(x) = \Psi^i(x) + \Delta$, and Δ satisfies

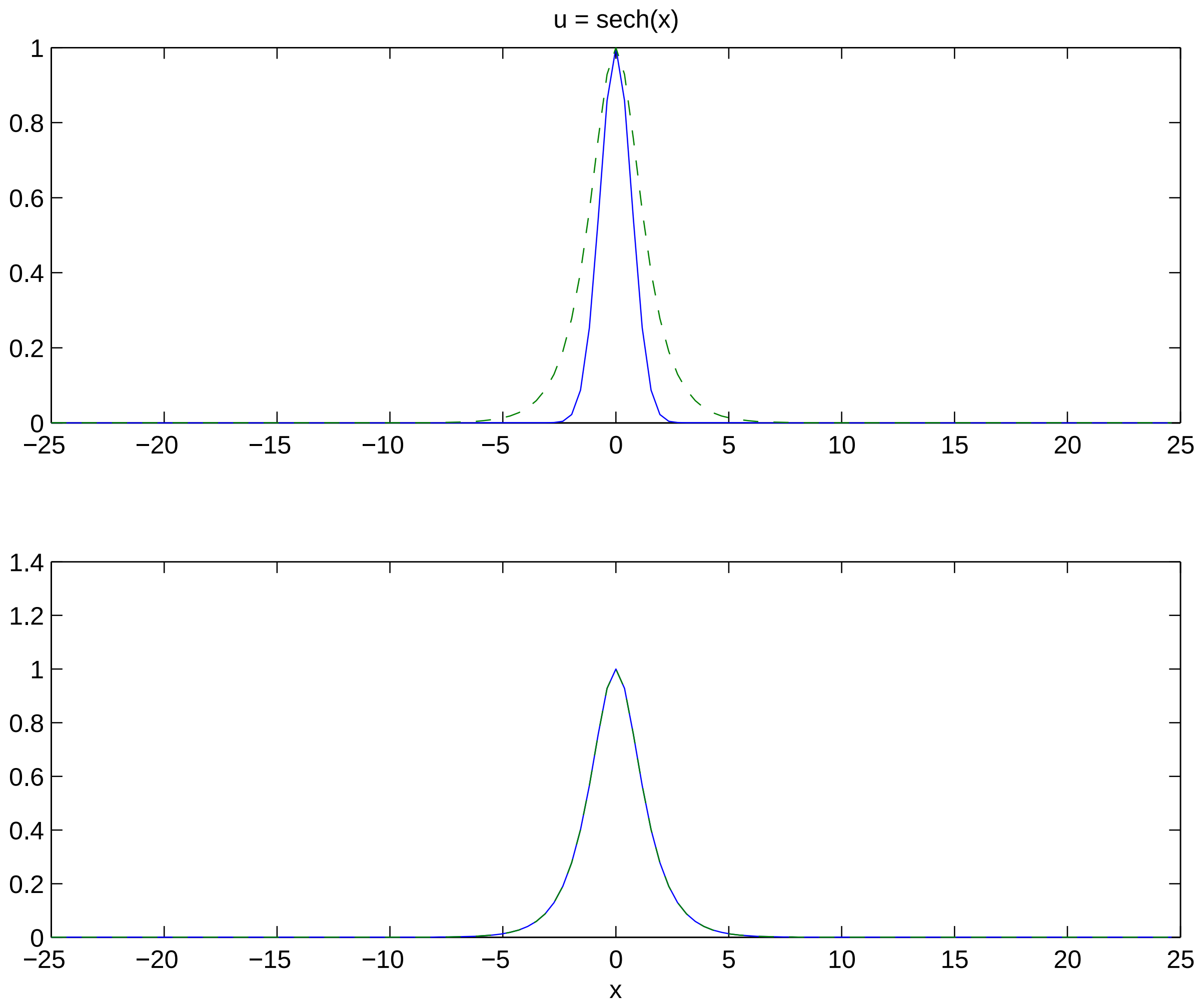
$$(\mu + \frac{\partial^2}{\partial x^2} + 2|\Psi^i|^2)\Delta + \Psi^{i2}\Delta^* = -(\mu\Psi^i(x) + \frac{\partial^2\Psi^i}{\partial x^2} + |\Psi^i|^2\Psi^i).$$

