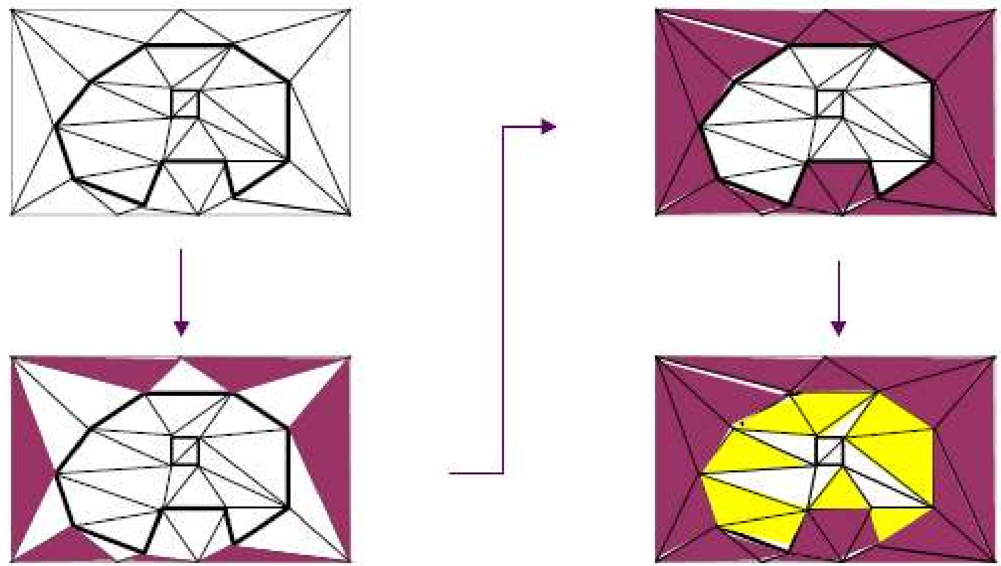
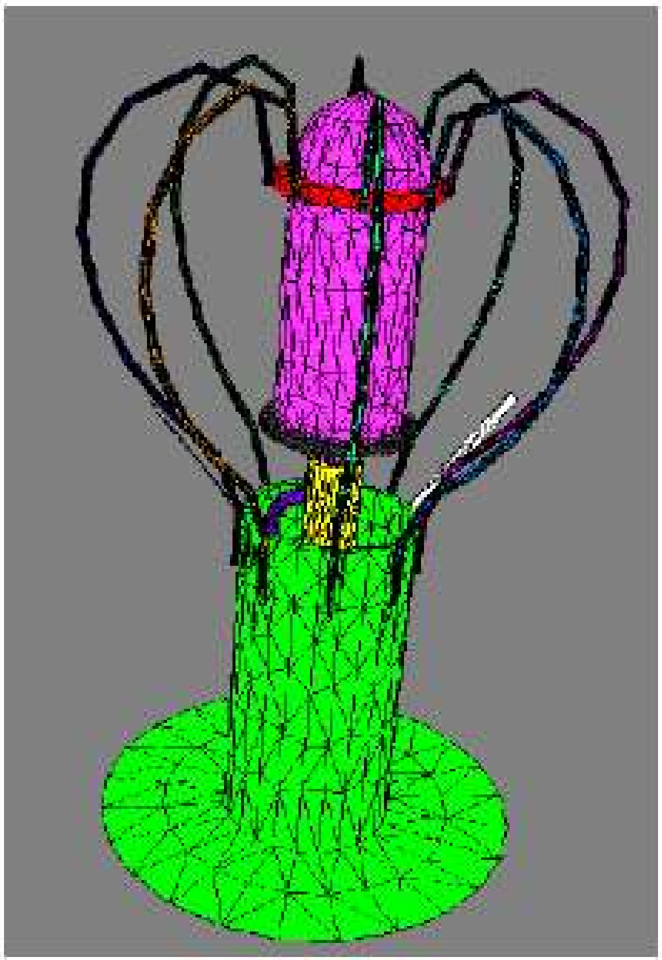
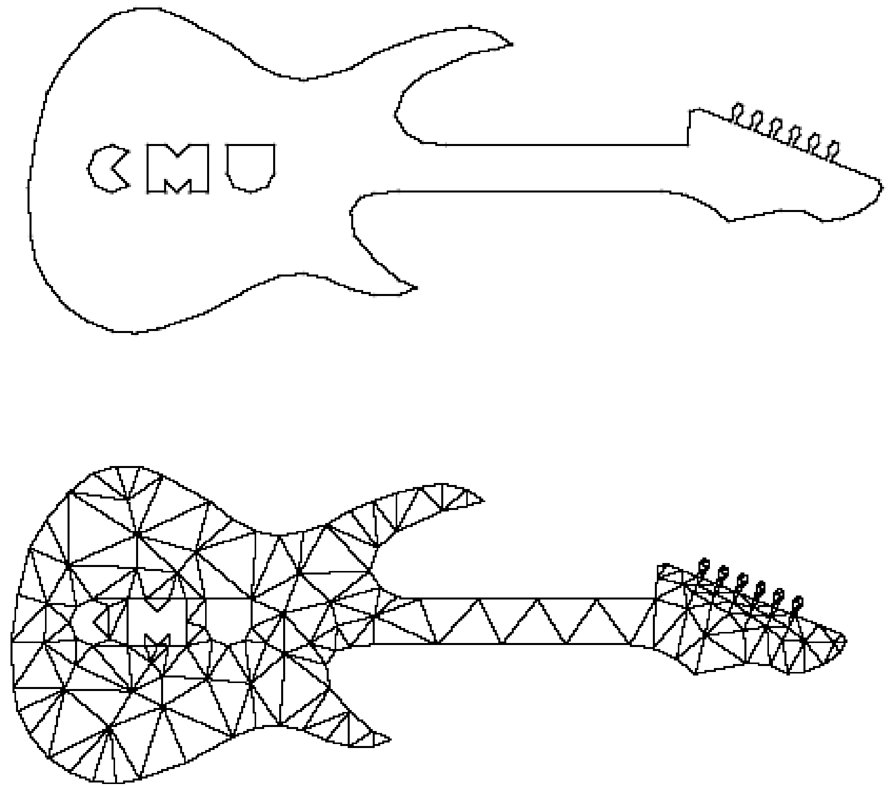
7, Finite Element Method



