4, Quantum Distribution Theory

1. Expansion in Number states
2. Expansion in Coherent states
3. Q-representation
4. Wigner-Weyl distribution
5. Master Equation
6. Stochastic Differential Equation

Ref:

Ch. 3 in *"Quantum Optics,"* by M. Scully and M. Zubairy.

Ch. 6 in *"Quantum Optics,"* by D. Wall and G. Milburn.

Ch. 8 in *"Mesoscopic Quantum Optics,"* by Y. Yamamoto and A. Imamoglu.

Ch. 4, 5 in *"Mathematical Methods of Quantum Optics,"* by R. Puri.

"Quantum Optics in Phase Space," by W. Schleich.

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Phase Space Probability Distribution Function

- A classical dynamical system may be described by a phase space probability distribution function,

$$f(\{q\}, \{p\}),$$

where

$$\{q\} \equiv q_1, q_2, \ldots, q_N; \quad \text{and} \quad \{p\} \equiv p_1, p_2, \ldots, p_N,$$

- the probability

$$f(\{q\}, \{p\})\mathrm{d}^N q \, \mathrm{d}^N p,$$

gives the description about the system in a volume element $\mathrm{d}^N q \, \mathrm{d}^N p$,

- in quantum mechanics, the phase coordinates q_i and p_i can not described definite values simultaneously,
- hence the concept of phase space distribution function does not exist for a quantum system,
- however, it's possible to construct a *quantum quasi-probability distribution* resembling the classical phase space distribution functions.

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Phase Space Distribution Function

- consider a one dimensional dynamical system, described classically by a phase space distribution function $f(q,p,t)$,

$$\langle A(q,p)\rangle_{\text{cl}} = \int \mathrm{d}q\,\mathrm{d}p A(q,p) f(q,p,t),$$

- for the quantum mechanical description, if we know that the system is in state $|\psi\rangle$, then an operator $\hat{O}$ has the expectation value,

$$\langle\hat{O}\rangle_{\text{qm}} = \langle\psi|\hat{O}|\psi\rangle,$$

- but we typically do not know that we are in state $|\psi\rangle$, then an ensemble average must be performed,

$$\langle\langle\hat{O}\rangle_{\text{qm}}\rangle_{\text{ensemble}} = \sum_{\psi} P_{\psi}\langle\psi|\hat{O}|\psi\rangle,$$

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Phase Space Distribution Function

- using completeness $\sum_n |n\rangle\langle n| = 1$,

$$\langle\langle\hat{O}\rangle_{\text{qm}}\rangle_{\text{ensemble}} = \sum_n \langle n|\hat{\rho}\hat{O}|n\rangle,$$

where the P_ψ is the probability of being in the state $|\psi\rangle$ and we introduce a density operator,

$$\hat{\rho} = \sum_\psi P_\psi |\psi\rangle\langle\psi|,$$

- the expectation value of any operator $\hat{A}$ is given by,

$$\langle\hat{A}(\hat{q},\hat{p})\rangle_{\text{qm}} = \text{Tr}[\hat{\rho}\hat{A}(\hat{q},\hat{p})],$$

where Tr stands for trace.

- the density operator $\hat{\rho}$ can be expanded in terms of the number states,

$$\hat{\rho} = \sum_n \sum_m |n\rangle\langle n|\hat{\rho}|m\rangle\langle m| = \sum_n \sum_m \rho_{nm}|n\rangle\langle m|,$$

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Expansion in Number States

- the density operator $\hat{\rho}$ can be expanded in terms of the number states,

$$\hat{\rho} = \sum_n \sum_m |n\rangle\langle n|\hat{\rho}|m\rangle\langle m| = \sum_n \sum_m \rho_{nm}|n\rangle\langle m|,$$

- the expansion coefficients ρ_{nm} are complex and there is an infinite number of them,
- for problems where the phase-dependent properties of EM field are important, this make the general expansion rather less useful,
- in certain case where only the photon number distribution is of interest, one may use

$$\hat{\rho} = \sum_n P_n|n\rangle\langle n|,$$

- for a chaotic field, $P_n = \frac{1}{1+\bar{n}}(\frac{\bar{n}}{1+\bar{n}})^n$,
- for a Poisson distribution of photons, $P_n = \frac{e^{-\bar{n}}}{n!}\bar{n}^n$,

