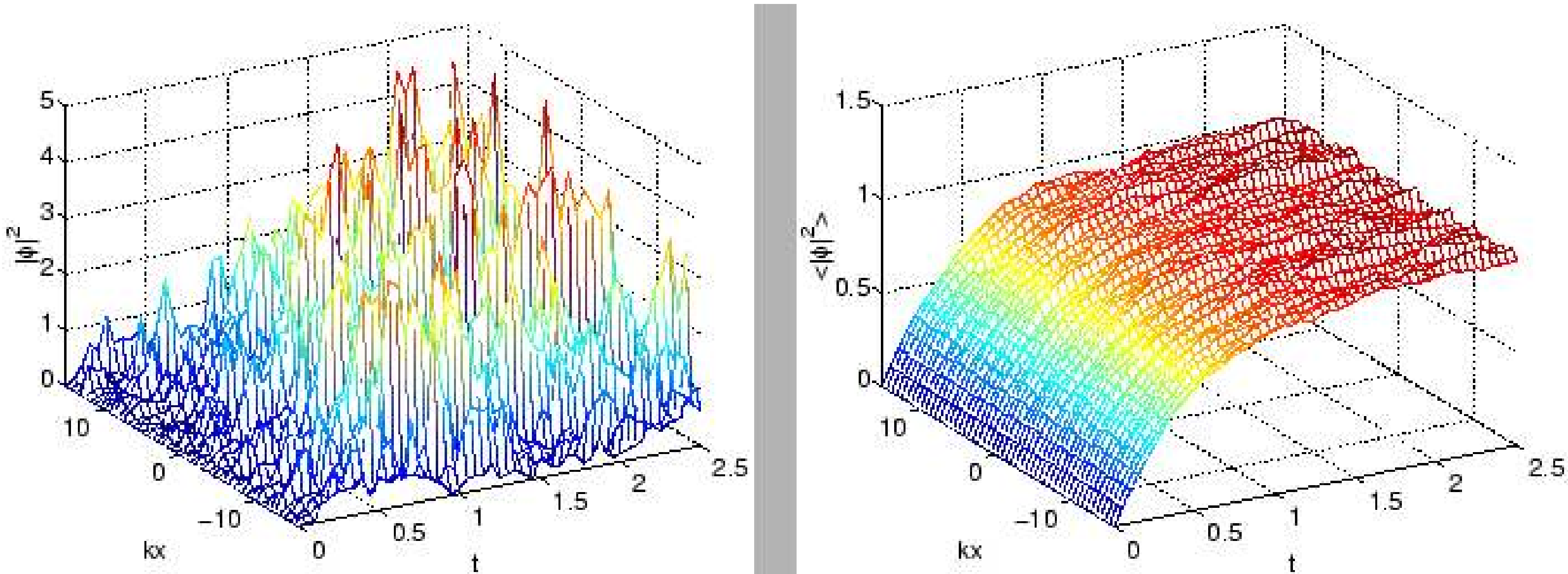
8, Monte Carlo Method



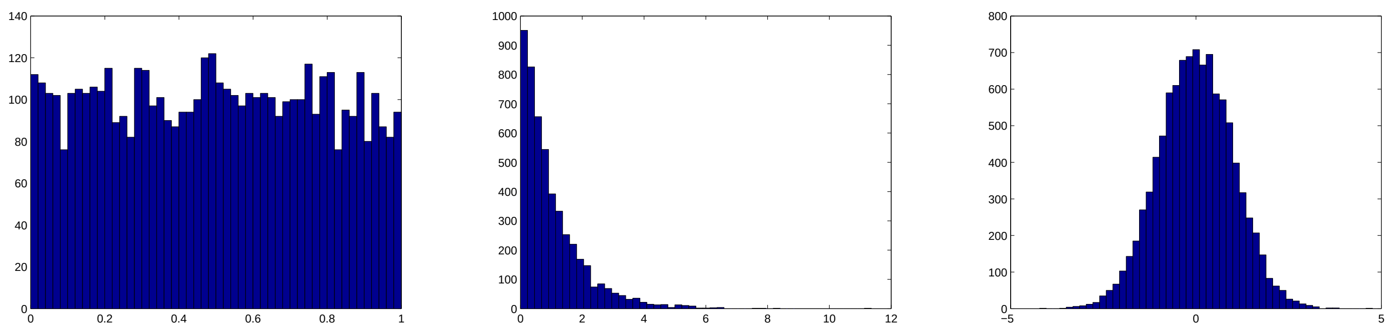
a) Single path   b) 1024 path mean

- Random numbers with uniform deviates
- Transformation method
- Rejection method
- Random bits
- Monte Carlo methods

Random Numbers

- Philosophers: Any program will produce output that is entirely preditable, hence not truly "*random*".
- Random number generators: *pseudo-random*.
- A good generator: to produce statistically the same results.

1. Uniform Deviates
2. Exponential Deviates
3. Normal Deviates



Uniform Deviates

Uniform deviates are just random numbers that line within a specified range (typically 0 to 1), with any one number in the range just as likely as any other.

- in C:
  void srand(unsigned seed);
  int rand(void);
- in Matlab
  s = rand('state');
  rand('state',s);

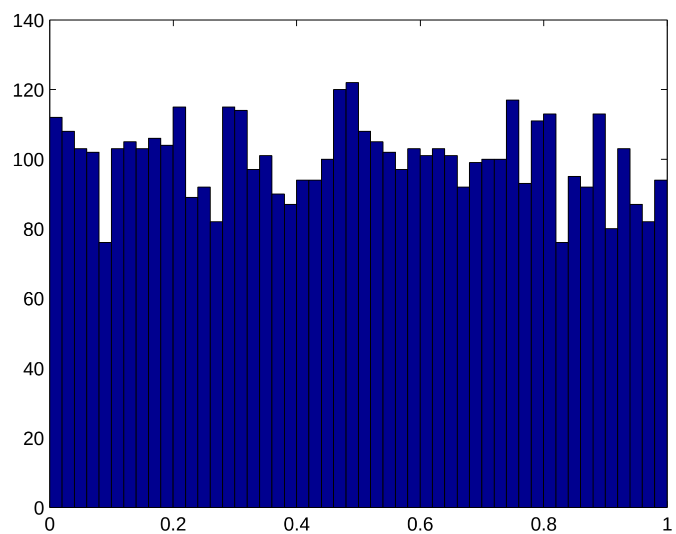
System-supplied **rand()**s are almost always *linear congruential generators*, which generate a sequence of integers $I_1, I_2, I_3, \ldots$, each between 0 and $m-1$ (e.g. $RAND_MAX$) by the recurrence relation,

$$I_{j+1} = aI_j + c \qquad (\text{mod} \quad m),$$

for example, $a = 1103515245$, $c = 12345$, and $m = 2^{32}$.

