1, A brief review about Quantum Mechanics

1. Basic Quantum Theory
2. Time-Dependent Perturbation Theory
3. Simple Harmonic Oscillator
4. Quantization of the Field
5. Canonical Quantization

Ref:

Ch. 2 in *"Mesoscopic Quantum Optics,"* by Y. Yamamoto and A. Imamoglu.

Ch. 2 in *"Introductory Quantum Optics,"* by C. Gerry and P. Knight.

Ch. 1 in *"Quantum Optics,"* by D. Wall and G. Milburn.

Ch. 4 in *"The Quantum Theory of Light,"* by R. Loudon.

Ch. 1, 2, 3, 6 in *"Mathematical Methods of Quantum Optics,"* by R. Puri.

Ch. 3 in *"Elements of Quantum Optics,"* by P. Meystre and M. Sargent III.

Ch. 9 in *"Modern Foundations of Quantum Optics,"* by V. Vedral.

Postulates of Quantum Mechanics

Postulate 1: An isolated quantum system is described by a vector in a Hilbert space. Two vectors differing only by a multiplying constant represent the same physical state.

- quantum state: $|\Psi\rangle = \sum_i \alpha_i |\psi_i\rangle$,
- completeness: $\sum_i |\psi_i\rangle\langle\psi_i| = I$,
- probability interpretation (projection): $\Psi(x) = \langle x|\Psi\rangle$,
- operator: $\hat{A}|\Psi\rangle = |\Phi\rangle$,
- representation: $\langle\phi|\hat{A}|\psi\rangle$,
- adjoint of $\hat{A}$: $\langle\phi|\hat{A}|\psi\rangle = \langle\psi|\hat{A}^\dagger|\phi\rangle^*$,
- hermitian operator: $\hat{H} = \hat{H}^\dagger$,
- unitary operator: $\hat{U}\hat{U}^\dagger = \hat{U}^\dagger\hat{U} = I$.

Ch. 1-5 in *"The Principles of Quantum Mechanics,"* by P. Dirac.

Ch. 1 in *"Mathematical Methods of Quantum Optics,"* by R. Puri.

Operators

- For a unitary operator, $\langle\psi_i|\psi_j\rangle = \langle\psi_i|\hat{U}^\dagger\hat{U}\psi_j\rangle$, the set of states $\hat{U}|\psi\rangle$ preserves the scalar product.
- $\hat{U}$ can be represented as $\hat{U} = \exp(i\hat{H})$ if $\hat{H}$ is hermitian.
- normal operator: $[\hat{A}, \hat{A}^\dagger] = 0$, the eigenstates of only a normal operator are *orthonormal*.
  i.e. hermitian and unitary operators are normal operators.
- The sum of the diagonal elements $\langle\phi|\hat{A}|\psi\rangle$ is call the *trace* of $\hat{A}$,

  $$\text{Tr}(\hat{A}) = \sum_i \langle\phi_i|\hat{A}|\phi_i\rangle,$$

  The value of the trace of an operator is independent of the basis.
- The eigenvalues of a hermitian operator are real, $\hat{H}|\Psi\rangle = \lambda|\Psi\rangle$, where λ is real.
- If $\hat{A}$ and $\hat{B}$ do not commute then they do not admit a common set of eigenvectors.

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Postulates of Quantum Mechanics

Postulate 2: To each dynamical variable there corresponds a unique hermitian operator.
Postulate 3: If $\hat{A}$ and $\hat{B}$ are hermitian operators corresponding to classical dynamical variables a and b, then the commutator of $\hat{A}$ and $\hat{B}$ is given by

$$[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A} = i\hbar\{a, b\},$$

where $\{a, b\}$ is the classical Poisson bracket.
Postulate 4: Each act of measurement of an observable $\hat{A}$ of a system in state $|\Psi\rangle$ collapses the system to an eigenstate $|\psi_i\rangle$ of $\hat{A}$ with probability $|\langle\phi_i|\Psi\rangle|^2$.
The average or the expectation value of $\hat{A}$ is given by

$$\langle\hat{A}\rangle = \sum_i \lambda_i |\langle\phi_i|\Psi\rangle|^2 = \langle\Psi|\hat{A}|\Psi\rangle,$$

where λ_i is the eigenvalue of $\hat{A}$ corresponding to the eigenstate $|\psi_i\rangle$.

Uncertainty relation

- Non-commuting observable do not admit common eigenvectors.
- Non-commuting observables can not have definite values simultaneously.
- Simultaneous measurement of non-commuting observables to an arbitrary degree of accuracy is thus *incompatible*.
- variance: $\Delta\hat{A}^2 = \langle\Psi|(\hat{A} - \langle\hat{A}\rangle)^2|\Psi\rangle = \langle\Psi|\hat{A}^2|\Psi\rangle - \langle\Psi|\hat{A}|\Psi\rangle^2$.

$$\Delta A^2 \Delta B^2 \geq \frac{1}{4}[\langle\hat{F}\rangle^2 + \langle\hat{C}\rangle^2],$$

where

$$[\hat{A}, \hat{B}] = i\hat{C}, \qquad \text{and} \qquad \hat{F} = \hat{A}\hat{B} + \hat{B}\hat{A} - 2\langle\hat{A}\rangle\langle\hat{B}\rangle.$$

- Take the operators $\hat{A} = \hat{q}$ (position) and $\hat{B} = \hat{p}$ (momentum) for a free particle,

$$[\hat{q}, \hat{p}] = i\hbar \rightarrow \langle\Delta\hat{q}^2\rangle\langle\Delta\hat{p}^2\rangle \geq \frac{\hbar^2}{4}.$$

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Uncertainty relation

- Schwarz inequality: $\langle\phi|\phi\rangle\langle\psi|\psi\rangle \geq \langle\phi|\psi\rangle\langle\psi|\phi\rangle$.
- Equality holds if and only if the two states are *linear dependent*, $|\psi\rangle = \lambda|\phi\rangle$, where λ is a complex number.
- uncertainty relation,

$$\Delta A^2 \Delta B^2 \geq \frac{1}{4}[\langle\hat{F}\rangle^2 + \langle\hat{C}\rangle^2],$$

where

$$[\hat{A}, \hat{B}] = i\hat{C}, \qquad \text{and} \qquad \hat{F} = \hat{A}\hat{B} + \hat{B}\hat{A} - 2\langle\hat{A}\rangle\langle\hat{B}\rangle.$$

- the operator $\hat{F}$ is a measure of correlations between $\hat{A}$ and$\hat{B}$.
- define two states,

$$|\psi_1\rangle = [\hat{A} - \langle\hat{A}\rangle]|\psi\rangle, \qquad |\psi_2\rangle = [\hat{B} - \langle\hat{B}\rangle]|\psi\rangle,$$

the uncertainty product is minimum, i.e. $|\psi_1\rangle = -i\lambda|\psi_2\rangle$,

$$[\hat{A} + i\lambda\hat{B}]|\psi\rangle = [\langle\hat{A}\rangle + i\lambda\langle\hat{B}\rangle]|\psi\rangle = z|\psi\rangle.$$

- the state $|\psi\rangle$ is a *minimum uncertainty state*.

Uncertainty relation

- if *Re*$(\lambda) = 0$, $\hat{A} + i\lambda\hat{B}$ is a normal operator, which have orthonormal eigenstates.
- the variances,

$$\Delta\hat{A}^2 = -\frac{i\lambda}{2}[\langle\hat{F}\rangle + i\langle\hat{C}\rangle], \qquad \Delta\hat{B}^2 = -\frac{i}{2\lambda}[\langle\hat{F}\rangle - i\langle\hat{C}\rangle],$$

- set $\lambda = \lambda_r + i\lambda_i$,

$$\Delta\hat{A}^2 = \frac{1}{2}[\lambda_i\langle\hat{F}\rangle + \lambda_r\langle\hat{C}\rangle], \qquad \Delta\hat{B}^2 = \frac{1}{|\lambda|^2}\Delta\hat{A}^2, \qquad \lambda_i\langle\hat{C}\rangle - \lambda_r\langle\hat{F}\rangle = 0.$$

- if $|\lambda| = 1$, then $\Delta\hat{A}^2 = \Delta\hat{B}^2$, *equal variance minimum uncertainty states*.
- if $|\lambda| = 1$ along with $\lambda_i = 0$, then $\Delta\hat{A}^2 = \Delta\hat{B}^2$ and $\langle\hat{F}\rangle = 0$, **uncorrelated** *equal variance minimum uncertainty states*.
- if $\lambda_r \neq 0$, then $\langle\hat{F}\rangle = \frac{\lambda_i}{\lambda_r}\langle\hat{C}\rangle$, $\quad \Delta\hat{A}^2 = \frac{|\lambda|^2}{2\lambda_r}\langle\hat{C}\rangle$, $\quad \Delta\hat{B}^2 = \frac{1}{2\lambda_r}\langle\hat{C}\rangle$.
  If $\hat{C}$ is a positive operator then the minimum uncertainty states exist only if $\lambda_r > 0$.