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Expansion in Coherent States

- likewise the expansion may be in terms of coherent states,

$$\hat{\rho} = \frac{1}{\pi^2} \int \int \mathrm{d}^2\alpha \mathrm{d}^2\beta |\alpha\rangle\langle\alpha|\hat{\rho}|\beta\rangle\langle\beta|,$$

  where $\frac{1}{\pi} \int |\alpha\rangle\langle\alpha| \mathrm{d}^2\alpha = 1$,

- the expectation value of any operator $\hat{A}$ is given by, $\langle \hat{A}(\hat{a}, \hat{a}^\dagger)\rangle_{\text{qm}} = \mathrm{Tr}[\hat{\rho}\hat{A}(\hat{a}, \hat{a}^\dagger)]$,
- quasi-probability distribution,

$$\begin{aligned}
\langle \hat{O}(\hat{a}, \hat{a}^\dagger)\rangle &= \int \mathrm{d}^2\alpha P(\alpha, \alpha^*) O_N(\alpha, \alpha^*), \quad \text{for normally ordering operators,} \\
&= \int \mathrm{d}^2\alpha Q(\alpha, \alpha^*) O_A(\alpha, \alpha^*), \quad \text{for antinormally ordering operators,} \\
&= \int \mathrm{d}^2\alpha W(\alpha, \alpha^*) O_S(\alpha, \alpha^*), \quad \text{for symmetric ordering operators,}
\end{aligned}$$

- classically phase space distribution function $f(q, p, t)$,

$$\langle A(q, p)\rangle_{\text{cl}} = \int \mathrm{d}q\, \mathrm{d}p A(q, p) f(q, p, t),$$

Phase Space Distribution Function

- rewrite classical distribution as,

$$
\begin{aligned}
f(q,p,t) &= \int \mathrm{d}q'\,\mathrm{d}p'\,\delta(q-q')\delta(p-p')f(q',p',t),\\
&= \frac{1}{4\pi^2}\int \mathrm{d}q'\,\mathrm{d}p'\,\mathrm{d}k\,\mathrm{d}l\exp\{i[k(q-q')+l(p-p')]\}f(q',p',t),\\
&= \frac{1}{4\pi^2}\int \mathrm{d}k\,\mathrm{d}l\exp(ikq)\exp(ilp)\int \mathrm{d}q'\,\mathrm{d}p'\exp(-ikq)\exp(-ilp')f(q',p',t),\\
&= \frac{1}{4\pi^2}\int \mathrm{d}k\,\mathrm{d}l\exp(ikq)\exp(ilp)\langle\exp(-ikq)\exp(-ilp)\rangle_{cl},
\end{aligned}
$$

with $\delta x = \frac{1}{2\pi}\int \mathrm{d}k\exp(ikx)$,

- for the quantum analog of $f(q,p,t)$,
  1. replace the c-numbers q, p by the operators $\hat{q}$, $\hat{p}$,
  2. replace the classical average by the quantum average,
  3. express the exponential under the average as a sum of products of the form $q^m p^n$,

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Quasiprobability Distribution Function

- due to non-commutativity of $\hat{q}$ and $\hat{p}$, there are several different operator forms of a c-number product $q^m p^n$, if $m, n \neq 0$,
- for example, $q^2 p$ may be represented by an of the forms: $\hat{q}^2$, $\hat{q}\hat{p}\hat{q}$, $\hat{p}\hat{q}^2$ or by their linear combination $c_1\hat{q}^2 + c_2\hat{q}\hat{p}\hat{q} + c_3\hat{p}\hat{q}^2$, where x_i are arbitrary subject to the condition $c_1 + c_2 + c_3 = 1$,
- in general, we formally represent a c-number product as an operator as,

$$q^m p^n \to \Omega(\hat{q}^m \hat{p}^n),$$

which defines a linear combination of m $\hat{q}$'s and n $\hat{p}$'s,

- for example,

$$\begin{aligned} \exp[\alpha_1\hat{a} + \alpha_2\hat{a}^\dagger] &= \exp[\alpha_2\hat{a}^\dagger]\exp[\alpha_1\hat{a}]\exp[\frac{1}{2}\alpha_1\alpha_2], \quad \text{normally ordering,} \\ &= \exp[\alpha_1\hat{a}]\exp[\alpha_2\hat{a}^\dagger]\exp[-\frac{1}{2}\alpha_1\alpha_2], \quad \text{antinormally ordering,} \end{aligned}$$