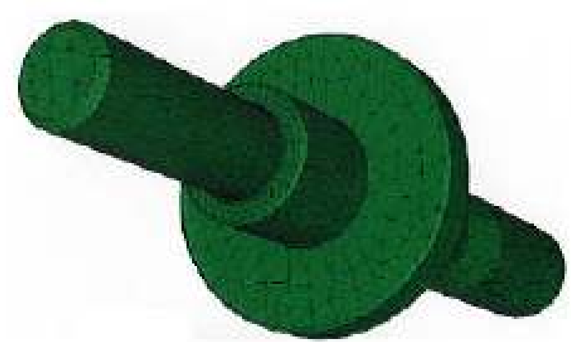
- Elements
- Mesh generation
- Element assembly
- Boundary condition

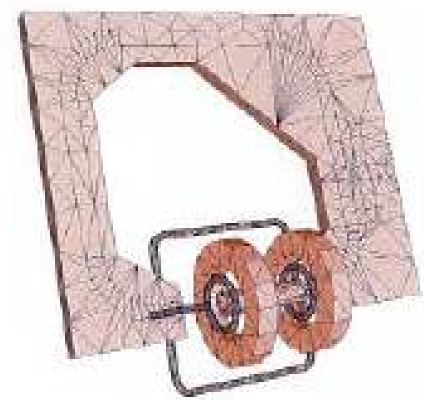
Examples

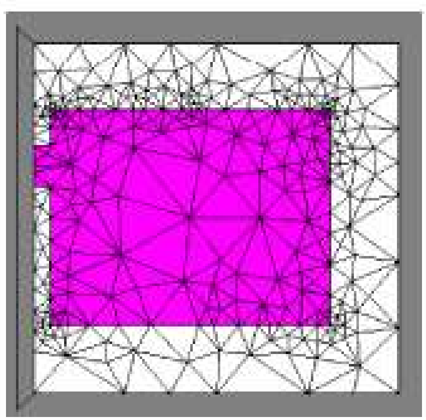




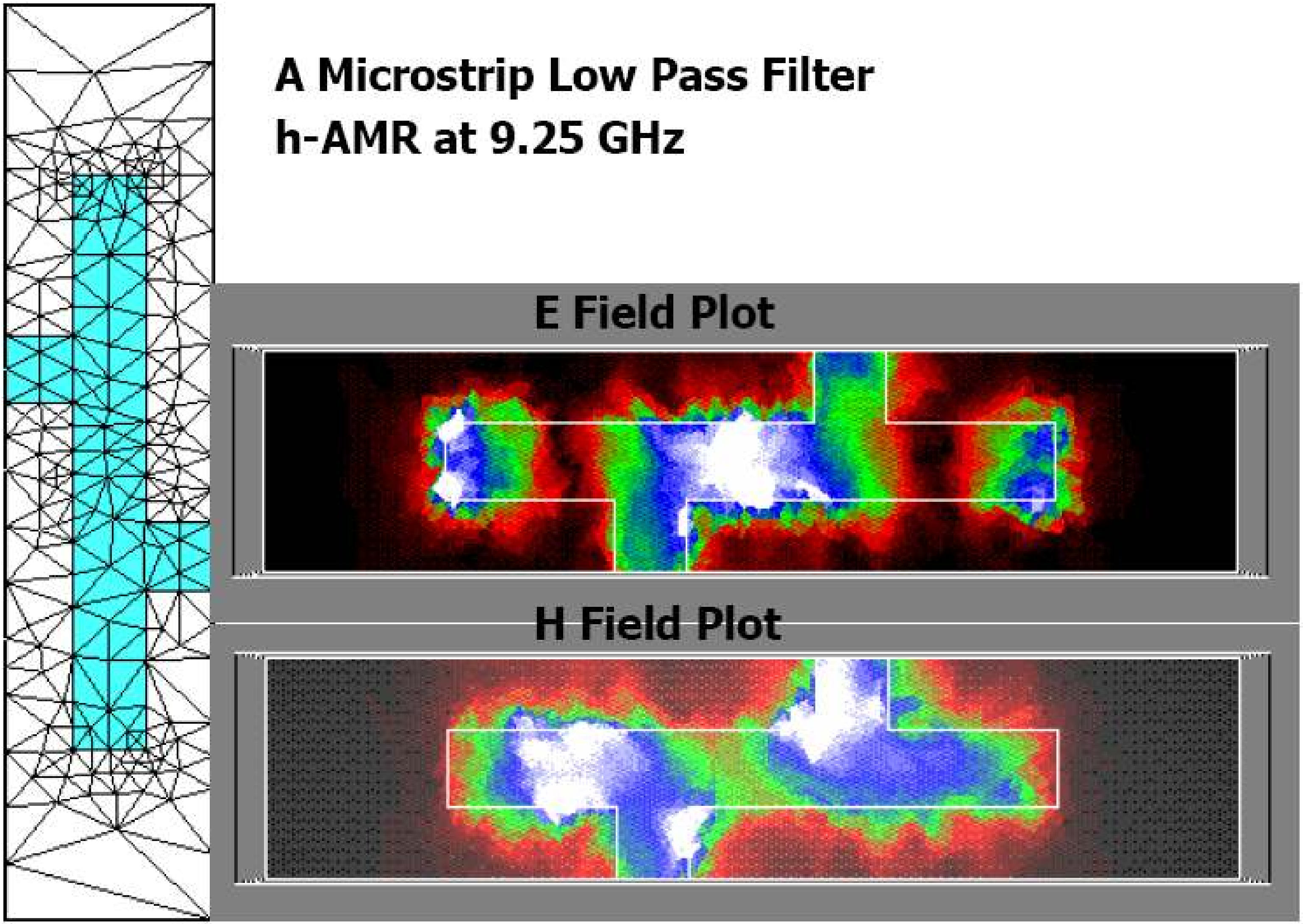
Mechanical problems







Microwave circuit



Ritz method

- For a boundary-value problem,

$$\mathcal{L}\phi = f,$$

where $\mathcal{L}$ is a differential operator, f is the excitation or forcing function, and ϕ is the unknown quantity.

- the solution of this kind of boundary-value problem can be obtained by minimizing the functional,

$$\mathbf{F}(\tilde{\phi}) = \frac{1}{2} < \mathcal{L}\tilde{\phi}, \tilde{\phi} > - \frac{1}{2} < \tilde{\phi}, f > - \frac{1}{2} < f, \tilde{\phi} >,$$

with respect to $\tilde{\phi}$, a trial function, and where

$$< \phi, \psi > = \int_{\Omega} \phi \psi^* \mathrm{d}\Omega.$$

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$$\begin{aligned} \delta\mathbf{F} &= \frac{1}{2}\int_{\Omega} \mathcal{L}\delta\tilde{\phi}\tilde{\phi}^* \mathrm{d}\Omega + \frac{1}{2}\int_{\Omega} \mathcal{L}\tilde{\phi}\delta\tilde{\phi}^* \mathrm{d}\Omega - \frac{1}{2}\int_{\Omega} \delta\tilde{\phi} f \mathrm{d}\Omega - \frac{1}{2}\int_{\Omega} f\delta\tilde{\phi}^* \Omega \\ &= 0 \end{aligned}$$

Ritz method

- Suppose that ϕ can be approximated by the expansion,

$$\tilde{\phi} = \sum_{j=1}^{N} c_j v_j = \{\mathbf{c}\}^T\{\mathbf{v}\} = \{\mathbf{v}\}^T\{\mathbf{c}\},$$

where the v_j are the chosen expansion functions defined over the entire domain and c_j are constant coefficients to be determined.