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Uncertainty relation for $\hat{q}$ and $\hat{p}$

- take the operators $\hat{A} = \hat{q}$ (position) and $\hat{B} = \hat{p}$ (momentum) for a free particle,

$$[\hat{q}, \hat{p}] = i\hbar \rightarrow \langle \Delta \hat{q}^2 \rangle \langle \Delta \hat{p}^2 \rangle \geq \frac{\hbar^2}{4}.$$

- define two states, $|\psi_1\rangle = [\hat{A} - \langle \hat{A} \rangle]|\psi\rangle \equiv \hat{\alpha}|\psi\rangle, \qquad |\psi_2\rangle = [\hat{B} - \langle \hat{B} \rangle]|\psi\rangle \equiv \hat{\beta}|\psi\rangle$.
- for uncorrelated minimum uncertainty states,

$$\hat{\alpha}|\psi\rangle = -i\lambda\hat{\beta}|\psi\rangle, \qquad \langle\psi|\hat{\alpha}\hat{\beta} + \hat{\beta}\hat{\alpha}|\psi\rangle = 0,$$

  where λ is a real number.
- if $\hat{A} = \hat{q}$ and $\hat{B} = \hat{p}$, we have $(\hat{q} - \langle \hat{q} \rangle)|\psi\rangle = -i\lambda(\hat{p} - \langle \hat{p} \rangle)|\psi\rangle$.
- the wavefunction in the q-basis is, i.e. $\hat{p} = -i\hbar\partial/\partial q$,

$$\psi(q) = \langle q|\psi\rangle = \frac{1}{(2\pi\langle\Delta\hat{q}^2\rangle)^{1/4}} \exp\left[\frac{i\langle\hat{p}\rangle q}{\hbar} - \frac{(q - \langle\hat{q}\rangle)^2}{4\langle\Delta\hat{q}^2\rangle}\right],$$

- in the p-basis, $\psi(p) = \langle p|\psi\rangle = \frac{1}{(2\pi\langle\Delta\hat{p}^2\rangle)^{1/4}} \exp[-\frac{i}{\hbar}(\langle\hat{q}\rangle(p - \langle\hat{p}\rangle) - \frac{(p-\langle\hat{p}\rangle)^2}{4\langle\Delta\hat{p}^2\rangle}]$.

Minimum Uncertainty State

- $(\hat{q} - \langle\hat{q}\rangle)|\psi\rangle = -i\lambda(\hat{p} - \langle\hat{p}\rangle)|\psi\rangle$
- if we define $\lambda = e^{-2r}$, then

$$(e^r\hat{q} + ie^{-r}\hat{p})|\psi\rangle = (e^r\langle\hat{q}\rangle + ie^{-r}\langle\hat{p}\rangle)|\psi\rangle,$$

- the minimum uncertainty state is defined as an *eigenstate* of a non-Hermitian operator $e^r\hat{q} + ie^{-r}\hat{p}$ with a c-number eigenvalue $e^r\langle\hat{q}\rangle + ie^{-r}\langle\hat{p}\rangle$.
- the variances of $\hat{q}$ and $\hat{p}$ are

$$\langle\Delta\hat{q}^2\rangle = \frac{\hbar}{2}e^{-2r}, \qquad \langle\Delta\hat{p}^2\rangle = \frac{\hbar}{2}e^{2r}.$$

- here r is referred as the *squeezing parameter*.

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Gaussian Wave Packets

- in the x-space,

$$\Psi(x) = \langle x|\Psi\rangle = [\frac{1}{\pi^{1/4}\sqrt{d}}]\exp[ikx - \frac{x^2}{2d^2}]$$

, which is a plane wave with wave number k and width d.

- the expectation value of $\hat{X}$ is zero for symmetry,

$$\langle\hat{X}\rangle = \int_{-\infty}^{\infty} \mathrm{d}x \langle\Psi|x\rangle\hat{X}\langle x|\Psi\rangle = 0.$$

- variation of $\hat{X}$, $\langle\Delta\hat{X}^2\rangle = \frac{d^2}{2}$.
- the expectation value of $\hat{P}$, $\langle\hat{P}\rangle = \hbar k$, i.e. $\langle x|\hat{P}|\Psi\rangle = -i\hbar\frac{\partial}{\partial x}\langle x|\Psi\rangle$.
- variation of $\hat{P}$, $\langle\Delta\hat{P}^2\rangle = \frac{\hbar^2}{2d^2}$.
- the Heisenberg uncertainty product is, $\langle\Delta\hat{X}^2\rangle\langle\Delta\hat{P}^2\rangle = \frac{\hbar^2}{4}$.
- a Gaussian wave packet is called a *minimum uncertainty wave packet*.

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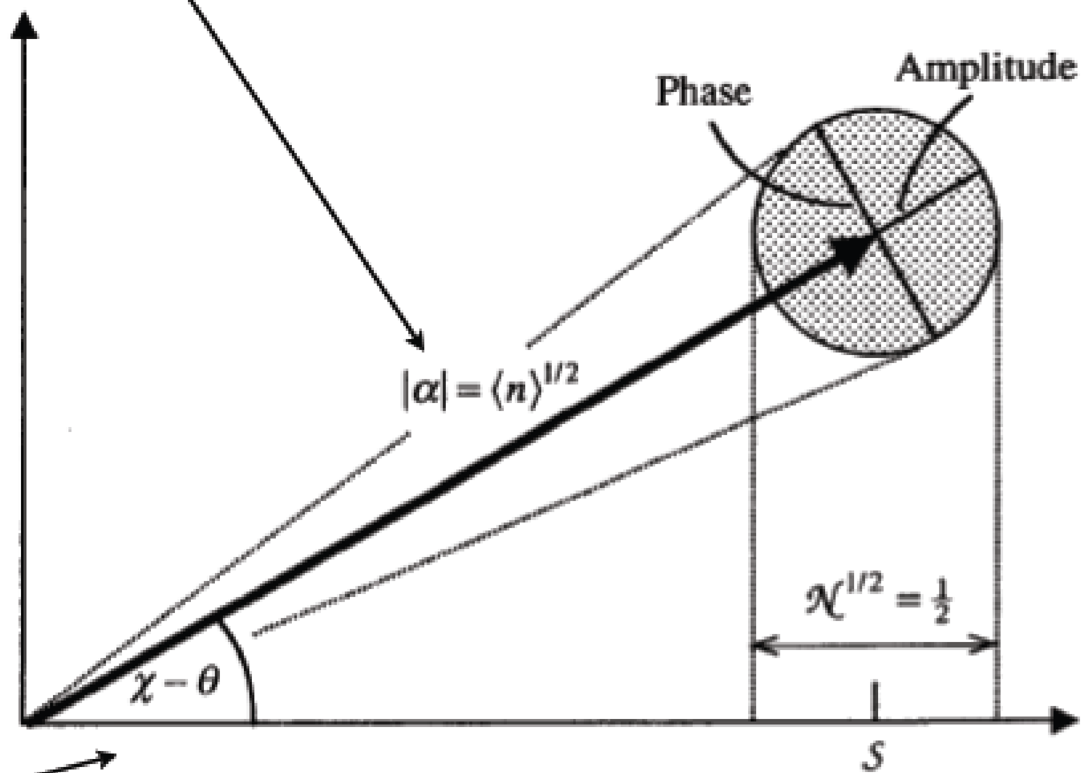
Phase diagram for coherent states