- with the Baker-Hausdorff relation, $e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{-\frac{1}{2}[\hat{A},\hat{B}]} = e^{+\frac{1}{2}[\hat{A},\hat{B}]}e^{\hat{B}}e^{\hat{A}}$, provided $[\hat{A},[\hat{A},\hat{B}]] = [[\hat{A},\hat{B}],\hat{B}] = 0$,

Quasiprobability Distribution Function

- the quantum analog of the classical phase space distribution function is then,

$$f^{\Omega}(q,p,t)=\frac{1}{4\pi^2}\int \mathrm{d}k\,\mathrm{d}l\,\mathrm{exp}(ikq)\mathrm{exp}(ilp)\langle\Omega[\mathrm{exp}(-ik\hat{q})\mathrm{exp}(-il\hat{p})]\rangle_{qm},$$

- different choices of the correspondence Ω lead to different $f^{\Omega}(q,p,t)$, each called a *quasi-probability distribution* function to emphasize that it is a mathematical construct and not a true phase space distribution function.

- the quantum analog of the classical phase space distribution function in terms of the creation and annihilation operators $\hat{a}$ and $\hat{a}^{\dagger}$ is,

$$f^{\Omega}(\alpha,\alpha^*)=\frac{1}{\pi^2}\int \mathrm{d}^2\xi\,\mathrm{exp}[i(\alpha\xi+\alpha^*\xi^*)]\mathrm{Tr}[\Omega\{\mathrm{exp}(-i\hat{\alpha}\xi)\mathrm{exp}(-i\hat{\alpha}^{\dagger}\xi^*)\}\hat{\rho}],$$

Quasiprobability Distribution Function

- now, let

$$\Omega\{\exp(-i\hat{\alpha}\xi)\exp(-i\hat{\alpha}^\dagger\xi^*)\} = \prod_{j=1}^{N}[\exp(-i\alpha_j\xi\hat{\alpha})\exp(-i\beta\xi^*\hat{\alpha}^\dagger], \\ = \exp(-\frac{s}{2}|\xi|^2)\exp[-i(\xi\hat{\alpha}+\xi^*\hat{\alpha}^\dagger)],$$

where s is a complex number related with products of the α_j and β_j,

- although, the exact expression of s in terms of the α_j and β_j may be derived, it is inessential.
- the ordering for $s = 0$ is called the *Weyl ordering*, or the *symmetric ordering*,
- the exponential operator may be put in the anitnormal or the normal ordering,

$$\exp[-i(\xi\hat{a}+\xi^*\hat{a}^\dagger)] = \exp(-i\xi^*\hat{a}^\dagger)\exp(-i\xi\hat{a})\exp(-\frac{1}{2}|\xi|^2), \quad \text{normally ordering,} \\ = \exp(-i\xi\hat{a})\exp(-i\xi^*\hat{a}^\dagger)\exp(\frac{1}{2}|\xi|^2), \quad \text{antinormally ordering,}$$