- the functional becomes

$$\mathbf{F} = \frac{1}{2}\{\mathbf{c}\}^T \int_\Omega \{\mathbf{v}\}\mathcal{L}\{\mathbf{v}\}^T \mathrm{d}\Omega\{\mathbf{c}\} - \{\mathbf{c}\}^T \int_\Omega \{\mathbf{v}\} f \mathbf{d}\Omega.$$

- To minimize $\mathbf{F}(\tilde{\phi})$,

$$\begin{aligned}
\frac{\partial \mathbf{F}}{\partial c_i} &= \frac{1}{2}\int_\Omega v_i\}\mathcal{L}\{\mathbf{v}\}^T \mathrm{d}\Omega\{\mathbf{c}\} + \frac{1}{2}\{\mathbf{c}\}^T \int_\Omega \{\mathbf{v}\}\mathcal{L} v_i \mathrm{d}\Omega - \int_\Omega v_i f \mathbf{d}\Omega \\
&= \frac{1}{2}\sum_{j=1}^{N} c_j \int_\Omega (v_i \mathcal{L} v_j + v_j \mathcal{L} v_i)\mathrm{d}\Omega - \int_\Omega v_i f \mathrm{d}\Omega \\
&= 0, \qquad i = 1, 2, 3, \ldots, N
\end{aligned}$$

Matrix form

$$\begin{aligned}\frac{\partial \mathbf{F}}{\partial c_i} &= \frac{1}{2}\int_\Omega v_i\}\mathcal{L}\{\mathbf{v}\}^T \mathrm{d}\Omega\{\mathbf{c}\} + \frac{1}{2}\{\mathbf{c}\}^T\int_\Omega \{\mathbf{v}\}\mathcal{L}v_i \mathrm{d}\Omega - \int_\Omega v_i f \mathbf{d}\Omega \\ &= \frac{1}{2}\sum_{j=1}^{N} c_j \int_\Omega (v_i\mathcal{L}v_j + v_j\mathcal{L}v_i)\mathrm{d}\Omega - \int_\Omega v_i f \mathrm{d}\Omega \\ &= 0, \qquad i = 1, 2, 3, \ldots, N\end{aligned}$$

- written in the matrix form

$$[\mathbf{S}]\{\mathbf{c}\} = \{\mathbf{b}\},$$

where the elements in $[\mathbf{S}]$ are given by

$$S_{ij} = \frac{1}{2}\int_\Omega (v_i\mathcal{L}v_j + v_j\mathcal{L}v_i)\mathrm{d}\Omega,$$

and the elements in $\{\mathbf{b}\}$ are given by

$$b_i = \int_\Omega v_i f \mathrm{d}\Omega.$$

Poisson equation

- Consider two infinite parallel plates, one located at $x = 0$ with the potential $\phi = 0$ and the other located at $x = 1$ with $\phi = 1$. The space between the plates is filled with a medium with a varying electric charge $\rho(x) = -(x + 1)$.
- Then we have
$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} = x + 1, \qquad 0 < x < 1,$$
in conjunction with the boundary conditions,
$$\begin{aligned} \phi|_{x=0} &= 0, \\ \phi|_{x=1} &= 1. \end{aligned}$$
- The exact solution to this problem is
$$\phi(x) = \frac{1}{6}x^3 + \frac{1}{2}x^2 + \frac{1}{3}x.$$

Solution via the Ritz method

- For the Poisson equation

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} = x + 1, \qquad 0 < x < 1,$$

- the corresponding functional is

$$\mathbf{F}(\tilde{\phi}) = \frac{1}{2}\int_0^1 (\frac{\mathrm{d}\tilde{\phi}}{\mathrm{d}x})^2 \mathrm{d}x + \int_0^1 (x+1)\tilde{\phi}\mathrm{d}x,$$

- the necessary condtion for **F** to be minimum when $\tilde{\phi}(x) = \phi(x)$is

$$\delta\mathbf{F}(\phi) = 0,$$

- with the integrating by parts, we obtain,

$$\delta\phi\frac{\mathrm{d}\phi}{\mathrm{d}x}|_{x=0}^{x=1} - \int_0^1 (\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} - x - 1)\delta\phi\mathrm{d}x = 0.$$

Solution via the Ritz method

- Expand $\tilde{\phi}$ in terms of polynomials,

$$\tilde{\phi}(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3,$$

where c_i $(i = 0, 1, 2, 3)$ are the unknown constants to be determined.

- With the boundary conditions, we have $c_0 = 0$ and $c_1 = 1 - c_2 - c_3$, then

$$\tilde{\phi}(x) = x + c_2(x^2 - x) + c_3(x^3 - x),$$

- the functional **F** becomes, $\mathbf{F} = \frac{2}{5}c_3^2 + \frac{1}{6}c_2^2 + \frac{1}{2}c_2 c_3 - \frac{23}{60}c_3 - \frac{1}{4}c_2 + \frac{4}{3}$, whose derivatives with respect to c_2 and c_3 are given by

$$\begin{aligned}
\frac{\partial \mathbf{F}}{\partial c_2} &= \frac{1}{3}c_2 + \frac{1}{2}c_3 - \frac{1}{4} = 0, \\
\frac{\partial \mathbf{F}}{\partial c_3} &= \frac{1}{2}c_2 + \frac{4}{5}c_3 - \frac{23}{60} = 0,
\end{aligned}$$

we have the solution $c_2 = \frac{1}{2}$ and $c_3 = \frac{1}{6}$.