Uniform Deviates

```
a = 1103515245;
c = 12345;
m = 2^32;
xj = 10000.0;
yj = xj/(m-1);
for indi = 1:1000
xj1 = mod(a*xj+c, m);
yj = [yj; xj1/(m-1)];
xj = xj1;
end
hist(yj);
```



Quick and Dirty Random Number Generators

Constants for Quick and Dirty Random Number Generators

overflow at	im	ia	ic
2^{20}	6075	106	1283
2^{21}	7875	211	1663
2^{22}	7875	421	1663
2^{23}	6075	1366	1283
	6655	936	1399
	11979	430	2531
2^{24}	14406	967	3041
	29282	419	6173
	53125	171	11213
2^{25}	12960	1741	2731
	14000	1541	2957
	21870	1291	4621
	31104	625	6571
	139968	205	29573
2^{26}	29282	1255	6173
	81000	421	17117
	134456	281	28411
2^{27}	86436	1093	18257
	121500	1021	25673
	259200	421	54773
2^{28}	117128	1277	24749
	121500	2041	25673
	312500	741	66037
2^{29}	145800	3661	30809
	175000	2661	36979
	233280	1861	49297
	244944	1597	51749
2^{30}	139968	3877	29573
	214326	3613	45289
	714025	1366	150889
2^{31}	134456	8121	28411
	259200	7141	54773
2^{32}	233280	9301	49297
	714025	4096	150889

Transformation method

- For a unifrom probability distribution,

$$p(x)\mathrm{d}x = \begin{cases} \mathrm{d}x, & 0 < x < 1; \\ 0, & \text{otherwise}; \end{cases}$$

with the normalization condition,

$$\int_{-\infty}^{\infty} p(x)\mathrm{d}x = 1.$$

- Transformation law of probabilities,

$$|p(y)\mathrm{d}y| = |p(x)\mathrm{d}x|,$$

or

$$p(y) = p(x)|\frac{\mathrm{d}y}{\mathrm{d}x}|.$$

Exponential Deviates

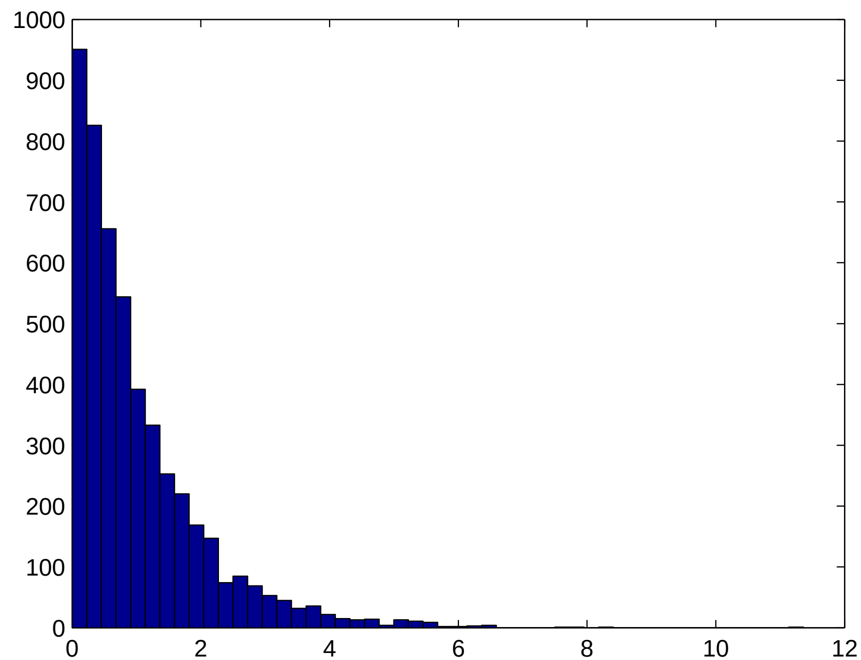
- Exponential deviates:

$$y(x) \equiv -\ln(x),$$

and $p(x)$ is a uniform deviate,

$$p(y)\mathrm{d}y = |\frac{\mathrm{d}y}{\mathrm{d}x}|\mathrm{d}y = e^{-y}\mathrm{d}y,$$

which is distributed exponentially.



Transformation method

- For the transformation method,

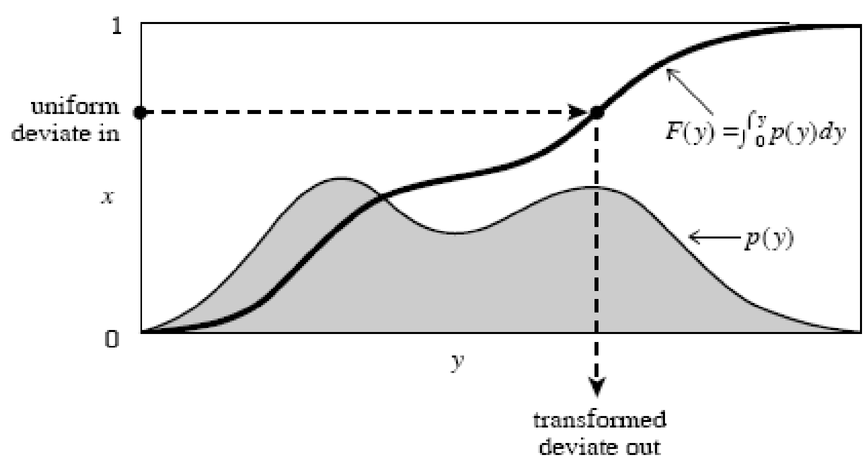
$$\frac{\mathrm{d}x}{\mathrm{d}y} = f(y),$$

  has the solution $x = F(y)$, where $F(y)$ is the indefinite integral of $f(y)$.

- To make a uniform deviate into one distributed as $f(y)$ is therefore,

$$y(x) = F^{-1}(x).$$

  The transformation method is to transform the *inverse function of the ingegral of* $f(y)$.



Normal (Gaussian) Deviates

- More than one dimension,

$$p(y_1, y_2, \ldots)\mathrm{d}y_1\mathrm{d}y_2 \cdots = p(x_1, x_2, \ldots)\left|\frac{\partial(x_1, x_2, \ldots)}{\partial(y_1, y_2, \ldots)}\right|\mathrm{d}y_1\mathrm{d}y_2 \ldots,$$

where $|\partial()/\partial()|$ is the Jacobian determinant of x's with respect to the y's.

- Box-Muller method for normal (Gaussian) distribution,

$$p(y)\mathrm{d}y = \frac{1}{\sqrt{2\pi}}e^{-y^2/2}\mathrm{d}y,$$

Consider the transformation between two uniform deviates on $(0, 1)$, x_1 and x_2,

$$\begin{aligned} y_1 &= \sqrt{-2\ln x_1}\cos 2\pi x_2, \\ y_2 &= \sqrt{-2\ln x_1}\sin 2\pi x_2, \end{aligned}$$

Box-Muller method

Box-Muller method for normal (Gaussian) distribution,

$$p(y)\mathrm{d}y = \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \mathrm{d}y,$$

- Equivalently, consider the transformation between two uniform deviates on $(0, 1)$, x_1 and x_2,

$$\begin{aligned} x_1 &= \exp[-\frac{1}{2}(y_1^2 + y_2^2)], \\ x_2 &= \frac{1}{2\pi} \arctan \frac{y_2}{y_1}, \end{aligned}$$

- The Jacobian determinant is

$$\frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = -[\frac{1}{2\pi} e^{-y_1^2/2}][\frac{1}{2\pi} e^{-y_2^2/2}]$$

i.e. $\frac{\mathrm{d}\arctan(x)}{\mathrm{d}x} = 1/(1 + x^2)$.