Quasiprobability Distribution Function

- the quantum analog of the classical phase space distribution function in the s-ordering is,

$$f^{(s)}(\alpha,\alpha^*) = \frac{1}{\pi^2}\int \mathrm{d}^2\xi \exp[i(\alpha\xi+\alpha^*\xi^*)]\exp(-\frac{s}{2}|\xi|^2)\mathrm{Tr}\{\exp[-i(\xi\hat{a}+\xi^*\hat{a}^\dagger)]\hat{\rho}\},$$

  this is some kind of two-dimensional Fourier transformation,

- define

$$\mathrm{Tr}\{\exp[-i(\xi\hat{a}+\xi^*\hat{a}^\dagger)]\hat{\rho}\} \equiv G(\xi,\xi^*)\exp(\frac{s}{2}|\xi|^2),$$

  then

$$f^{(s)}(\alpha,\alpha^*) = \frac{1}{\pi^2}\int \mathrm{d}^2\xi G(\xi,\xi^*)\exp[i(\alpha\xi+\alpha^*\xi^*)],$$

  and by the inverse Fourier transformation,

$$G(\xi,\xi^*) = \int \mathrm{d}^2\alpha f^{(s)}(\alpha,\alpha^*)\exp[-i(\alpha\xi+\alpha^*\xi^*)]$$

Quasiprobability Distribution Function

- for antinormal form of the exponential,

$$\begin{aligned}\mathrm{Tr}[\exp[-i(\xi\hat{a}+\xi^*\hat{a}^\dagger)\hat{\rho}] &= \mathrm{Tr}[\exp(-i\xi\hat{a})\exp(-i\xi^*\hat{a}^\dagger)\hat{\rho}](\frac{1}{2}|\xi|^2),\\ &= \mathrm{Tr}[\exp(-i\xi^*\hat{a}^\dagger)\hat{\rho}\exp(-i\xi\hat{a})](\frac{1}{2}|\xi|^2),\\ &= \frac{1}{\pi}\int \mathrm{d}^2\alpha\exp[-i(\alpha\xi+\alpha^*\xi^*)](\frac{1}{2}|\xi|^2)\langle\alpha|\hat{\rho}|\alpha\rangle,\\ &= G(\xi,\xi^*)\exp(\frac{s}{2}|\xi|^2),\end{aligned}$$

- for the density matrix in the coherent state representation,

$$\langle\alpha|\hat{\rho}|\alpha\rangle = \frac{1}{\pi}\int \mathrm{d}^2\xi G(\xi,\xi^*)\exp(\frac{s-1}{2}|\xi|^2)\exp[i(\alpha\xi+\alpha^*\xi^*)]$$

Quasiprobability Distribution Function

- and the relationship between the density operator and its various phase space representation through $G(\xi, \xi^*)$ is,

$$\begin{aligned}\hat{\rho} &= \frac{1}{\pi}\int \mathrm{d}^2\xi G(\xi,\xi^*)\exp(\frac{s-1}{2}|\xi|^2)\exp(i\xi^*\hat{a}^\dagger)\exp(i\xi\hat{a}), \quad \text{for antinormally ordering}\\ &= \frac{1}{\pi}\int \mathrm{d}^2\xi G(\xi,\xi^*)\exp(\frac{s}{2}|\xi|^2)\exp[i(\xi\hat{a}+\xi^*\hat{a}^\dagger)], \quad \text{for symmetric ordering,}\\ &= \frac{1}{\pi}\int \mathrm{d}^2\xi G(\xi,\xi^*)\exp(\frac{s+1}{2}|\xi|^2)\exp(i\xi\hat{a})\exp(i\xi^*\hat{a}^\dagger), \quad \text{for normally ordering,}\end{aligned}$$

- the relation between different phase space representation $f^{(s)}$ and $f^{(t)}$ is,

$$f^{(s)}(\alpha,\alpha^*) = \frac{2}{\pi(s-t)}\int \mathrm{d}^2\beta \exp[-\frac{2|\alpha-\beta|^2}{s-t}]f^{(t)}(\beta,\beta^*),$$

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Expectation value of the operator

- the phase space distribution function in the s-ordering is,

$$f^{(s)}(\alpha, \alpha^*) = \frac{1}{\pi^2} \int \mathrm{d}^2\xi \exp[i(\alpha\xi + \alpha^*\xi^*)] \exp(-\frac{s}{2}|\xi|^2) \mathrm{Tr}[\exp[-i(\xi\hat{a} + \xi^*\hat{a}^\dagger)\hat{\rho}],$$