3. Start with a good initial guess, $\Psi^0(x)$, then iterate Ψ^{i+1} by *Step 2* until $\Delta = 0$.

4. In spectral method, $\frac{\partial^2}{\partial x^2}$ is approximated by a differential matrix D_2.

N=1 solitons in NLSE

Initial guess, $\Psi(x) = Ae^{-x^2/x_c^2}$, ($A = x_c = 1$, $\mu = 1.0$).

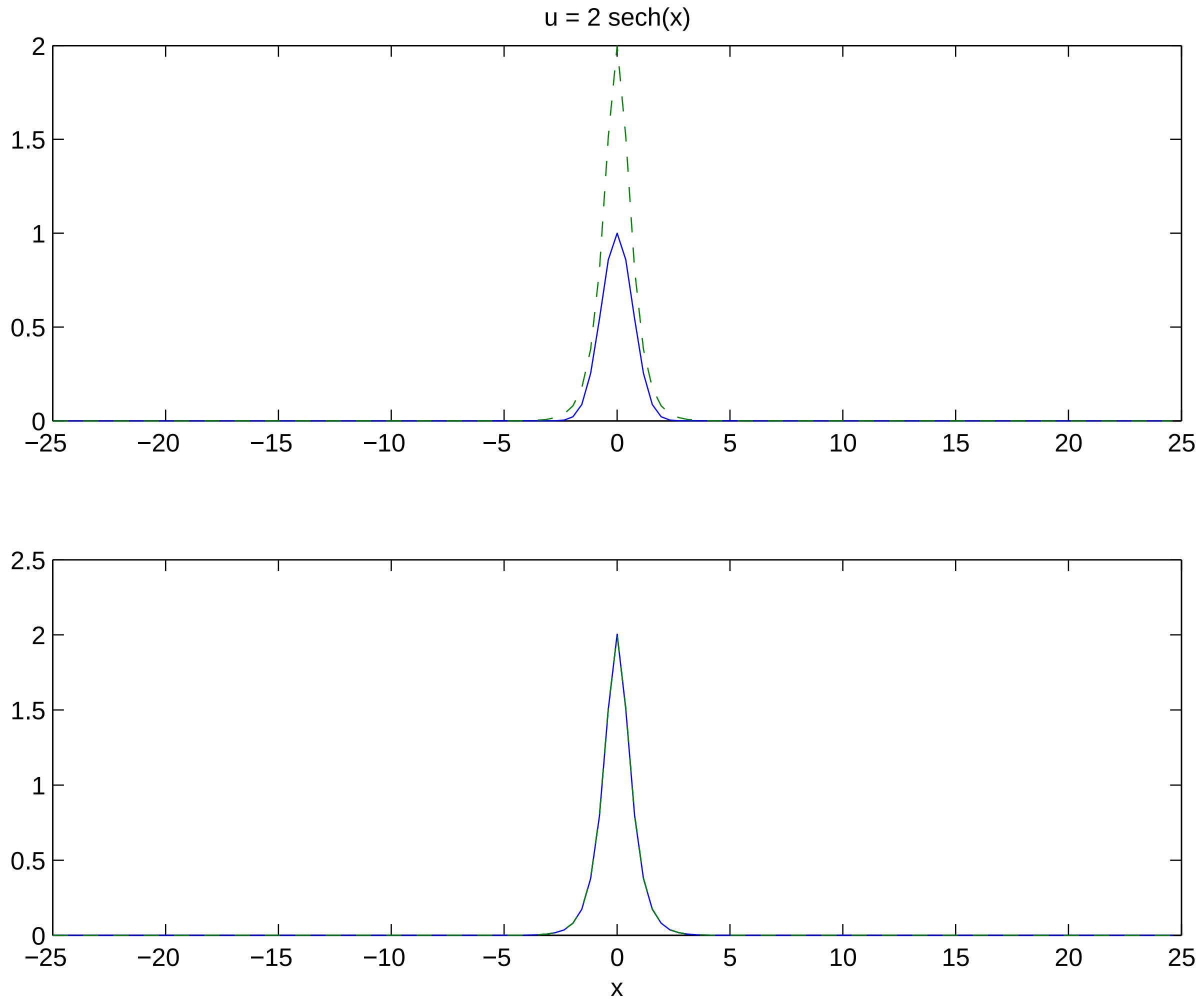