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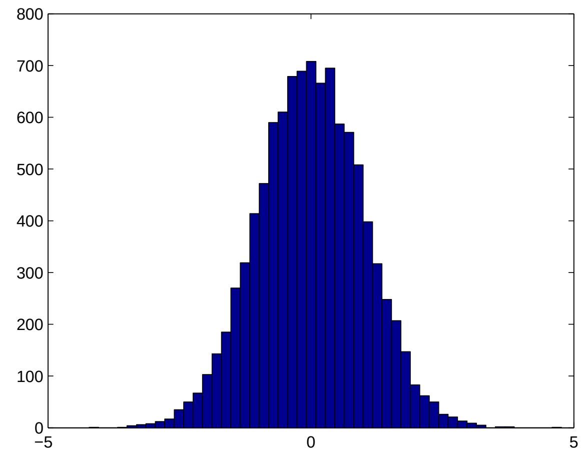
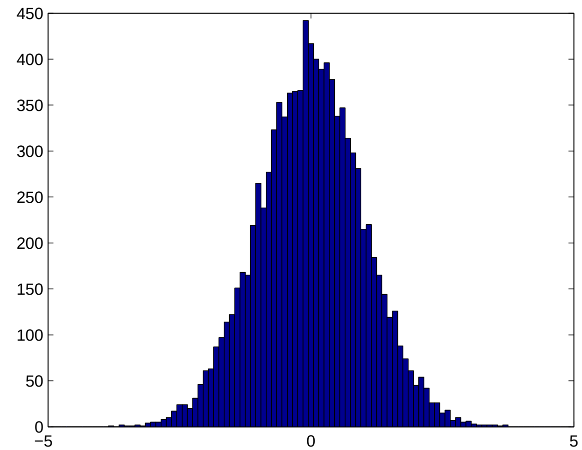
Normal (Gaussian) Deviats

By Box-Muller method





Using Matlab build-in **randn**

Entangled sources for Quantum Information Science

In QIS, you need *non-classical* states as *qbits*.

- Low-intensity limit:
  Single photon sources, with definite *photon number* but largest fluctuation in phase, which is intrinsic *non-classical* states.

- High-intensity limit:
  Squeezed states, which are *macroscopic*, continuous-variables, i.e.

$$\hat{M} = M_0 + \Delta\hat{M},$$

where M_0 is the classical (mean-field) variables, such as *photon-number*, *phase*, *position*, and *momentum* etc.

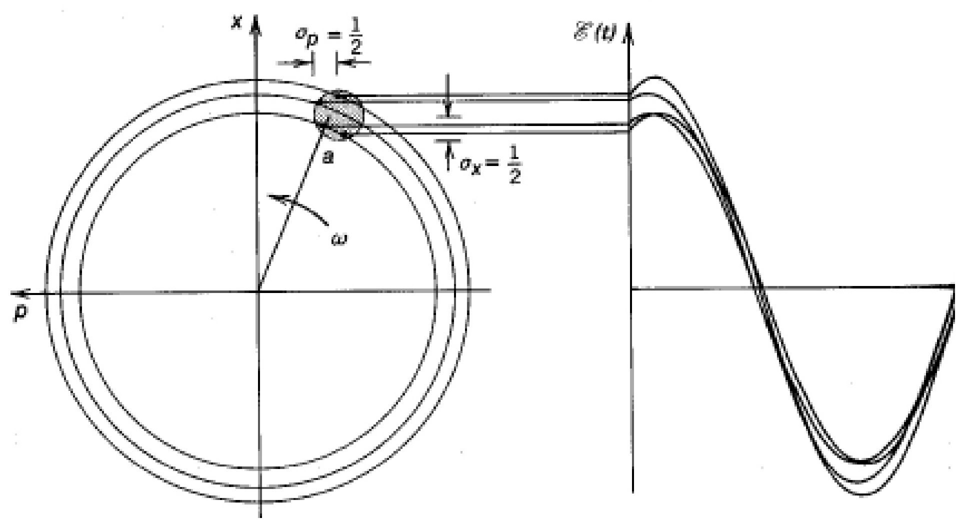
Phase diagram for EM waves

Electromagnetic waves can be represented by

$$\hat{E}(t) = E_0[\hat{X}_1 \sin(\omega t) - \hat{X}_2 \cos(\omega t)]$$

where

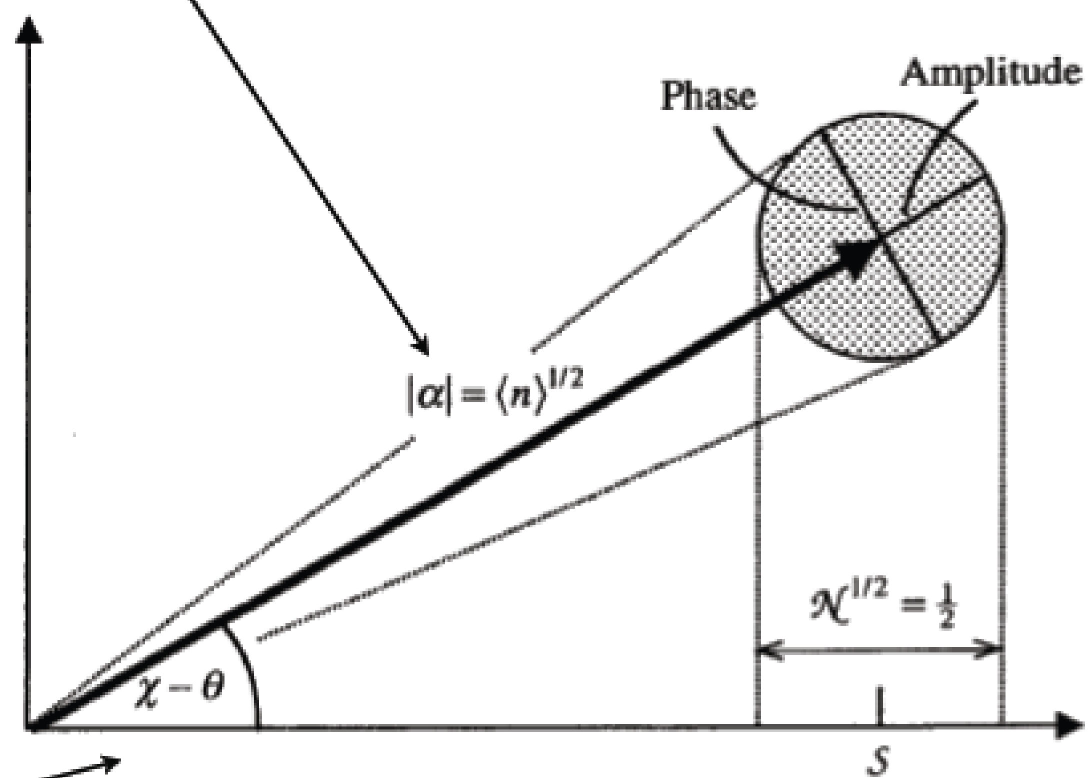
$$\hat{X}_1 = \text{amplitude quadrature}$$
$$\hat{X}_2 = \text{phase quadrature}$$



Phase diagram for coherent states

mean number of photons

$$< \hat{N} >=< \alpha|\hat{N}|\alpha >=< \alpha|\hat{a}^\dagger \hat{a}|\alpha >= |\alpha|^2$$



phase of the field

$$\alpha = |\alpha| \exp(i\theta)$$

2005 Nobel Laureates

Glauber(Harvard)   Hall(JILA)   Hänsch(MPI)

Roy J. Glauber: "for his contribution to the quantum theory of optical coherence,"

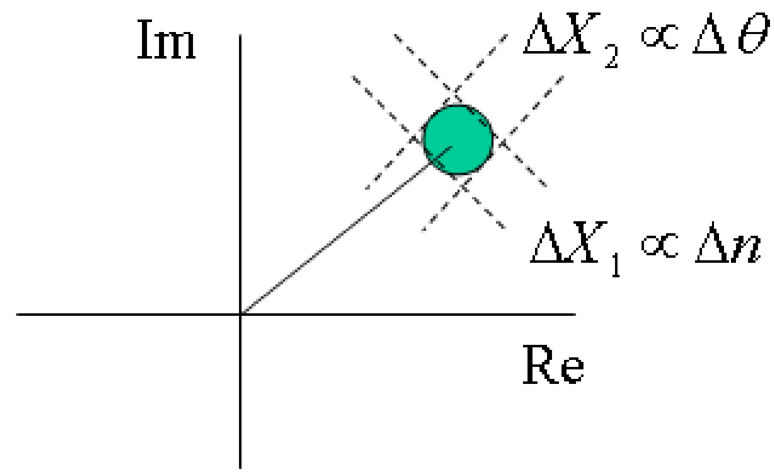
John L. Hall and Theodor W. Hänsch: "for their contributions to the development of laser-based precision spectroscopy, including the optical frequency comb technique."