Wrong trial function for Ritz method

- Expand $\tilde{\phi}$ in terms of polynomials,

$$\tilde{\phi}(x) = c_0 + c_1 x + c_2 x^2,$$

where c_i $(i = 0, 1, 2)$ are the unknown constants to be determined.

- With the boundary conditions, we have $c_0 = 0$ and $c_1 = 1 - c_2$, then

$$\tilde{\phi}(x) = x + c_2(x^2 - x),$$

- the functional **F** becomes,

$$\mathbf{F} = \frac{2}{3}c_2^2 - \frac{3}{4}c_2 + \frac{8}{6},$$

whose derivative with respect to c_2 is given by

$$\frac{\partial \mathbf{F}}{\partial c_2} = \frac{4}{3}c_2 - \frac{3}{4} = 0,$$

we have the solution $c_2 = \frac{9}{16}$.

Galerkin's method

- Assume that $\tilde{\phi}$ is an approximation solution, and substitute of $\tilde{\phi}$ for ϕ would then result in a nonzero residual,

$$\mathbf{r} = \mathcal{L}\tilde{\phi} - f \neq 0,$$

- The best approximation for $\tilde{\phi}$ will be the one that reduce the residula **r** to the least value at all points of Ω.
- The weighted residual method enforce the condition,

$$\mathbf{R}_i = \int_\Omega w_i \mathbf{r} \mathrm{d}\Omega = 0,$$

  where $\mathbf{R}_i$ denote weighted residual integrals and w_i are chosen weighting functions.
- In Galerkin's method, the weighting function are selected to be the same as those used for expansion of the approximated solution,

$$w_i = v_i, \qquad i = 1, 2, 3, \ldots, N$$

- the functional becomes $\mathbf{R}_i = \int_\Omega (v_i \mathcal{L}\{\mathbf{v}\}^T\{\mathbf{c}\} - v_i f)\mathrm{d}\Omega, \qquad i = 1, 2, 3, \ldots, N$

Using subdomain expansion functions

- The important step in the Ritz and Galerkin methods is the selection of rial functions defined over the entire solution domain that can represent the true solution, at least approximately.
- To alleviate the difficulty, we can divide the entrie domain into small subdomains and use trial functions defined over each subdomain.
- For example, we divide the entire solution domain $(0, 1)$ into tree subdomains defined by (x_1, x_2), (x_2, x_3), and (x_3, x_4), where $x_1 = 0$ and $x_4 = 1$ being the endpoints.
- a linear variation of $\phi(x)$ over the subdomain is defined

$$\tilde{\phi}(x) = \phi_i \frac{x_{i+1} - x}{x_{i+1} - x_i} + \phi_{i+1} \frac{x - x_{i+1}}{x_i - x_{i+1}},$$

for $x_i \le x \le x_{i+1}$ and $i = 1, 2, 3$, where the ϕ_i are the unknown constants to be determined.

Using subdomain expansion functions

- From the boundary condition, we find $\phi_1 = 0$ and $\phi_4 = 1$.
- Apply the Ritz method,

$$\mathbf{F} = \sum_{i=1}^{3} [\frac{1}{2} \int_{x_i}^{x_{i+1}} (\frac{\phi_{i+1} - \phi_i}{x_{i+1} - x_i})^2 \mathrm{d}x + \int_{x_i}^{x_{i+1}} (x+1)(\phi_i \frac{x_{i+1} - x}{x_{i+1} - x_i} + \phi_{i+1} \frac{x - x_i}{x_{i+1} - x_i}) \mathrm{d}x],$$

- Evaluate the integrals, we obtain,

$$\begin{aligned} \mathbf{F} &= \sum_{i=1}^{3} \frac{1}{2}(x_{i+1} - x_i)[(\frac{\phi_{i+1} - \phi_i}{x_{i+1} - x_i})^2 + \phi_{i+1}(\frac{2}{3}x_{i+1} + \frac{1}{3}x_i + 1) + \phi_i(\frac{2}{3}x_i + \frac{1}{3}x_{i+1} \\ &= 3\phi_2^2 + 3\phi_3^2 - 3\phi_2\phi_3 + \frac{4}{9}\phi_2 - \frac{22}{9}\phi_3 + \frac{49}{27}. \end{aligned}$$