simulated by FS (Fourier spectral)+ NK (Newton-Kantorovich) methods,

N_x = 128, no. of iterations = 5; nlseground.m.

N=2 solitons in NLSE

Initial guess, $\Psi(x) = Ae^{-x^2/x_c^2}$, ($A = x_c = 1$, $\mu = 4.0$).



simulated by FS (Fourier spectral)+ NK (Newton-Kantorovich) methods,

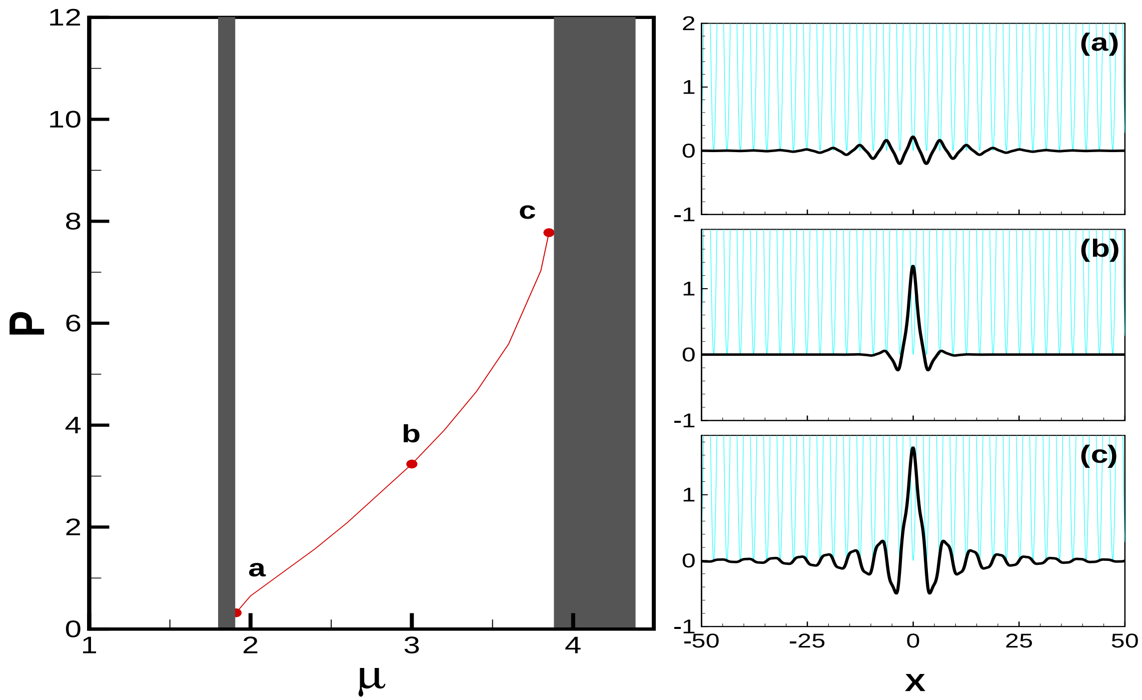
N_x = 128, no. of iterations = 9; nlsen2.m.