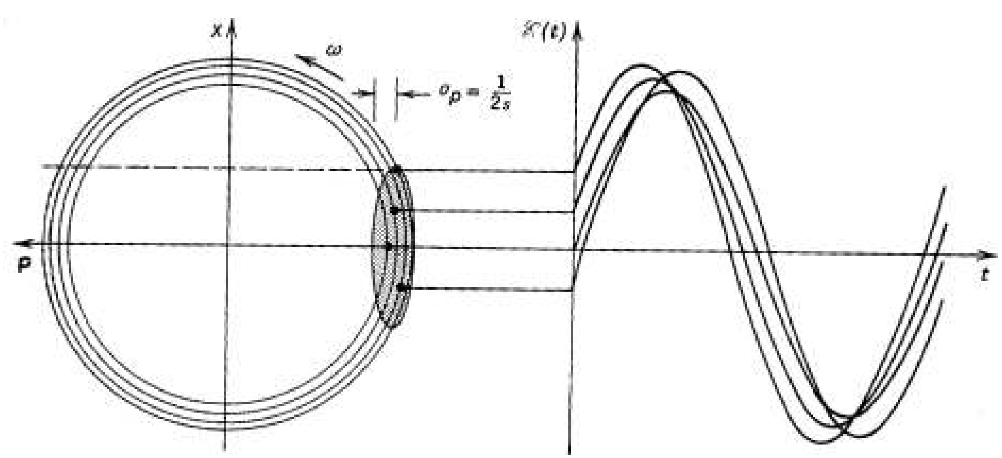
from: http://nobelprize.org/

Coherent and Squeezed States

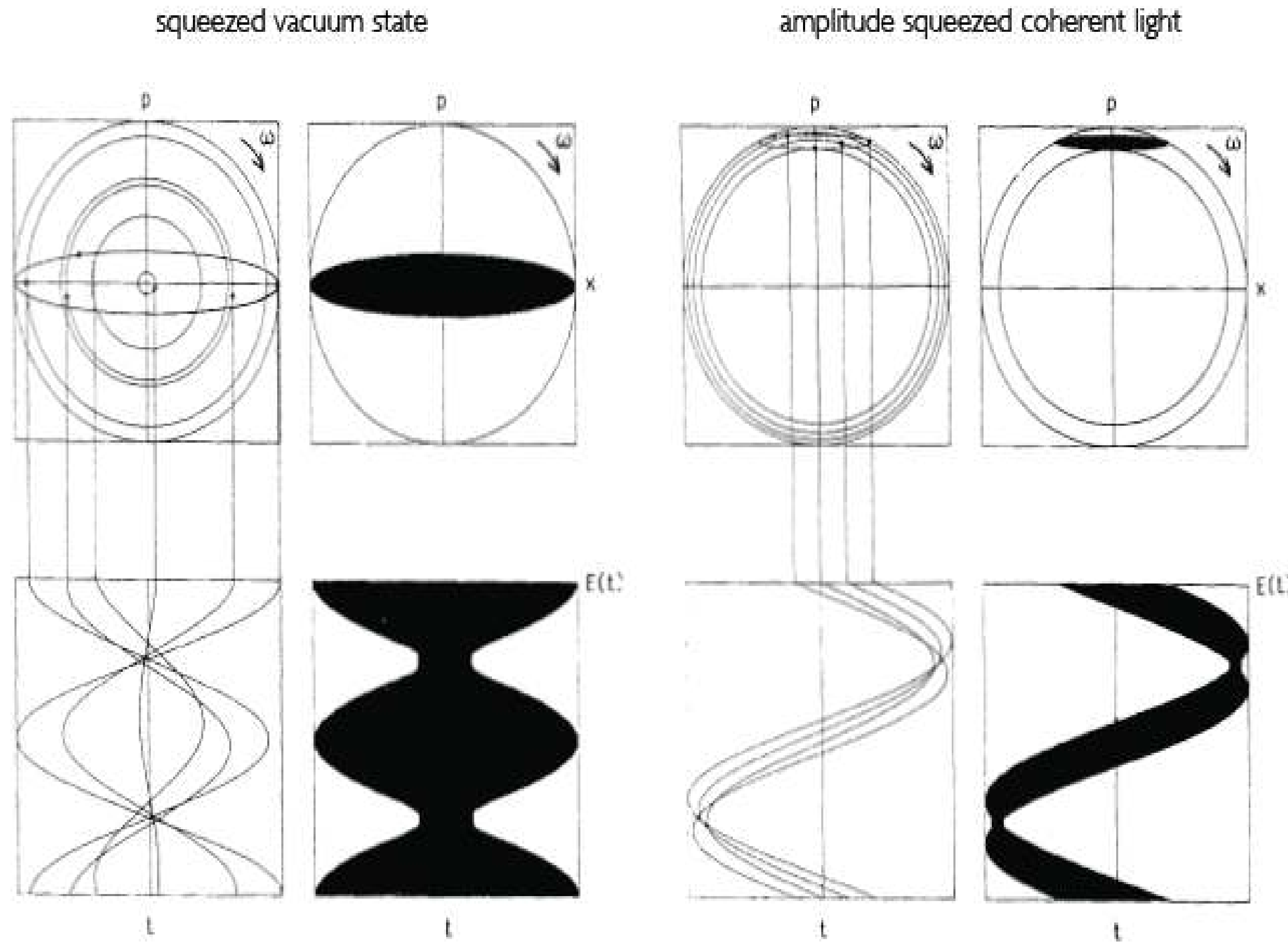
Uncertainty Principle: $\Delta\hat{X}_1\Delta\hat{X}_2 \geq 1$.

1. Coherent states: $\Delta\hat{X}_1 = \Delta\hat{X}_2 = 1$,
2. Amplitude squeezed states: $\Delta\hat{X}_1 < 1$,
3. Phase squeezed states: $\Delta\hat{X}_2 < 1$,
4. Quadrature squeezed states.