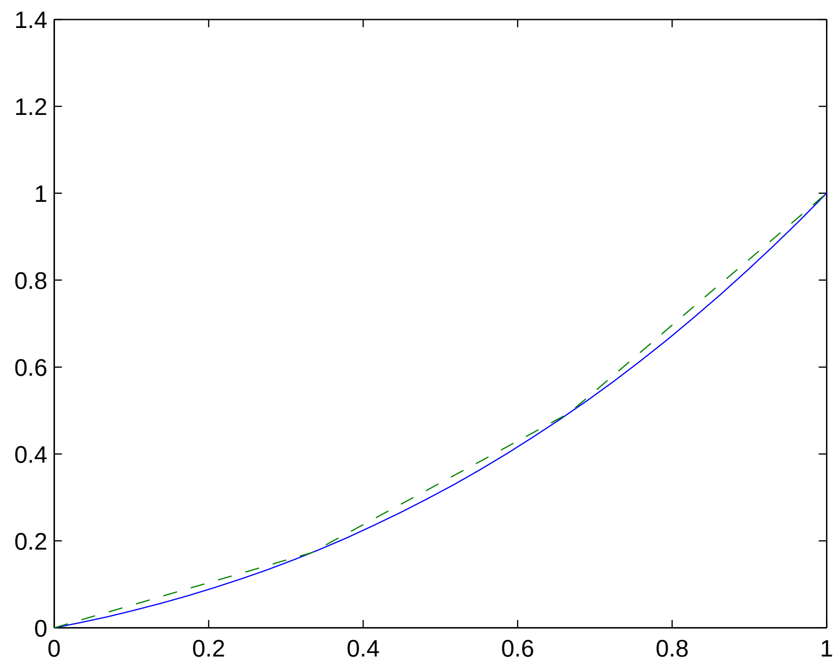
- Minimize **F**, we obtain $\phi_2 = \frac{14}{81}$ and $\phi_3 = \frac{40}{81}$.

FEM method for Poisson equation

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} = x + 1, \qquad 0 < x < 1,$$

in conjunction with the boundary conditions,

$$\begin{aligned} \phi|_{x=0} &= 0, \\ \phi|_{x=1} &= 1. \end{aligned}$$



Linear Elements for 1D

- Linear elements: within the i-th element $\phi(x)$ may be approximated by

$$\phi^i(x) = a^i + b^i x,$$

where a^i and b^i are the constants to be determined. At the two nodes,

$$\begin{aligned} \phi_1^i &= a^i + b^i x_1^i, \\ \phi_2^i &= a^i + b^i x_2^i, \end{aligned}$$

subsitute a^i and b^i back to $\phi^i(x)$, one has,

$$\phi^i(x) = \sum_{j=1}^{2} N_j^i(x)\phi_j^i,$$

where

$$N_1^i(x) = \frac{x_2^i - x}{\Delta x}, \quad \text{and} \quad N_2^i(x) = \frac{x - x_1^i}{\Delta x},$$

denote the interpolation or basis function, and $N_j^i(x_k^i) = \delta_{jk}$.

Quadratic Elements for 1D

- Quadratic elements: also known as second-order elements, within the i-th element $\phi(x)$ may be approximated by $\phi^i(x) = a^i + b^i x + c^i x^2$, where a^i and b^i are the constants to be determined. At the two nodes,

$$\begin{aligned}
\phi_1^i &= a^i + b^i x_1^i + c^i (x_1^i)^2, \\
\phi_2^i &= a^i + b^i x_2^i + c^i (x_2^i)^2, \\
\phi_3^i &= a^i + b^i x_1^i + c^i (x_3^i)^2,
\end{aligned}$$

subsitute a^i, b^i, and c^i back to $\phi^i(x)$, one has, $\phi^i(x) = \sum_{j=1}^{3} N_j^i(x)\phi_j^i$, where

$$\begin{aligned}
N_1^i(x) &= \frac{(x - x_2^i)(x - x_3^i)}{(x_1^i - x_2^i)(x_1^i - x_3^i)}, \\
N_2^i(x) &= \frac{(x - x_1^i)(x - x_3^i)}{(x_2^i - x_1^i)(x_2^i - x_3^i)}, \\
N_3^i(x) &= \frac{(x - x_1^i)(x - x_2^i)}{(x_3^i - x_1^i)(x_3^i - x_1^i)},
\end{aligned}$$

denote the interpolation or basis function, and $N_j^i(x_k^i) = \delta_{jk}$.