Gap solitons in optical lattices

1-D Gross-Pitaevskii equation with periodic potentials, $V(x) = V_0 \sin^2(k_0 x)$,

$$i\hbar\frac{\partial}{\partial t}\Phi_0(t,x) = -\frac{1}{2}\frac{\partial^2}{\partial x^2}\Phi_0(t,x) + V(x)\Phi_0(t,x) + g_{1D}|\phi_0(t,x)|^2\phi_0(t,x)$$

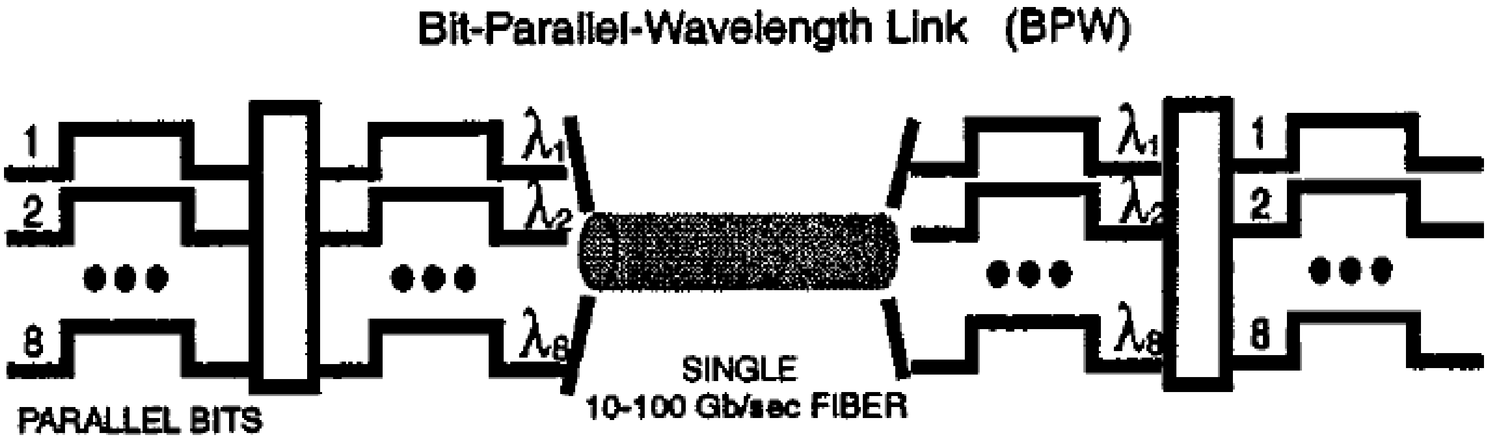
which has gap soliton solutions.



by FS + NK methods, N_x = 512, no. of iterations < 10; gpeol.m.

R.-K. Lee, E. A. Ostrovskaya, Yu. S. Kivshar, and Y. Lai, *Phys. Rev. A* **72**, 033607 (2005).

Multichannel solitons in Bit-Parallel-Wavelength links



N incoherently coupled NLSE,

$$i\left(\frac{\partial}{\partial z} + \delta_j \frac{\partial}{\partial t}\right)A_j = \frac{\alpha_j}{2}\frac{\partial^2 A_j}{\partial t^2} - \gamma_j(|A_j|^2 + S_{mj})A_j$$

where

$$S_{mj} = 2\sum_{m\neq j}^{N-1}(\gamma_m/\gamma_j)|A_m|^2$$

Stationary soliton solutions

By the transformation:

$$A_j(t,z) = u_j(t) exp(i2\delta_j \alpha_j^{-1} t + i\lambda_j z),$$

BPW system becomes a coupled NLS equations:

$$\frac{1}{2}\frac{d^2 u_0}{dt^2} + (u_0^2 + 2\sum_{n=1}^{N-1} |\gamma_n| u_n^2) u_0 = \frac{1}{2} u_0,$$