mean number of photons

$$< \hat{N} >=< \alpha|\hat{N}|\alpha >=< \alpha|\hat{a}^{\dagger}\hat{a}|\alpha >= |\alpha|^2$$



phase of the field

$$\alpha = |\alpha| \exp(i\theta)$$

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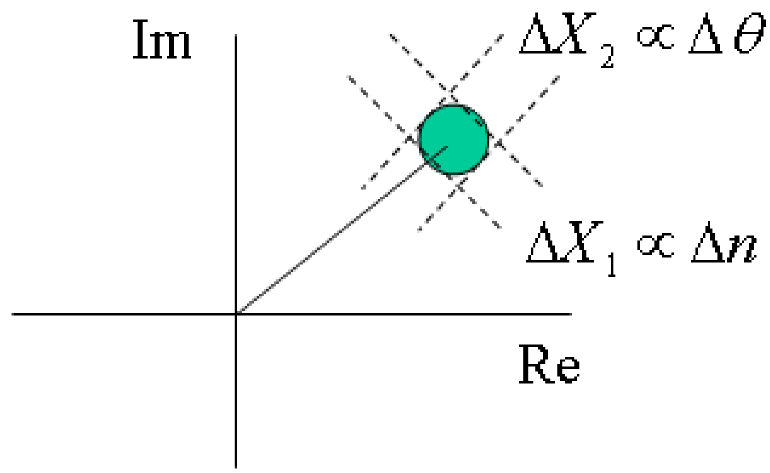
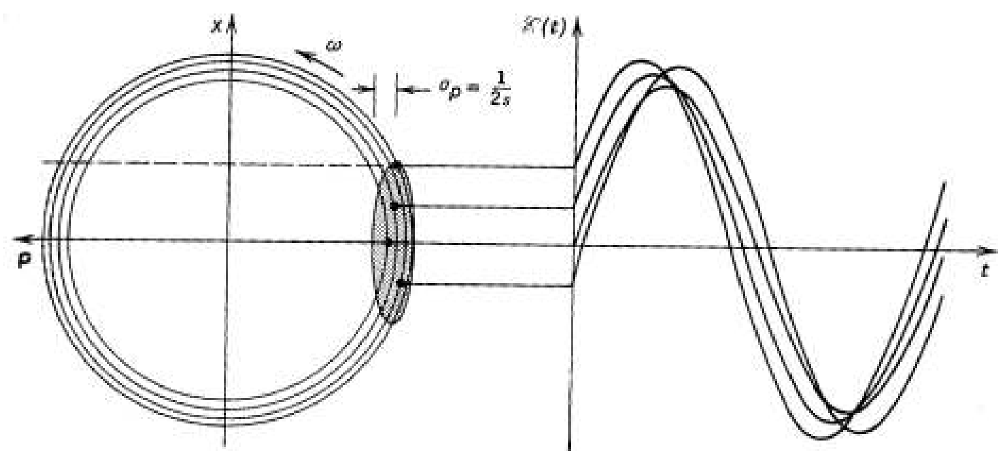
Coherent and Squeezed States

Uncertainty Principle: $\Delta\hat{X}_1\Delta\hat{X}_2 \geq 1$.

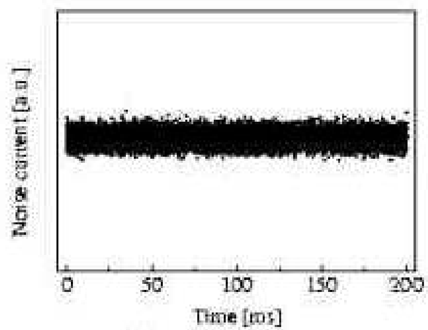
1. Coherent states: $\Delta\hat{X}_1 = \Delta\hat{X}_2 = 1$,
2. Amplitude squeezed states: $\Delta\hat{X}_1 < 1$,
3. Phase squeezed states: $\Delta\hat{X}_2 < 1$,
4. Quadrature squeezed states.