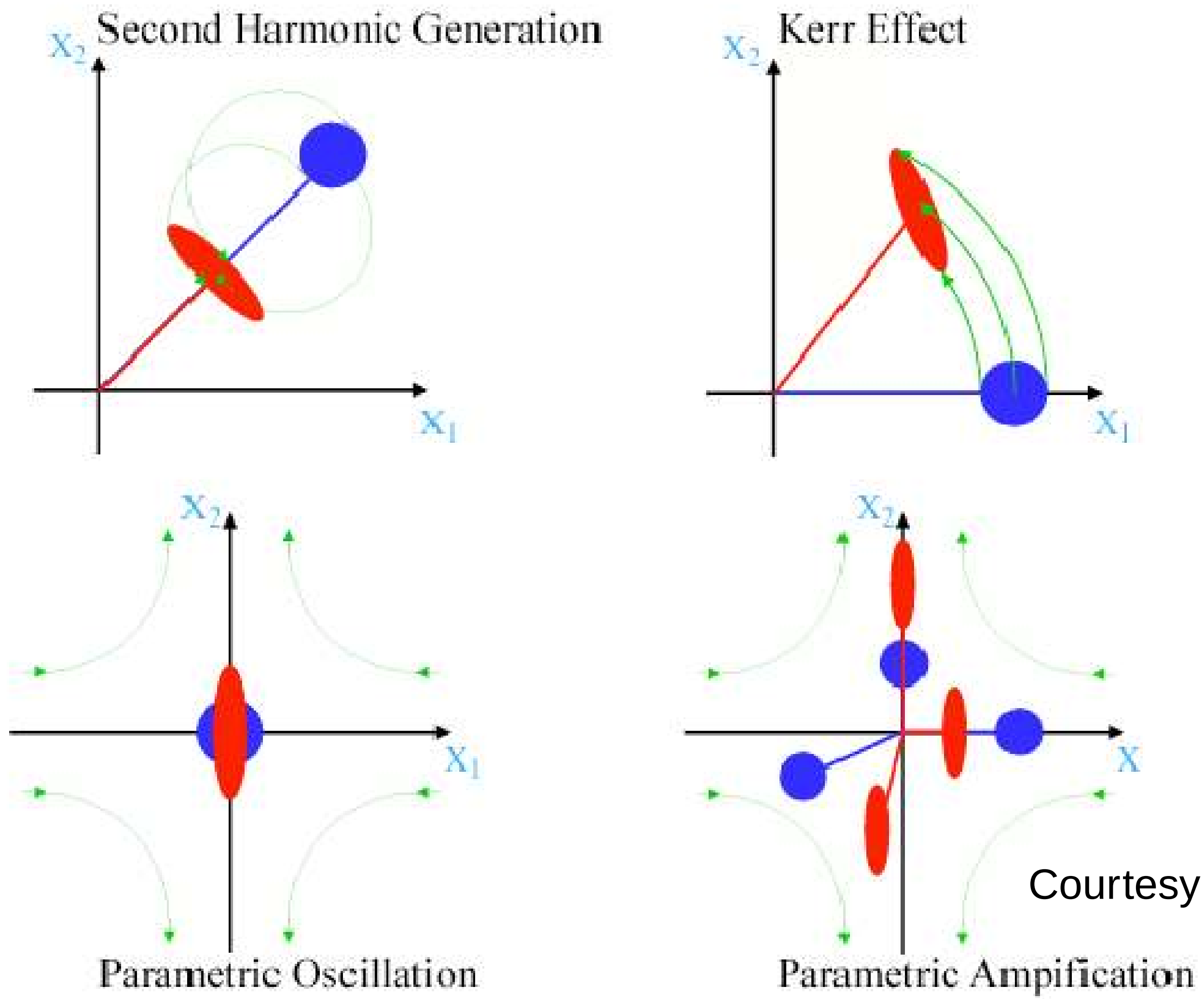


Phase diagram for squeezed states



Generations of Squeezed States

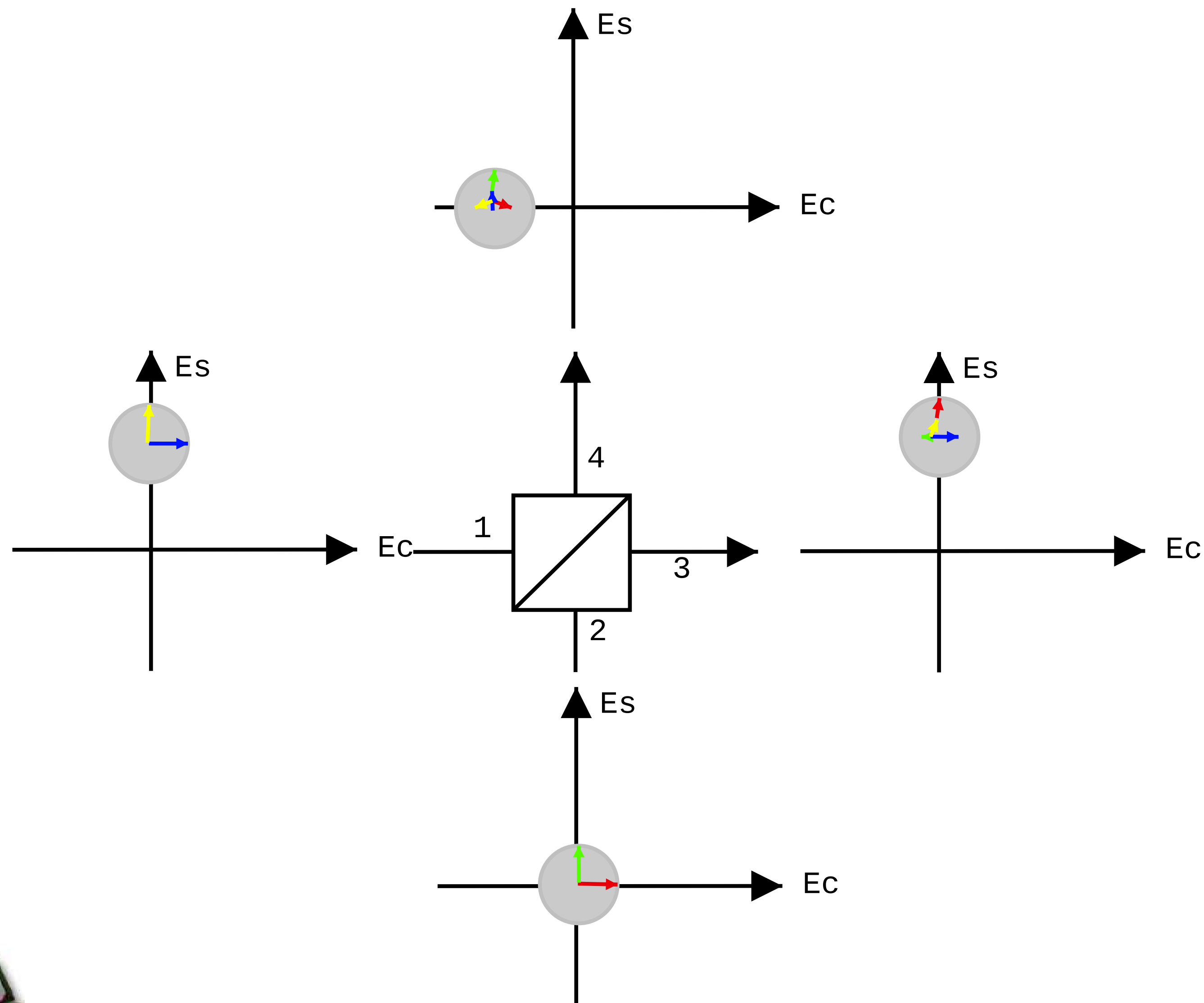
Nonlinear optics:



Courtesy of P. K. Lam

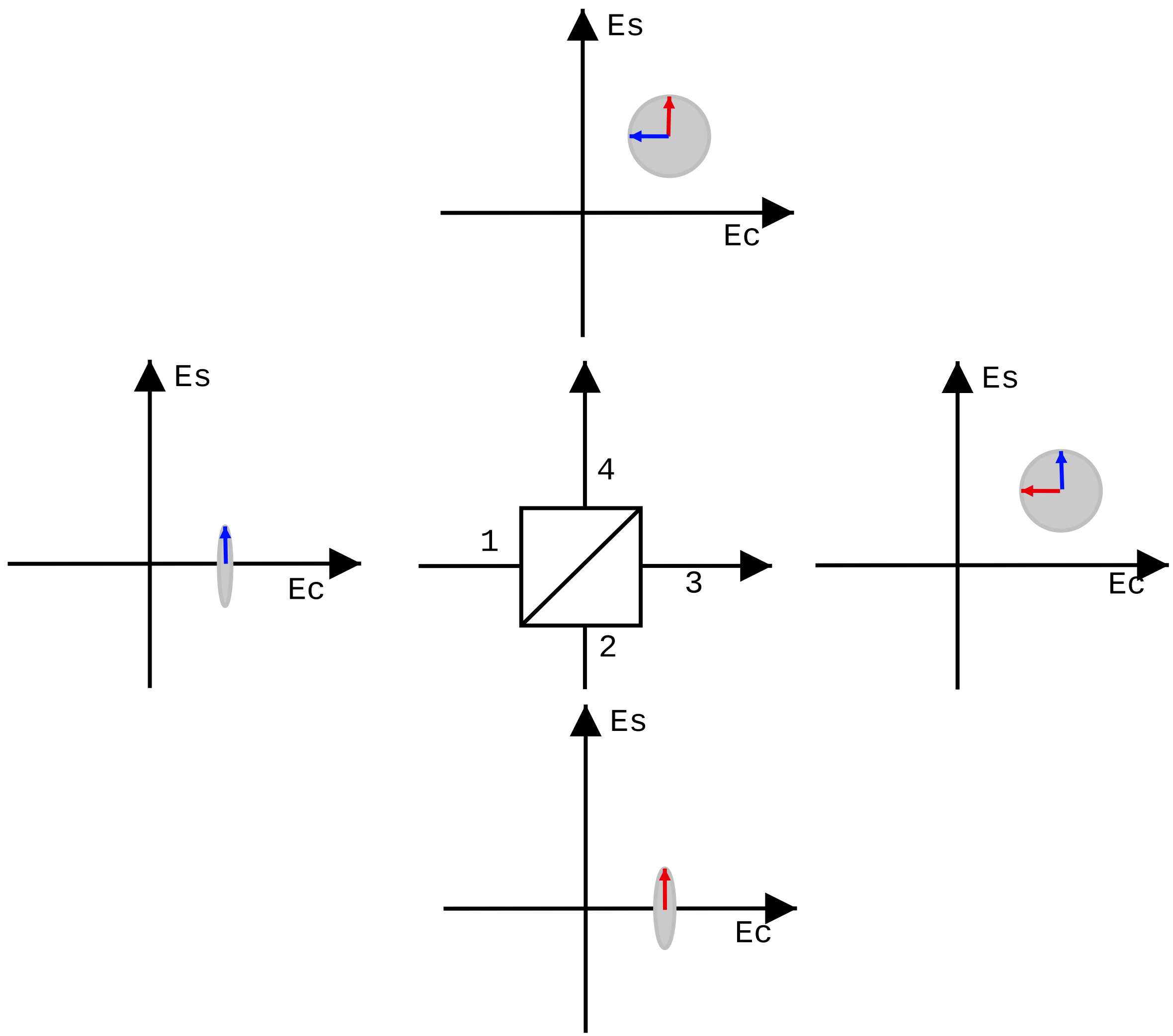
Interference of Coherent States

Coherent States



Generation of Continuous Variables Entanglement

Preparation EPR pairs by Squeezed Sates



$$\delta\hat{n}_3 = -\delta\hat{n}_4, \delta\hat{\theta}_3 = \delta\hat{\theta}_4.$$

Definition of Squeezing and Correlation

Squeezing Ratio

$$\hat{M} = M + \Delta\hat{M}$$

$$\text{SR} = \frac{\langle \Delta\hat{M}^2 \rangle}{\langle \Delta\hat{M}^2 \rangle_{\text{c.s.}}}$$

$\text{SR} < 1 : Squeezing$

$\text{SR} > 1 : Anti-Squeezing$

Correlation

$$C = \frac{\langle : \Delta\hat{A}\Delta\hat{B} : \rangle}{\sqrt{\langle \Delta\hat{A}^2 \rangle \langle \Delta\hat{B}^2 \rangle}}$$

$0 \leq C \leq 1$: *Positive Correlation*

$C = 0$: *No Correlation*

$-1 \leq C \leq 0$: *Negative Correlatior*

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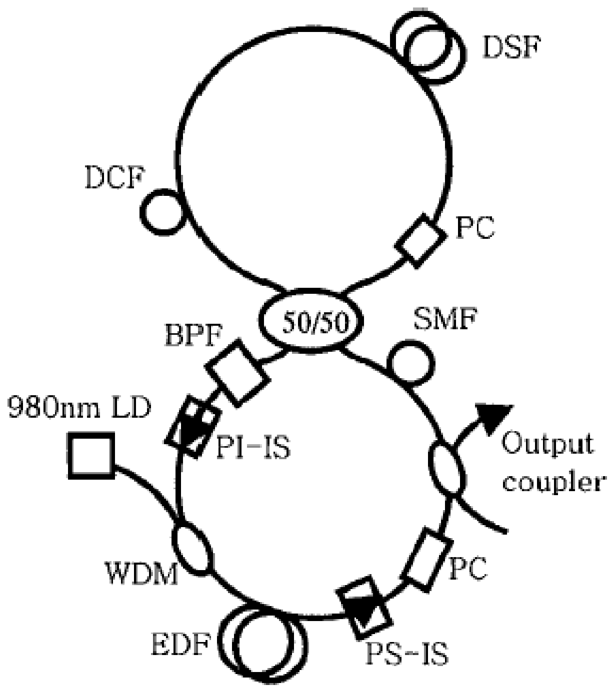
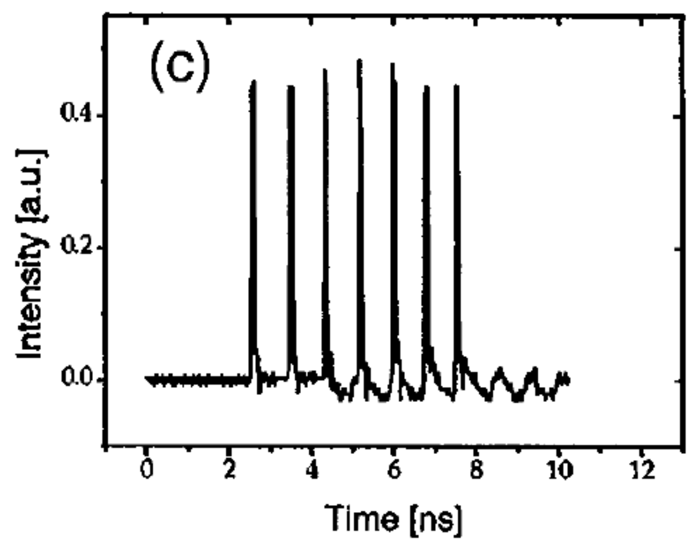
Bound-soliton pairs in fiber lasers

Recently, formation of stable double-, triple-, and multi-soliton bound states has been observed experimentally in various passively mode-locked fiber lasers.





N. H. Seong and Dug Y. Kim, *Opt. Lett.* **27**, 1321 (2002); (Source of figures.)
D. Y. Tang, W. S. Man, H. Y. Tam, and P. D. Drummond, *Phys. Rev. A* **64**, 033814 (2001).
P. Grelu, F. Belhache, F. Gutty, and J. M. Soto-Crespo, *J. Opt. Soc. Am. B* **20**, 863 (2003).