Elements for 2D

- Cubic elements for 1D:

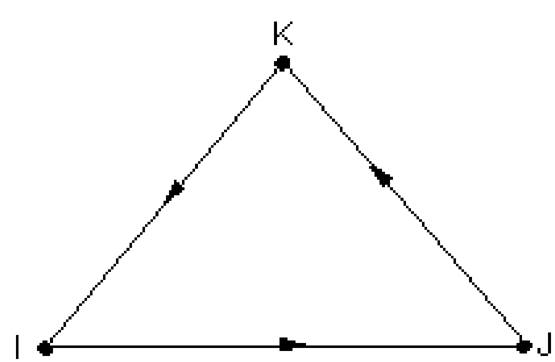
$$\phi^i(x) = a^i + b^i x + c^i x^2 + d^i x^3,$$

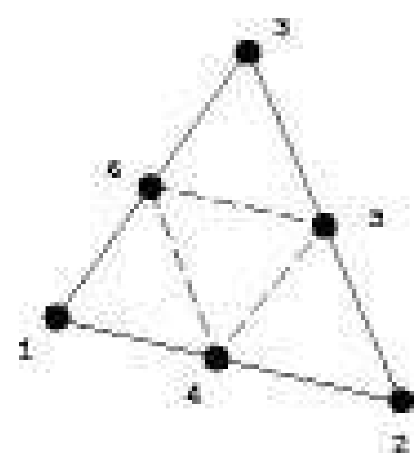
- Linear triangular element for 2D:

$$\phi_i(x, y) = \sum_{j=1}^{3} N_j^i(x, y)\phi_j^i,$$

where

$$N_j^i(x, y) = \frac{1}{2\Delta^i}(a_j^i + b_j^i x + c_j^i y).$$





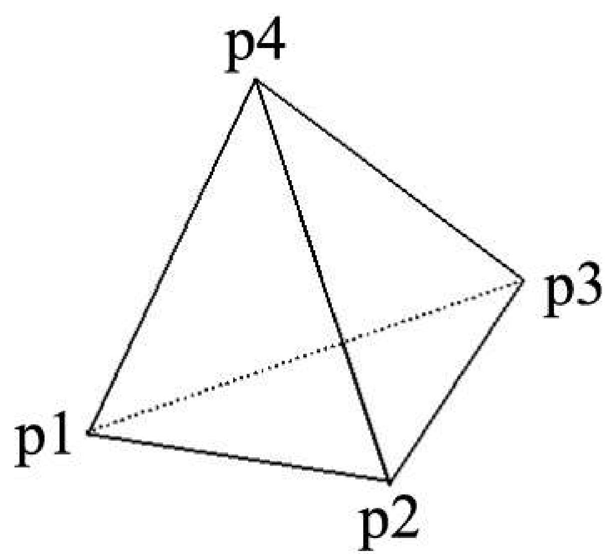
Elements for 3D

- Linear tetrahedral element

$$\phi_i(x,y) = \sum_{j=1}^{4} N_j^i(x,y,z)\phi_j^i,$$

where

$$N_j^i(x,y,z) = \frac{1}{6\mathbf{V}^i}(a_j^i + b_j^i x + c_j^i y + d_j^i z).$$

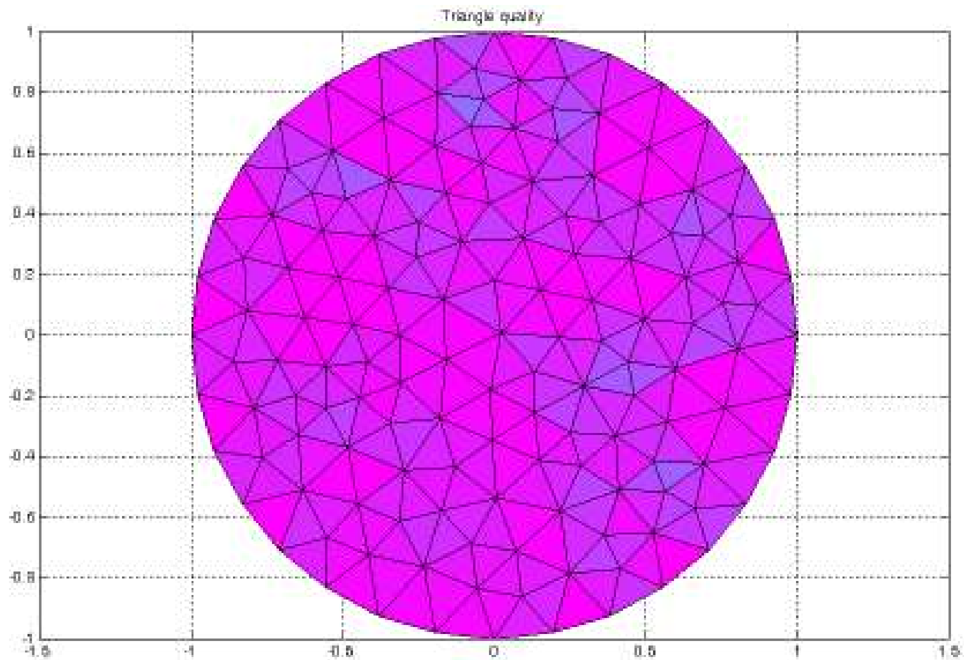


PDETOOL in Matlab

For the Poisson's equation on unit disk,

$$\begin{aligned} \nabla^2 U &= 1, \qquad \text{in} \quad \Omega, \\ U &= 0, \qquad \text{on the boundary,} \quad \partial\Omega \end{aligned}$$

where Ω is the unit disk.