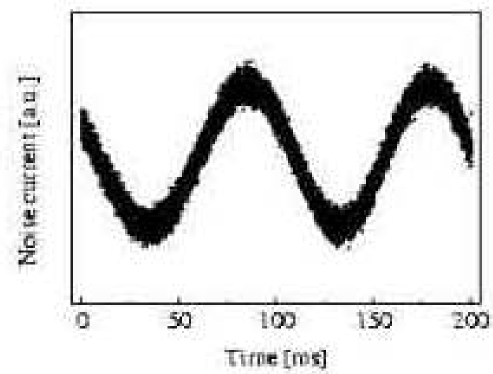


Vacuum, Coherent, and Squeezed states

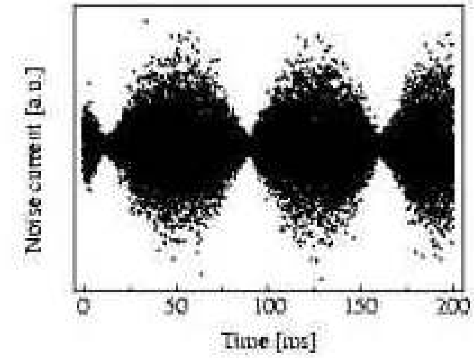
vacuum

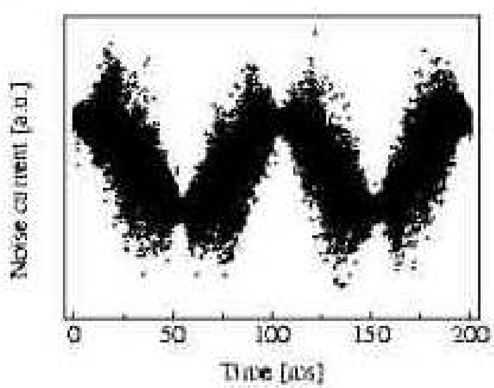


coherent

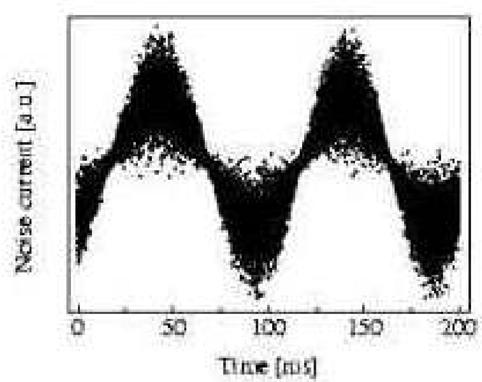


squeezed-vacuum

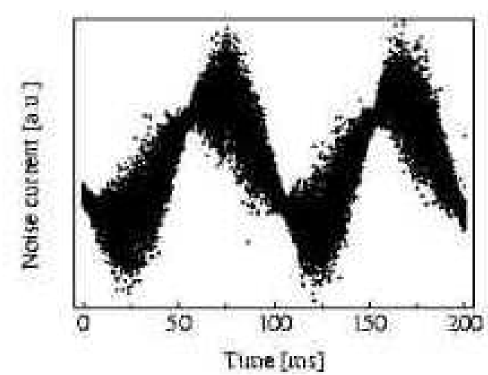




amp-squeezed



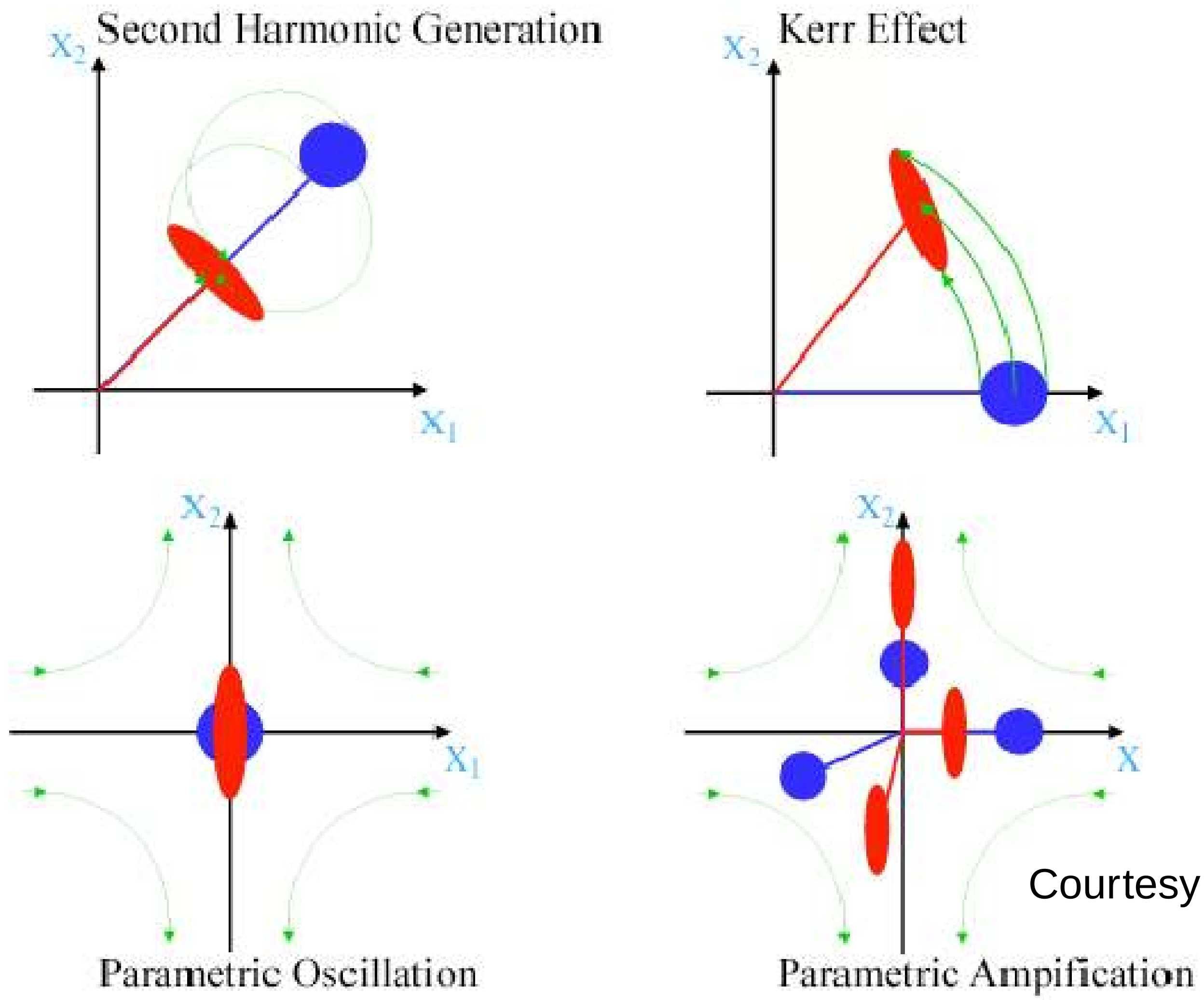
phase-squeezed



quad-squeezed

Generations of Squeezed States

Nonlinear optics:

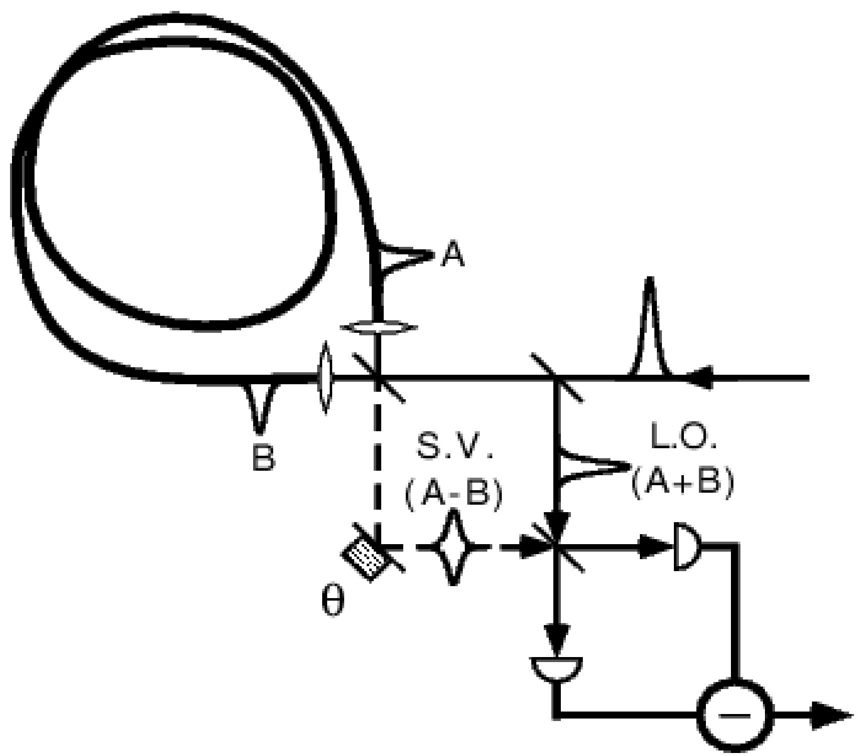


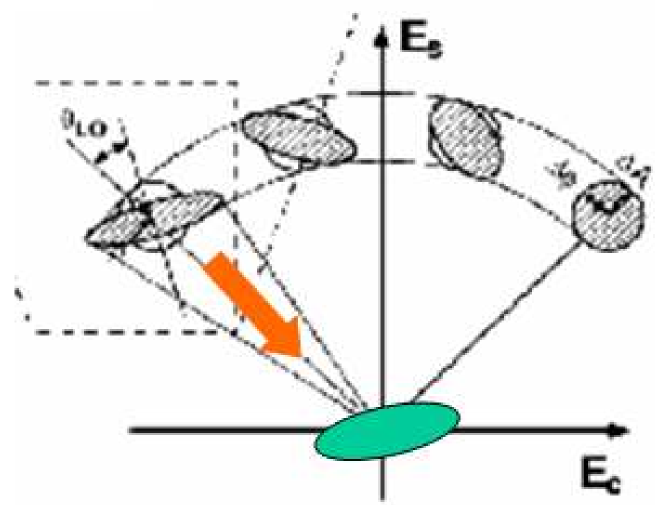
Courtesy of P. K. Lam

Generation and Detection of Squeezed Vacuum

1. Balanced Sagnac Loop (to cancel the mean field),

2. Homodyne Detection.





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M. Rosenbluh and R. M. Shelby, *Phys. Rev. Lett.* **66**, 153(1991).

Schrödinger equation

Postulate 5: The time evolution of a state $|\Psi\rangle$ is governed by the Schrödinger equation,

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\Psi(t)\rangle = \hat{H}(t)|\Psi(t)\rangle,$$

where $\hat{H}(t)$ is the Hamiltonian which is a hermitian operator associated with the total energy of the system.
The solution of the Schrödinger equation is,

$$|\Psi(t)\rangle = \overleftarrow{T}\exp[-\frac{i}{\hbar}\int_{t_0}^{t}\mathrm{d}\tau\hat{H}(\tau)]|\Psi(0)\rangle \equiv \hat{U}_S(t,t_0)|\Psi(t_0)\rangle,$$

where $\overleftarrow{(}T)$ is the time-ordering operator.

Schrödinger picture:

$$|\Psi(r,t)\rangle = \sum_i \alpha_i(t)|\psi_i(r)\rangle.$$

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Time Evolution of a Minimum Uncertainty State

- the Hamiltonian for a free particle, $\hat{H} = \frac{\hat{p}^2}{2m}$, then

$$\hat{U} = \text{exp}(-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}t).$$

- the Schrödinger wavefunction,

$$\begin{aligned}\Psi(q,t) = \langle q|\hat{U}|\Psi(0)\rangle &= \int_{-\infty}^{\infty} \text{d}p\langle|p\rangle\Psi(p,0)\text{exp}(-\frac{i}{\hbar}\frac{p^2}{2m}t), \\ &= \frac{1}{(2\pi)^{1/4}(\Delta q + i\hbar t/2m\Delta q)^{1/2}}\text{exp}[-\frac{q^2}{4(\Delta q)^2 + 2i\hbar t/m}],\end{aligned}$$

where $\Delta q = \hbar/2\langle\hat{p}^2\rangle^{1/2}$, and $\langle q|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}\text{exp}(\frac{ipq}{\hbar})$.