Complex Ginzburg-Landau equation

Multiple-pulse generation in the passively mode-lock fiber lasers is quite accurately described by the quintic Complex Ginzburg-Landau equation (CGLE):

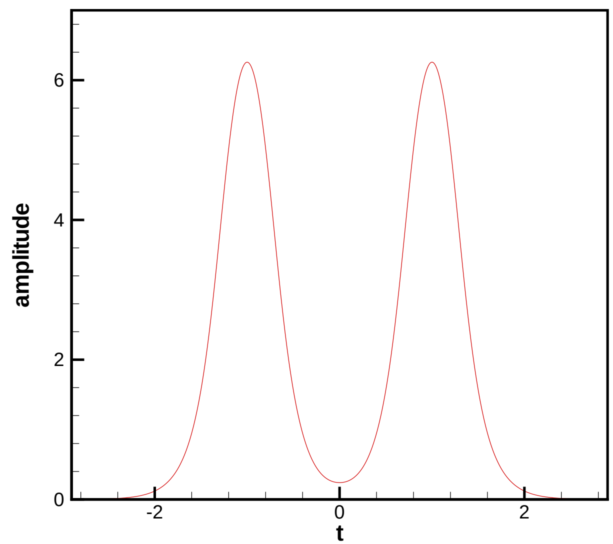
$$\begin{aligned} iU_z + \frac{D}{2}U_{tt} + |U|^2 U &= i\delta U + i\epsilon |U|^2 U + i\beta U_{tt} \\ &+ i\mu |U|^4 U - \nu |U|^4 U, \end{aligned}$$

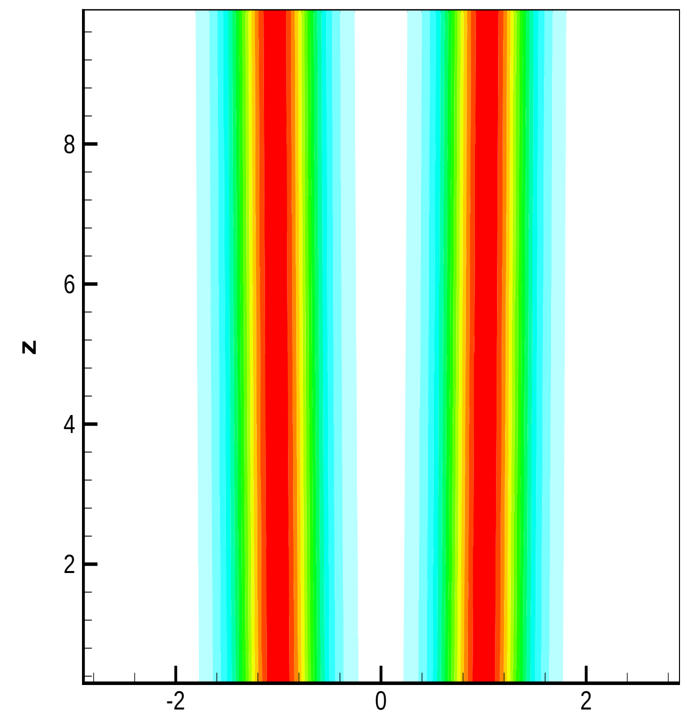
- U is the local amplitude,
- D corresponds to dispersion ($+1$ for anomalous),
- ν account for the quintic correction to the Kerr effect,
- δ, μ, and ϵ are linear, cubic, and quintic loss/gain,
- and β accounts spectral filtering.

Degenerate bound-state soliton pair solutions

There exist three bound pair solutions with the same separation and amplitude but different relative phases, i.e. $\theta = 0$ (in-phase), $\pi/2$, and $\theta = \pi$ (out-of-phase).

$$U(z,t) = U_0(z,t+\rho) + U_0(z,t-\rho)e^{i\theta},$$





Simulation parameters: $D = 1$, $\delta = -0.01$, $\epsilon = 1.8$, $\beta = 0.5$, $\mu = -0.05$, and $\nu = 0$.

N. N. Akhmediev, A. Ankiewicz, and J. M. Soto-Crespo, *Phys. Rev. Lett.* **79**, 4047 (1997).

Quantum theory of bound-soliton in CGLE model, $1/2$

1, we linearize the equation around the classical solution, i.e. $\hat{U}(z,t) = U_0 + \hat{u}(z,t)$, for the photon number in fiber lasers are large enough,

$$\frac{d}{dz}\hat{u}(z,t) = \mathcal{P}_1(z,t)\hat{u}(z,t) + \mathcal{P}_2(z,t)\hat{u}^\dagger(z,t) + \hat{n}(z,t),$$

where $\mathcal{P}_1$ and $\mathcal{P}_2$ are two special operators defined as follows,

$$\begin{aligned}
\mathcal{P}_1(z,t) &= i\frac{D}{2}\frac{\partial^2}{\partial t^2} + 2i|U_0|^2 + \delta + 2\epsilon|U_0|^2 + \beta\frac{\partial^2}{\partial t^2} \\
&+ 3\mu|U_0|^4 + 3i\nu|U_0|^4, \\
\mathcal{P}_2(z,t) &= iU_0^2 + \epsilon U_0^2 + 2\mu U_0^3 U_0^* + 2i\nu U_0^3 U_0^*.
\end{aligned}$$

Quantum theory of bound-soliton in CGLE model, $2/2$

2, to make perturbed quantum field, $\hat{u}$, satisfies the Bosonic communication relations in the linearized equation,

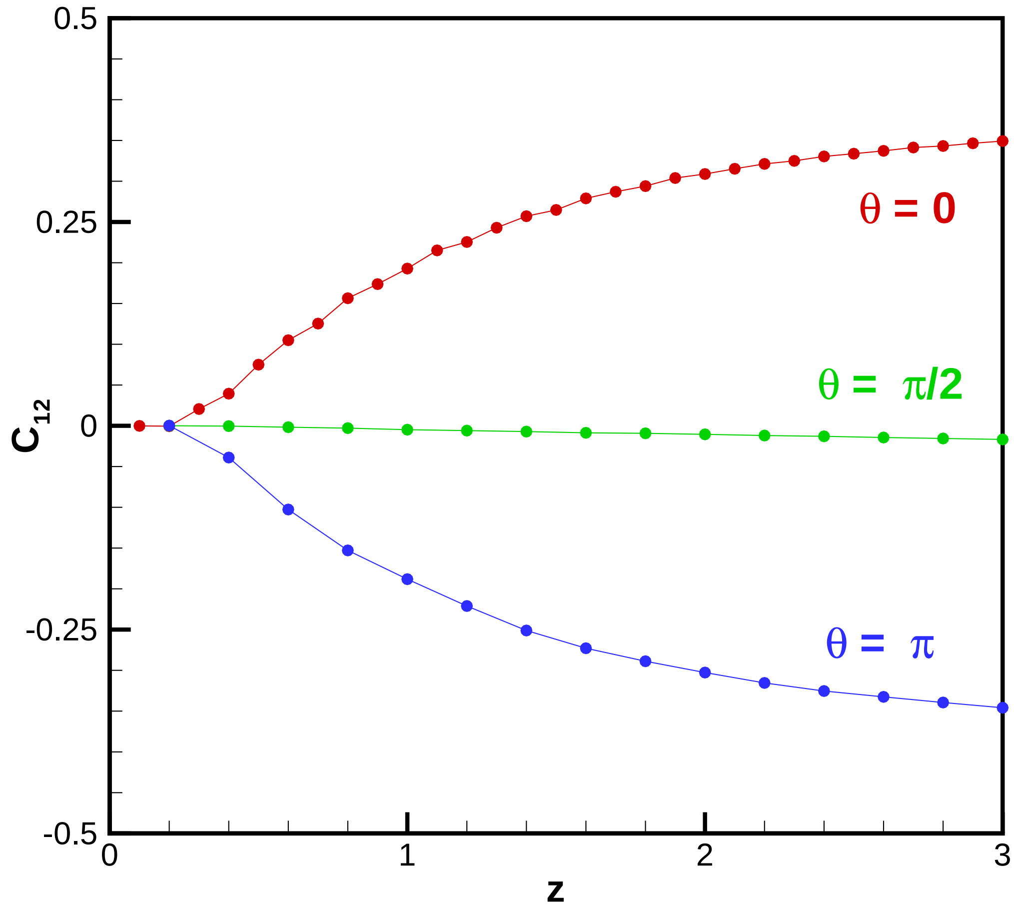
$$[\hat{u}(z,t_1), \hat{u}^\dagger(z,t_2)] = \delta(t_1 - t_2),$$
$$[\hat{u}(z,t_1), \hat{u}(z,t_2)] = [\hat{u}^\dagger(z,t_1), \hat{u}^\dagger(z,t_2)] = 0$$