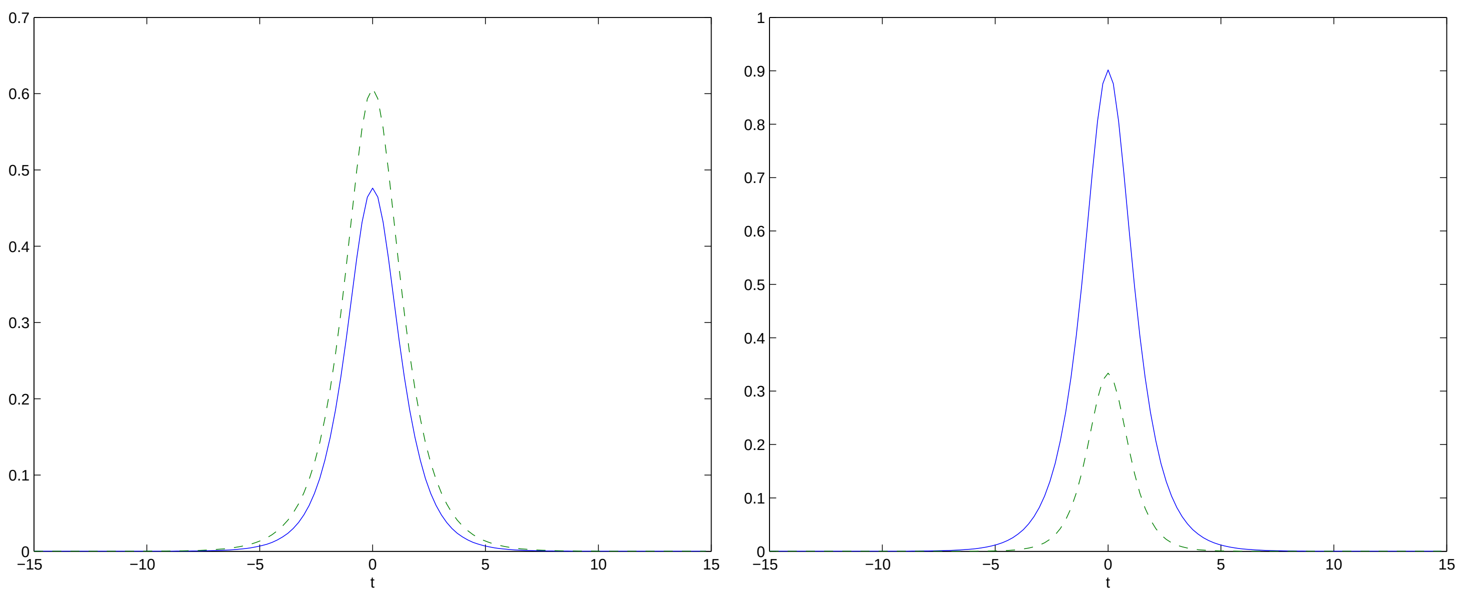
$$\frac{\alpha_n}{2}\frac{d^2 u_n}{dt^2} + \gamma_n (u_n^2 + 2\sum_{m\neq n}^{N-1} \frac{\gamma_m}{\gamma_n} |u_m|^2) u_n = \lambda_n u_n.$$

Two-channel BPW solitons

For $N = 2$ (u_0 and u_1),

$$\frac{1}{2}\frac{d^2u_0}{dt^2} + (u_0^2 + 2\gamma_1 u_1^2)u_0 = \frac{1}{2}u_0,$$

$$\frac{\alpha_1}{2}\frac{d^2u_1}{dt^2} + (\gamma_1|u_1|^2 + 2\gamma_0|u_0|^2)u_1 = \lambda u_1.$$