- even though the momentum uncertainty $\langle\hat{p}^2\rangle$ is preserved,
- the position uncertainty increases as time develops,

$$\langle\Delta\hat{q}^2(t)\rangle = (\Delta\hat{q})^2 + \frac{\hbar^2 t^2}{4m^2(\Delta q)^2}$$

Gaussian Optics

- Wave equation: In free space, the vector potential, A, is defined as $A(r,t) = \vec{n}\psi(x,y,z)e^{j\omega t}$, which obeys the vector wave equation,

$$\nabla^2\psi + k^2\psi = 0.$$

- The paraxial wave equation: $\psi(x,y,z) = u(x,y,z)e^{-jkz}$, one obtains

$$\nabla_T^2 u - 2jk\frac{\partial u}{\partial z} = 0,$$

where $\nabla_T \equiv \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}$.

- This solution is proportional to the impulse response function (Fresnel kernel),

$$h(x,y,z) = \frac{j}{\lambda z}e^{-jk[(x^2+y^2)/2z]},$$

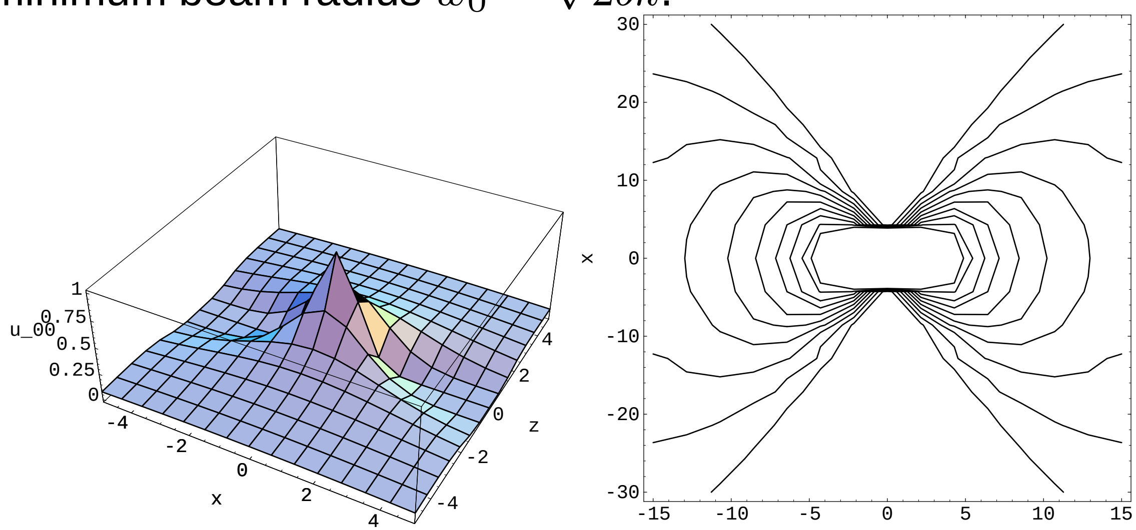
i.e. $\nabla_T^2 h(x,y,z) - 2jk\frac{\partial h}{\partial z} = 0$.

Gaussian Optics

- The solution of the scalar paraxial wave equation is,

$$u_{00}(x,y,z) = \frac{\sqrt{2}}{\sqrt{\pi}w} exp(j\phi) exp(-\frac{x^2+y^2}{w^2}) exp[-\frac{jk}{2R}(x^2+y^2],$$

- beam width $w^2(z) = \frac{2b}{k}(1+\frac{z^2}{b^2} = w_0^2[1+(\frac{\lambda z}{\pi w_0^2})^2]$,
- radius of phase front $\frac{1}{R(z)} = \frac{z}{z^2+b^2} = \frac{z}{z^2+(\pi w_0^2/\lambda)^2}$,
- $phasedelay \tan\phi = \frac{z}{b} = \frac{z}{\pi w_0^2/\lambda}$,
- with the minimum beam radius $w_0 = \sqrt{2bk}$.



Time Evolution of a Minimum Uncertainty State

- Uncertainty relation and Fourier Transform,
- Minimum Uncertainty State and Gaussian beams,
- Minimum Uncertainty State and Chirpless optical short pulse,
- Non-classical state,

Heisenberg equation

- The solution of the Schrödinger equation is,
$|\Psi(t)\rangle = \overleftarrow{T}\text{exp}[-\frac{i}{\hbar}\int_{t_0}^{t} \mathrm{d}\tau \hat{H}(\tau)]|\Psi(0)\rangle \equiv \hat{U}_S(t,t_0)|\Psi(t_0)\rangle$.

- The quantities of physical interest are the expectation values of operators,

$$\langle\Psi(t)|\hat{A}|\Psi(t)\rangle = \langle\Psi(t_0)|\hat{A}(t)|\Psi(t_0)\rangle,$$

  where

$$\hat{A}(t) = \hat{U}_S^\dagger(t,t_0)\hat{A}\hat{U}_S(t,t_0).$$

- The time-dependent operator $\hat{A}(t)$ evolves according to the Heisenberg equation,

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\hat{A}(t) = [\hat{A},\hat{H}(t)].$$

- Schrödinger picture: time evolution of the states.

- Heisenberg picture: time evolution of the operators.

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Interaction picture

- Consider a system described by $|\Psi(t)\rangle$ evolving under the action of a hamiltonian $\hat{H}(t)$ decomposable as,

$$\hat{H}(t) = \hat{H}_0 + \hat{H}_1(t),$$

where $\hat{H}_0$ is time-independent.

- Define

$$|\Psi_I(t)\rangle = \mathrm{exp}(i\hat{H}_0 t/\hbar)|\Psi(t)\rangle,$$

then $|\Psi_I(t)\rangle$ evolves accords to

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\Psi_I(t)\rangle = \hat{H}_I(t)|\Psi_I(t)\rangle,$$

where

$$\hat{H}_I(t) = \mathrm{exp}(i\hat{H}_0 t/\hbar)\hat{H}_1(t)\mathrm{exp}(-i\hat{H}_0 t/\hbar).$$

- The evolution is in the **interaction picture** generated by $\hat{H}_0$.

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Similarity Transformations

- The Heisenberg picture is a "natural" picture in the sense that the observables (electric fields, dipole moment, etc.) are time-dependent, exactly as in classical physics.
- In the interaction picture, we have eliminated the part of the problem whose solution we already knew.
- The interaction picture is particularly helpful in visualizing the response of a two-level atom to light.

$$\hat{H}_I(t) = \text{exp}(i\hat{H}_0 t/\hbar)\hat{H}_1(t)\text{exp}(-i\hat{H}_0 t/\hbar).$$

- similarity transformation, $\hat{S}^{-1}\hat{A}\hat{S} = \hat{B}$, where $\hat{S}$ is a non-singular operator,
- consider the similarity transformation,

$$\hat{A}(\theta) \equiv \text{exp}(-\theta\hat{Z})\hat{A}\text{exp}(\theta\hat{Z}),$$

then the differentiation of this equation with respect to θ yields,

$$\frac{\text{d}}{\text{d}\theta}\hat{A}(\theta) = \text{exp}(-\theta\hat{Z})[\hat{A}, \hat{Z}]\text{exp}(\theta\hat{Z}).$$

Paradoxes of Quantum Theory

- Geometric phase
- Measurement theory
- Schrödinger's Cat paradox
- Einstein-Podolosky-Rosen paradox
- Local Hidden Variables theory

Quantum Zeno effect (watchdog effect)

- multi-time joint probability: $P(\{|\phi_i\rangle, t_i\})$, the probability that a system in a state $|\phi_0(t_0)\rangle$ at t_0 is found in the state $|\phi_i\rangle$ at t_i, where $i = 1, \ldots, n$.
- at t_1: the state is $\hat{U}_S(t_1, t_0)|\phi_0(t_0)\rangle$.
- projection on $|\phi_1\rangle$ is

$$|\phi_1(t_1)\rangle = |\phi_1\rangle\langle\phi_1|\hat{U}_S(t_1, t_0)|\phi_0(t_0)\rangle.$$