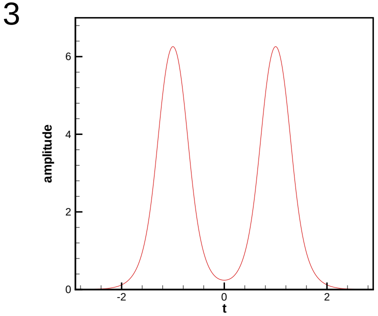
3, we introduce a zero-mean additional noise operator, $\hat{n}(z,t)$, which satisfy following commutation relations,

$$\begin{aligned} [\hat{n}(z,t_1), \hat{n}^\dagger(z',t_2)] &= \{-\mathcal{P}_1(z,t_1) - \mathcal{P}_1^*(z',t_2)\}\delta(z-z')\delta(t_1 - t_2 \\ [\hat{n}(z,t_1), \hat{n}(z',t_2)] &= [\hat{n}^\dagger(z,t_1), \hat{n}^\dagger(z',t_2)] = 0. \end{aligned}$$

R.-K. Lee, Y. Lai, and B. A. Malomed, *Phys. Rev. A* **70**, 063817 (2004).

Photon-number correlation parameters

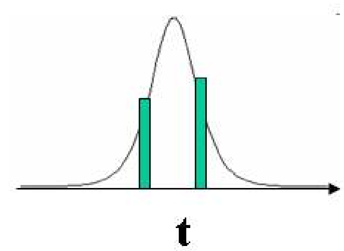


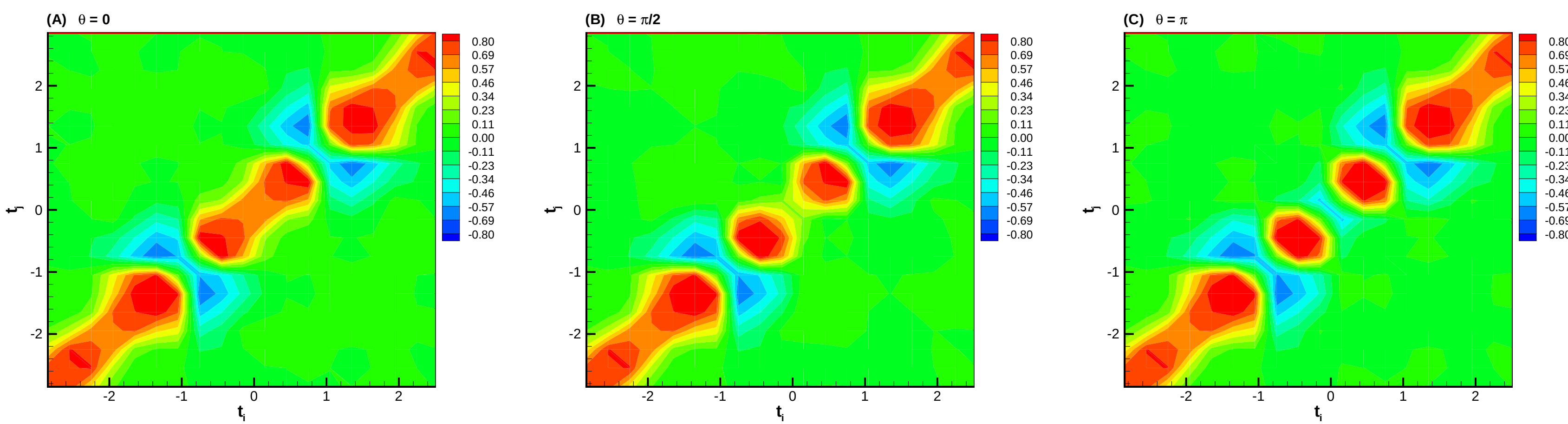


$$C_{12} = \frac{\langle : \Delta\hat{N}_1 \Delta\hat{N}_2 : \rangle}{\sqrt{\langle \Delta\hat{N}_1^2 \rangle \langle \Delta\hat{N}_2^2 \rangle}}.$$

Photon-number correlation spectra in t-domain

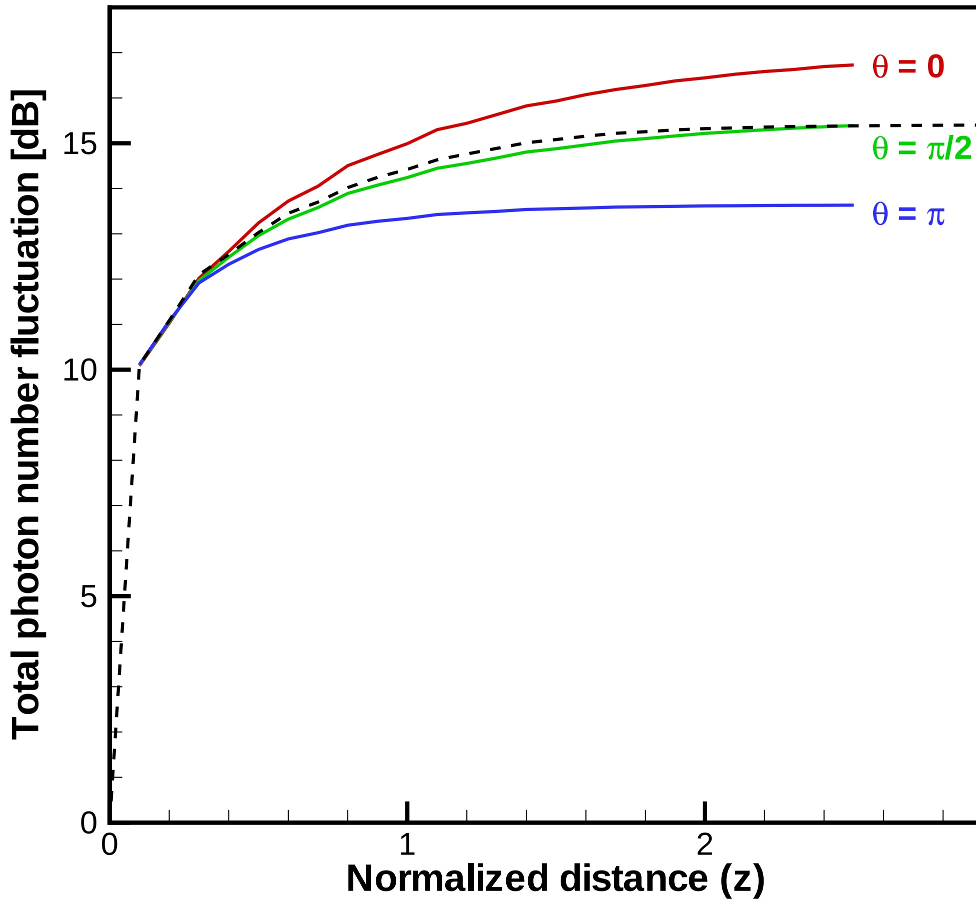
$$\eta_{ij} \equiv \frac{\langle : \Delta\hat{n}_i \Delta\hat{n}_j : \rangle}{\sqrt{\Delta\hat{n}_i^2 \Delta\hat{n}_j^2}},$$





$\theta = 0$ $\theta = \pi/2$ $\theta = \pi$

Total photon-number fluctuations



R.-K. Lee, Y. Lai, and B. A. Malomed, *Opt. Lett.* **34**, 3084 (2005).