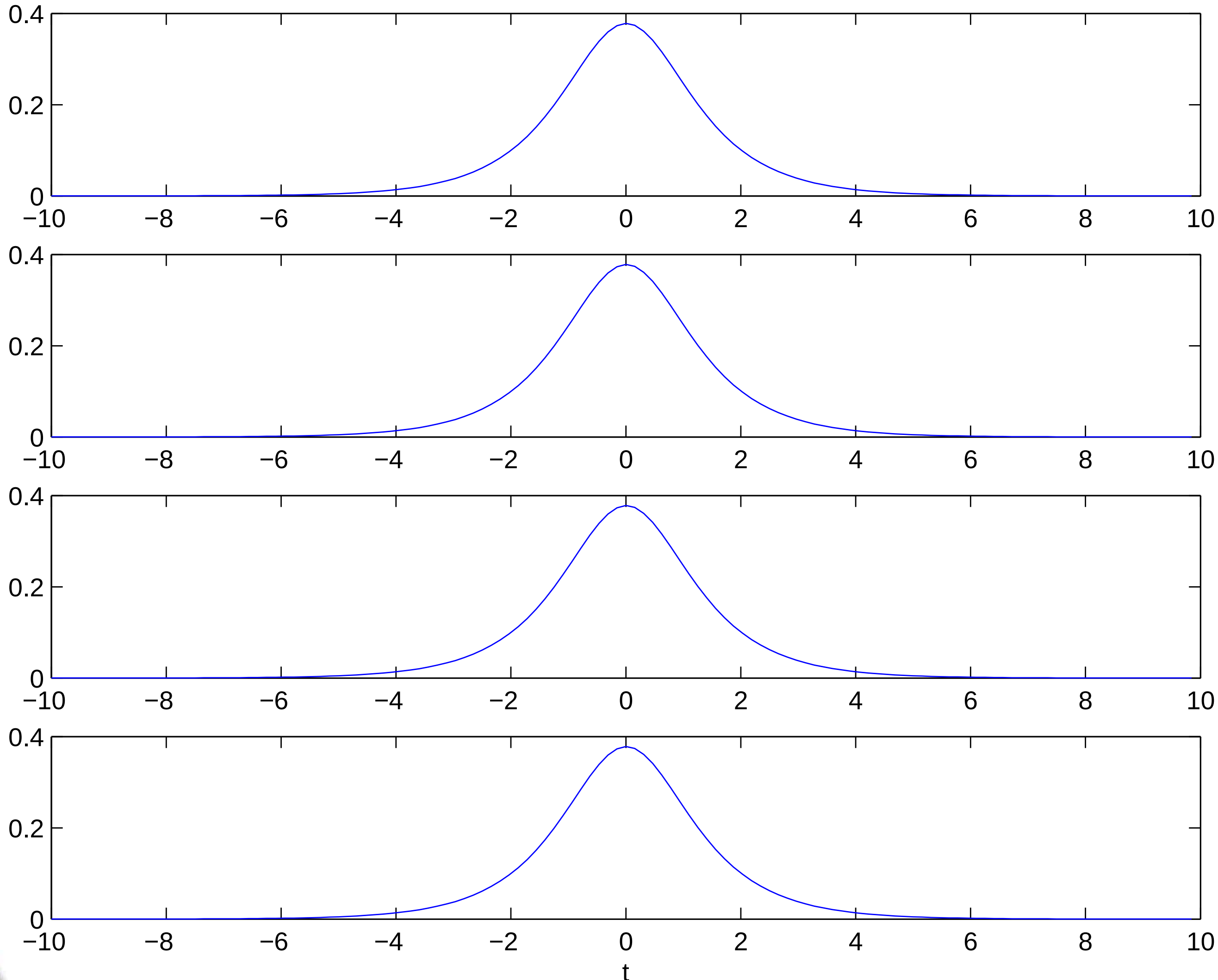


Left: $\lambda = 0.4$, Right $\lambda = 1.0$; solid-line: u_0, dashed-line: u_1;

by FS + NK methods, N_x = 128, no. of iterations = 9, (16) for $\lambda = 0.4$, (1.0); shepherd.m.