- the sate $|\phi_1(t_1)\rangle$ then evolves till time t_2 to $\hat{U}_S(t_2, t_1)|\phi_1(t_1)\rangle$, with the projection,

$$|\phi_2(t_2)\rangle = |\phi_2\rangle\langle\phi_2|\hat{U}_S(t_2, t_1)|\phi_1(t_1)\rangle.$$

- continuing till time t_n,

$$P(\{|\phi_i\rangle, t_i\}) = |\prod_{i=1}^{n} \langle\phi_i|\hat{U}_S(t_i, t_{i-1})|\phi_{i-1}\rangle|^2.$$

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Quantum Zeno effect (watchdog effect)

- consider a time-independent hamiltonian, $\hat{U}_S(t_i, t_j) = \exp[-i\hat{H}(t_i - t_j)/\hbar]$.
- let the observation be spaced at equal time intervals, $t_i - t_{i-1} = t/n$.
- the probability that at each time t_i the system is observed in its initial state $|\phi_0\rangle$ is,

$$P(\{|\phi_0\rangle, t_i\}) = |\langle\phi_0|\exp[-i\hat{H}t/n\hbar]|\phi_0\rangle|^{2n}.$$

- let $t/n \ll 1$,

$$|\langle\phi_0|\exp[-i\hat{H}t/n\hbar]|\phi_0\rangle|^2 \approx 1 - (\frac{t}{n\hbar})^2\Delta\hat{H}^2,$$

  where $\Delta\hat{H}^2 = \langle\phi_0|\hat{H}^2|\phi_0\rangle - \langle\phi_0|\hat{H}|\phi_0\rangle^2$.

Quantum Zeno effect (watchdog effect)

- the joint probability for n equally spaced observations becomes,

$$P(\{|\phi_0\rangle, t_i\}) = [1 - (\frac{t}{n\hbar})^2 \Delta\hat{H}^2]^n.$$

- for *unobserved in between*, the probability is,

$$P(\{|\phi_0\rangle, t\}) = 1 - (\frac{t^2}{\hbar^2})\Delta\hat{H}^2.$$

- the probability of finding the system in its initial state at a given time is **increased** if it is observed repeatedly at intermediate times.
- for $n \gg 1$,

$$P(\{|\phi_0\rangle, t_i\}) = [1 - (\frac{t}{n\hbar})^2 \Delta\hat{H}^2]^n \approx \text{exp}[-t^2\Delta\hat{H}^2/n\hbar^2],$$

the system under observation does not evolve.

- this effect was invoked to predict the inhibition of decay of an unstable system.

Quantum Zeno effect (watchdog effect)

- Quantum Zeno effect
- Quantum Anti-Zeno effect
- Quantum Super-Zeno effect

Time-dependent perturbation theory

- with the interaction picture, $\hat{H} = \hat{H}_0 + \hat{H}_1$.
- the state, $\Psi(r,t) = \sum_n C_n(t) u_n(r) e^{-i\omega_n t}$ with the energy eigenvalue $\hat{H}_0 u_n(r) = \hbar\omega_n u_n(r)$.
- the wavefunction has the initial value, $\Psi(r,0) = u_i(r)$, i.e. $C_i(0) = 1, C_{n\neq i} = 0$.
- the equation of motion for the probability amplitude $C_n(t)$ is,

$$\begin{aligned} \dot{C_n}(t) &= -\frac{i}{\hbar}\sum_m \langle n|\hat{H}_1|m\rangle e^{i\omega_{nm}t} C_m(t), \\ &\approx \dot{C_n}^{(1)}(t) = -i\hbar^{-1}\langle n|\hat{H}_1|i\rangle e^{i\omega_{ni}t}. \end{aligned}$$

- if $\hat{H}_1 = V_0$ time independent, we have

$$C_n(t) \approx C_n{}^{(1)}(t) = -i\hbar^{-1}\langle n|\hat{H}_1|i\rangle \frac{e^{i\omega_{ni}t}-1}{i\omega_{ni}} = -i\hbar^{-1}\langle n|\hat{H}_1|i\rangle e^{i\omega_{ni}t/2}\frac{\sin(\omega_{ni}t/2)}{\omega_{ni}/2}$$

Ch. 3 in *"Elements of Quantum Optics,"* by P. Meystre and M. Sargent III.

Ch. 5 in *"Modern Quantum Mechanics,"* by J. Sakurai.

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Rotational-Wave Approximation

- if $\hat{H}_1 = V_0 \cos \nu t$, we have

$$C_n(t) \approx C_n{}^{(1)}(t) = -i \frac{V_{ni}}{2\hbar} [\frac{e^{i(\omega_{ni}+\nu)t} - 1}{i(\omega_{ni} + \nu)} + \frac{e^{i(\omega_{ni}-\nu)t} - 1}{i(\omega_{ni} - \nu)}],$$

  where $V_{ni} = \langle n|\hat{H}_1|i\rangle$.

- if near resonance $\omega_{ni} \approx \nu$, we can neglect the terms with $\omega_{ni} + \nu$. This is called the **rotational-wave approximation.**

- making the rotational-wave approximation,

$$|C_n^{(1)}|^2 = \frac{|V_{ni}|^2}{4\hbar^2} \frac{\sin^2[(\omega_{ni} - \nu)t/2]}{(\omega_{ni} - \nu)^2/4}.$$

- we have the same transition probability as the dc case, provided we substitute $\omega_{ni} - \nu$ for ω_{ni}.

Fermi-Golden rule

- the total transition probability from an initial state to the final state is,

$$P_T \approx \int D(\omega)|C_n^{(1)}|^2 \mathrm{d}\omega,$$

where $D(\omega)$ is the density of state factor.

- Fermi-Golden rule,

$$P_T = \int \mathrm{d}\omega D(\omega) \frac{|V(\omega)|^2}{4\hbar^2} t^2 \frac{\sin^2[(\omega_{ni} - \nu)t/2]}{[(\omega_{ni} - \nu)t/2]^2}.$$

- consider resonance condition $\omega = \nu$,

$$\begin{aligned} P_T &\approx D(\nu) \frac{|V(\nu)|^2}{4\hbar^2} t^2 \int \mathrm{d}\omega \frac{\sin^2[(\omega_{ni} - \nu)t/2]}{[(\omega_{ni} - \nu)t/2]^2}, \\ &= \frac{\pi}{2\hbar^2} D(\nu)|V(\nu)|^2 t. \end{aligned}$$

- the transition rate, $\Gamma = \frac{\mathrm{d}P_T}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}|C_n^{(1)}|^2 = \frac{\pi}{2\hbar^2} D(\nu)|V(\nu)|^2$, which is a constant in time.

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Phase-Matching condition

- Second-Harmonic Generation

Simple Harmonic Oscillator in Schrödinger picture

- one-dimensional harmonic oscillator, $\hat{H} = \frac{p^2}{2m} + \frac{1}{2}kx^2$,
- Schrödinger equation,

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x) + \frac{2m}{\hbar^2}[E - \frac{1}{2}kx^2]\psi(x) = 0,$$

with dimensionless coordinates $\xi = \sqrt{m\omega/\hbar}x$ and dimensionless quantity $\epsilon = 2E/\hbar\omega$, we have

$$\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}\psi(x) + [\epsilon - \xi^2]\psi(x) = 0,$$

which has Hermite-Gaussian solutions,

$$\psi(\xi) = \mathrm{H}_n(\xi)e^{-\xi^2/2}, \qquad E = \frac{1}{2}\hbar\omega\epsilon = \hbar\omega(n + \frac{1}{2}),$$

where $n = 0, 1, 2, \ldots$

Ch. 7 in *"Quantum Mechanics,"* by A. Goswami.

Ch. 2 in *"Modern Quantum Mechanics,"* by J. Sakurai.