Laplace's equation

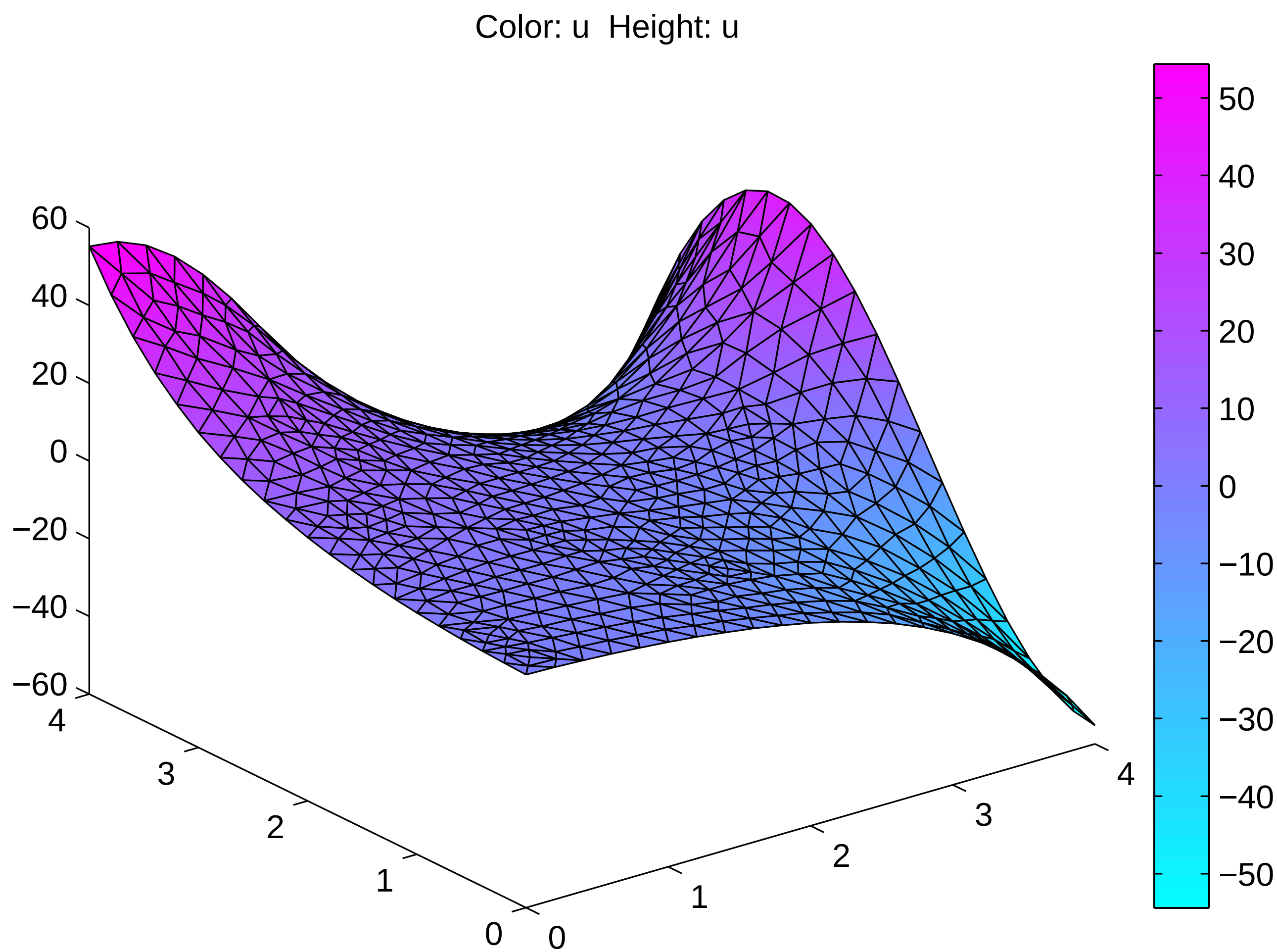
Consider the Laplace's equation:

$$\nabla^2 u(x,y) = \frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0, \qquad \text{for} \quad 0 \le x \le 4, 0 \le y \le 4,$$

with the following conditions

$$\begin{aligned} u(0,y) &= e^y - \cos y, \\ u(4,y) &= e^y \cos 4 - e^4 \cos y, \\ u(x,0) &= \cos x - e^x, \\ u(x,4) &= e^4 \cos x - e^x \cos 4. \end{aligned}$$

Laplace's equation



Photonic Crystals