Four-channel BPW solitons: the same profiles

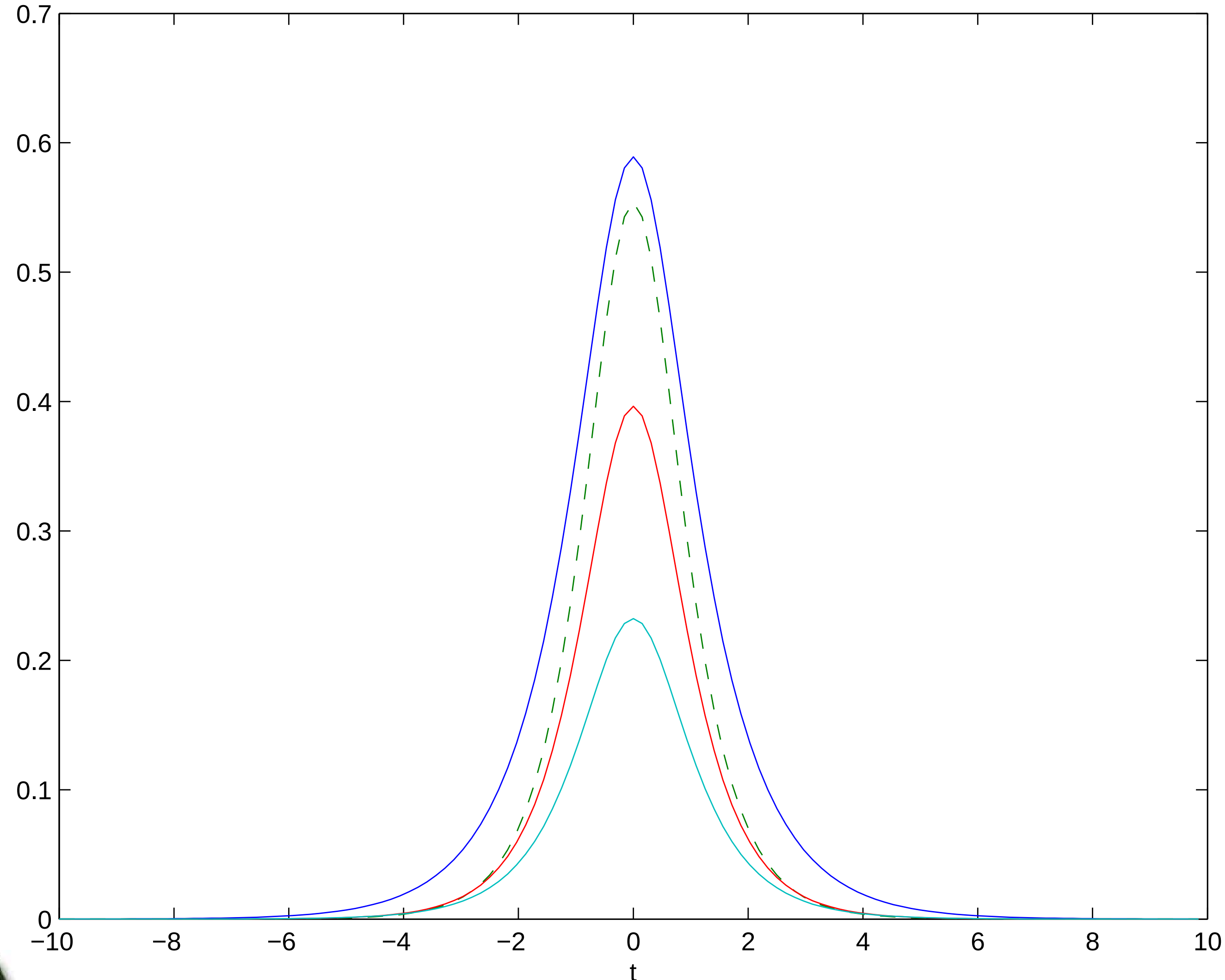
For $N = 4$ (u_0, u_1, u_2, and u_3),



$\lambda_i = 0.5$, $\alpha_i = \gamma_i = 1.0$; shec4t2.m.

Four-channel BPW solitons: the different profiles

For $N = 4$ (u_0, u_1, u_2, and u_3),



$\lambda_i = 0.7$, $\gamma_0 = \alpha_3 = \gamma_3 = 1.0$, $\alpha_1 = \gamma_1 = 0.65$, $\alpha_2 = \gamma_2 = 0.81$; shec4v4.m.

Laplacian eq. in a disk

Eigenmodes of Laplacian equations, $[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2}]u(x,y) = f(x,y)$.