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Simple Harmonic Oscillator: operator method

- one-dimensional harmonic oscillator, $\hat{H} = \frac{p^2}{2m} + \frac{1}{2}kx^2$, where $[\hat{x}, \hat{p}] = i\hbar$
- define *annihilation* operator (destruction, lowering, or step-down operators):

$$\hat{a} = \sqrt{m\omega/2\hbar}\hat{x} + i\hat{p}/\sqrt{2m\hbar\omega}.$$

- define *creation* operator (raising, or step-up operators):

$$\hat{a}^\dagger = \sqrt{m\omega/2\hbar}\hat{x} - i\hat{p}/\sqrt{2m\hbar\omega}.$$

- note that $\hat{a}$ and $\hat{a}^\dagger$ are not hermitian operators, but $(\hat{a}^\dagger)^\dagger = \hat{a}$.
- the commutation relation for $\hat{a}$ and $\hat{a}^\dagger$ is $[\hat{a}, \hat{a}^\dagger] = 1$.
- the oscillator Hamiltonian can be written as,

$$\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}) = \hbar\omega(\hat{N} + \frac{1}{2}),$$

where $\hat{N}$ is called the number operator, which is hermitian.

Simple Harmonic Oscillator: operator method

- the number operator, $\hat{N} = \hat{a}^\dagger \hat{a}$,
- $[\hat{H}, \hat{a}] = -\hbar\omega\hat{a}$, and $[\hat{H}, \hat{a}^\dagger] = \hbar\omega\hat{a}^\dagger$.
- the eigen-energy of the system, $\hat{H}|\Psi\rangle = E|\Psi\rangle$, then

$$\hat{H}\hat{a}|\Psi\rangle = (E - \hbar\omega)\hat{a}|\Psi\rangle, \qquad \hat{H}\hat{a}^\dagger|\Psi\rangle = (E + \hbar\omega)\hat{a}^\dagger|\Psi\rangle.$$

- for any hermitian operator, $\langle\Psi|\hat{Q}^2|\Psi\rangle = \langle\hat{Q}\Psi|\hat{Q}\Psi\rangle \geq 0$.
- thus $\langle\Psi|\hat{H}|\Psi\rangle \geq 0$.
- ground state (lowest energy state), $\hat{a}|\Psi_0\rangle = 0$.
- energy of the ground state, $\hat{H}|\Psi_0\rangle = \frac{1}{2}\hbar\omega|\Psi_0\rangle$.
- excited state, $\hat{H}|\Psi_n\rangle = \hat{H}(\hat{a}^\dagger)^n|\Psi_0\rangle = \hbar\omega(n + \frac{1}{2})(\hat{a}^\dagger)^n|\Psi_0\rangle$.
- eigen-energy for excited state, $E_n = (n + \frac{1}{2})\hbar\omega$.

Simple Harmonic Oscillator: operator method

- normalization of the eigenstates, $(\hat{a}^\dagger)^n|\Psi_0\rangle = c_n|\Psi_n\rangle$, where $c_n = \sqrt{n}$.
- $\hat{a}|\Psi_n\rangle = \sqrt{n}|\Psi_{n-1}\rangle$,
- $\hat{a}^\dagger|\Psi_n\rangle = \sqrt{n+1}|\Psi_{n+1}\rangle$,
- x-representation, $\Psi_n(x) = \langle x|\Psi_n\rangle$.
- ground state, $\langle x|\hat{a}|\Psi_0\rangle = 0$, i.e.

$$[\sqrt{\frac{m\omega}{2\hbar}}x + \hbar\frac{1}{\sqrt{2m\hbar\omega}}\frac{\mathrm{d}}{\mathrm{d}x}]\Psi_0(x) = 0,$$

- define a dimensionless variable $\xi = \sqrt{m\omega/hbar}x$, we obtain

$$(\xi + \frac{\mathrm{d}}{\mathrm{d}\xi})\Psi_0 = 0,$$

with the solution $\Psi_0(\xi) = c_0\mathrm{exp}(-\xi^2/2)$.

Maxwell's equations in Free space

- Faraday's law:

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B},$$

- Ampére's law:

$$\nabla \times \mathbf{H} = \frac{\partial}{\partial t}\mathbf{D},$$

- Gauss's law for the electric field:

$$\nabla \cdot \mathbf{D} = 0,$$

- Gauss's law for the magnetic field:

$$\nabla \cdot \mathbf{B} = 0,$$

- the constitutive relation: $\mathbf{B} = \mu_0 \mathbf{H}$ and $\mathbf{D} = \epsilon_0 \mathrm{E}$.

Mode Expansion of the Field

- A single-mode field, polarized along the x-direction, in the cavity:

$$\mathbf{E}(r,t) = \hat{x}E_x(z,t) = \sum_j \left(\frac{2m_j\omega_j^2}{V\epsilon_0}\right)^{1/2} q_j(t)\sin(k_j z),$$

  where $k = \omega/c$, $\omega_j = c(j\pi/L)$, $j = 1, 2, \ldots$, V is the effective volume of the cavity, and $q(t)$ is the normal mode amplitude with the dimension of a length (acts as a canonical position, and $p_j = m_j\dot{q}_j$ is the canonical momentum).

- the magnetic field in the cavity:

$$\mathbf{H}(r,t) = \hat{y}H_y(z,t) = \left(m_j\frac{2\omega_j^2}{V\epsilon_0}\right)^{1/2}\left(\frac{\dot{q}_j(t)\epsilon_0}{k_j}\right)\cos(k_j z),$$

- the classical Hamiltonian for the field:

$$\begin{aligned} H &= \frac{1}{2}\int_V \mathrm{d}V[\epsilon_0 E_x^2 + \mu_0 H_y^2], \\ &= \frac{1}{2}\sum_j [m_j\omega_m^2 q_j^2 + m_j\dot{q}_j^2] = \frac{1}{2}\sum_j [m_j\omega_m^2 q_j^2 + \frac{p_j^2}{m_j}]. \end{aligned}$$

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Quantization of the Electromagnetic Field

- Like simple harmonic oscillator, $\hat{H} = \frac{p^2}{2m} + \frac{1}{2}kx^2$, where $[\hat{x}, \hat{p}] = i\hbar$,
- For EM field, $\hat{H} = \frac{1}{2}\sum_j [m_j\omega_m^2 q_j^2 + \frac{p_j^2}{m_j}]$, , where $[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}$,
- *annihilation* and *creation* operators:

$$\hat{a}_j e^{-i\omega_j t} = \frac{1}{\sqrt{2m_j\hbar\omega_j}}(m_j\omega_j\hat{q}_j + i\hat{p}_j),$$
$$\hat{a}_j^\dagger e^{i\omega_j t} = \frac{1}{\sqrt{2m_j\hbar\omega_j}}(m_j\omega_j\hat{q}_j - i\hat{p}_j),$$

- the Hamiltonian for EM fields becomes: $\hat{H} = \sum_j \hbar\omega_j(\hat{a}_j^\dagger\hat{a}_j + \frac{1}{2})$,
- the electric and magnetic fields become,

$$\hat{E}_x(z,t) = \sum_j (\frac{\hbar\omega_j}{\epsilon_0 V})^{1/2}[\hat{a}_j e^{-i\omega_j t} + \hat{a}_j^\dagger e^{i\omega_j t}]\sin(k_j z),$$
$$\hat{H}_y(z,t) = -i\epsilon_0 c\sum_j (\frac{\hbar\omega_j}{\epsilon_0 V})^{1/2}[\hat{a}_j e^{-i\omega_j t} - \hat{a}_j^\dagger e^{i\omega_j t}]\cos(k_j z),$$

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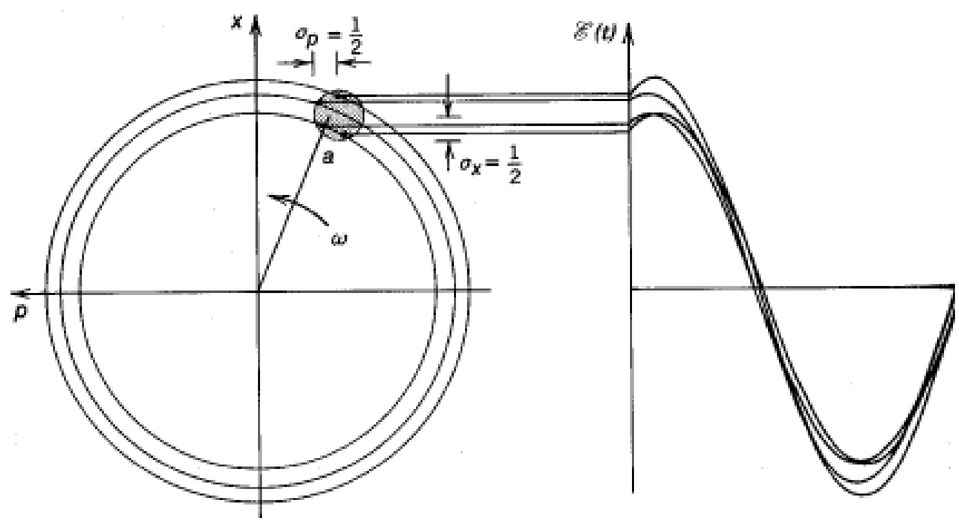
Phase diagram for EM waves

Electromagnetic waves can be represented by

$$\hat{E}(t) = E_0[\hat{X}_1 \sin(\omega t) - \hat{X}_2 \cos(\omega t)]$$

where

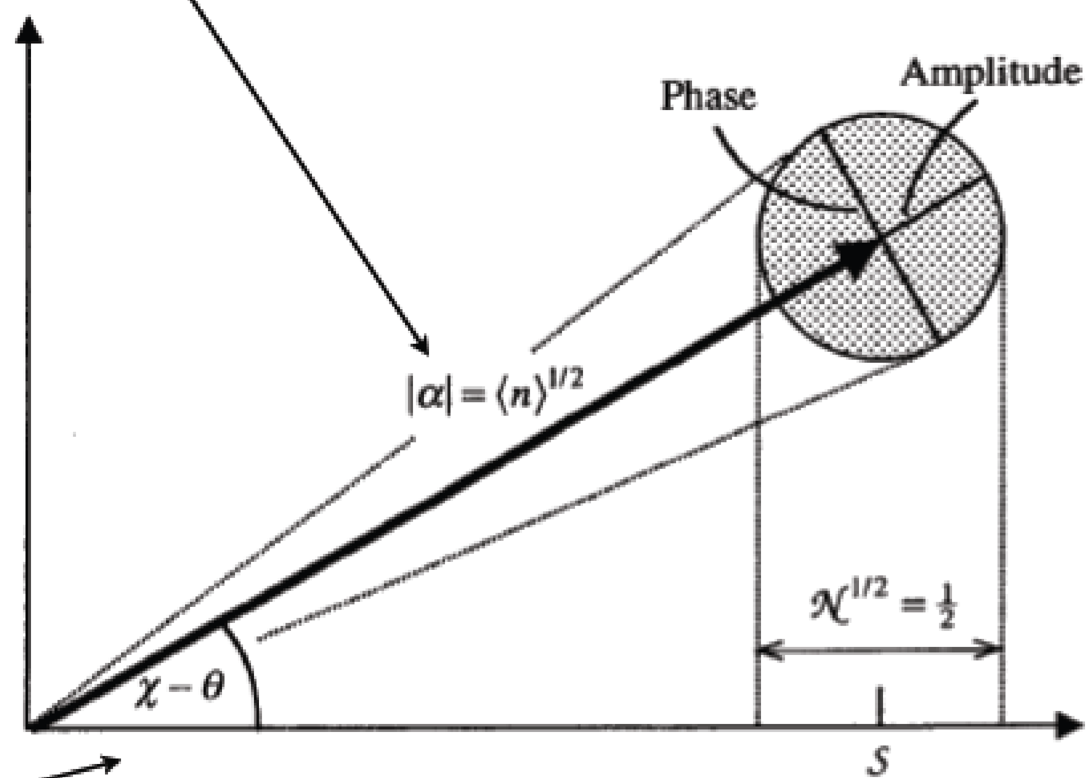
$$\begin{aligned} \hat{X}_1 &= \text{amplitude quadrature} \\ \hat{X}_2 &= \text{phase quadrature} \end{aligned}$$



Phase diagram for coherent states

mean number of photons

$$< \hat{N} >=< \alpha|\hat{N}|\alpha >=< \alpha|\hat{a}^\dagger \hat{a}|\alpha >= |\alpha|^2$$



phase of the field

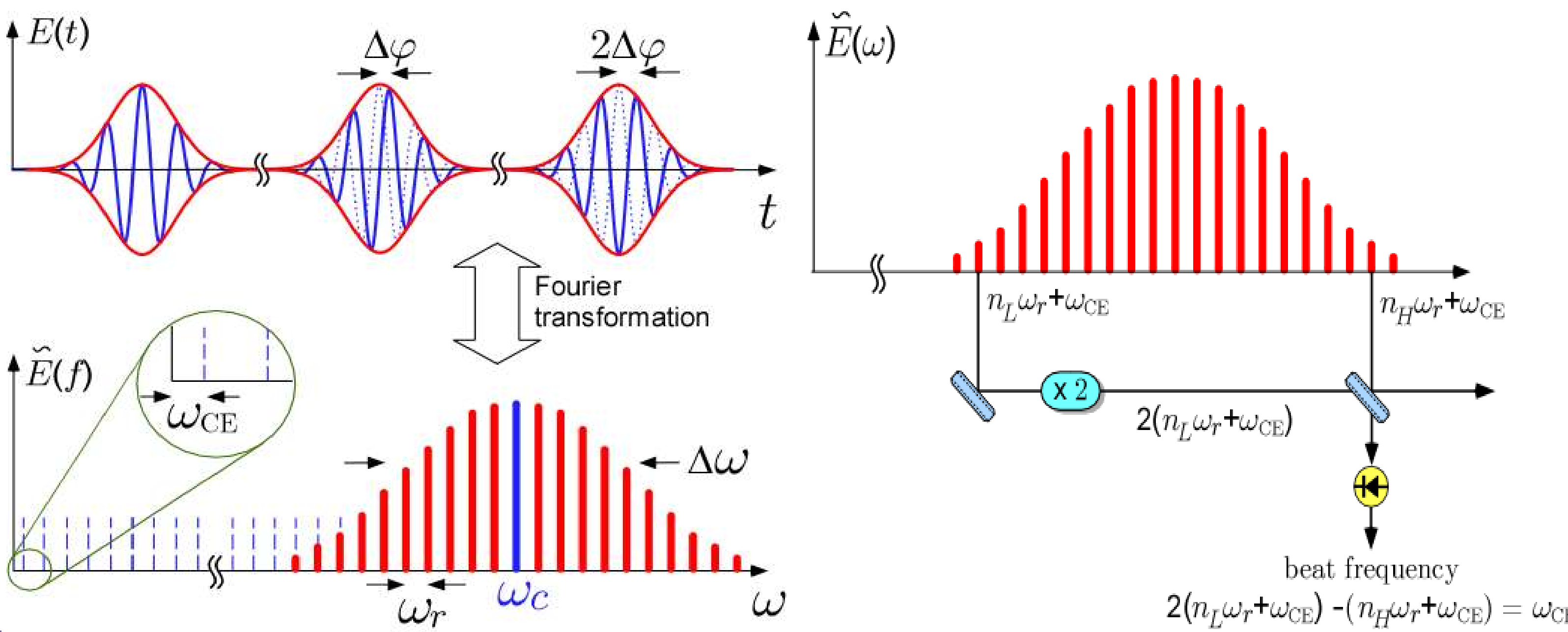
$$\alpha = |\alpha| \exp(i\theta)$$

Coherent states and Comb lasers

coherent Glauber state:

$$|\alpha> = \sum_{n=0} \alpha^n \frac{e^{-\frac{|\alpha|^2}{2}}}{\sqrt{n!}} |n>$$

Self referencing of frequency combs:



from: http://www.mpq.mpg.de/ haensch/

Quantum Fluctuations and Zero Point Energy

- divergence of the vacuum energy
- Casimir effect
- Lamb shift
- spontaneous emission