- the phase space representation of any operator $\hat{A}$ is similar,

$$A^{(s)}(\alpha, \alpha^*) = \frac{1}{\pi^2} \int \mathrm{d}^2\xi \exp[i(\alpha\xi + \alpha^*\xi^*)] \exp(-\frac{s}{2}|\xi|^2) \mathrm{Tr}[\exp[-i(\xi\hat{a} + \xi^*\hat{a}^\dagger)\hat{A}],$$

- the expectation value of $\hat{A}$ is,

$$\begin{aligned} \mathrm{Tr}[\hat{A}\hat{\rho}] &= \frac{1}{\pi} \int \mathrm{d}^2\xi G(\xi, \xi^*) \exp(\frac{s}{2}|\xi|^2) \mathrm{Tr}\{\hat{A}\exp[i(\xi\hat{a} + \xi^*\hat{a}^\dagger)]\}, \\ &= \frac{1}{\pi} \int\int \mathrm{d}^2\xi \mathrm{d}^2\alpha f^{(s)}(\alpha, \alpha^*) \exp[-i(\xi\alpha + \xi^*\alpha^*)] \exp(\frac{s}{2}|\xi|^2) \mathrm{Tr}\{\hat{A}\exp[i(\xi\hat{a} + \\ &= \pi \int \mathrm{d}^2\alpha f^{(s)}(\alpha, \alpha^*) A^{(-s)}(\alpha, \alpha^*), \end{aligned}$$

- the expectation value of an operator is TE phase space integral of the product of its phase space function with its conjugate representation of the density operator.

P-representation, normally ordering

- the density operator $\hat{\rho}$ can be expanded in terms of the number states,

$$\hat{\rho} = \sum_n \sum_m |n\rangle\langle n|\hat{\rho}|m\rangle\langle m| = \sum_n \sum_m \rho_{nm}|n\rangle\langle m|,$$

- likewise the expansion may be in terms of coherent states,

$$\hat{\rho} = \frac{1}{\pi^2}\int\int \mathrm{d}^2\alpha \mathrm{d}^2\beta|\alpha\rangle\langle\alpha|\hat{\rho}|\beta\rangle\langle\beta|,$$

- only the photon number distribution is of interest, one may use

$$\hat{\rho} = \sum_n P_n|n\rangle\langle n|,$$

- P-representation of a density operator,

$$\hat{\rho} = \int \mathrm{d}^2\alpha P(\alpha,\alpha^*)|\alpha\rangle\langle\alpha|,$$

P-representation, normally ordering

- P-representation of a density operator,

$$\hat{\rho} = \int \mathrm{d}^2\alpha P(\alpha, \alpha^*)|\alpha\rangle\langle\alpha|,$$

substitute into

$$f^{(s)}(\alpha, \alpha^*) = \frac{1}{\pi^2}\int \mathrm{d}^2\xi \exp[i(\alpha\xi + \alpha^*\xi^*)]\exp(-\frac{s}{2}|\xi|^2)\mathrm{Tr}[\exp[-i(\xi\hat{a} + \xi^*\hat{a}^\dagger)]\hat{\rho}],$$

with $s = -1$ and the exponential operator in the normal-ordering, we have

$$\begin{aligned} f^{(-1)}(\alpha, \alpha^*) &= \frac{1}{\pi^2}\int \mathrm{d}^2\xi \int \mathrm{d}^2\beta P(\beta, \beta^*)\exp[i(\alpha\xi + \alpha^*\xi^*)]\mathrm{Tr}[e^{(-i\xi\hat{a})}|\beta\rangle\langle\beta|e^{(-i\xi^*\hat{a} \\ &= \frac{1}{\pi^2}\int \mathrm{d}^2\xi \int \mathrm{d}^2\beta P(\beta, \beta^*)\exp\{i[(\alpha - \beta)\xi + (\alpha^* - \beta^*)\xi^*]\}, \\ &= P(\alpha, \alpha^*), \end{aligned}$$

- the phase space representation for $s = -1$ is thus the P-function,

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P-representation, normally ordering

- P-representation of a density operator,

$$\hat{\rho} = \int \mathrm{d}^2\alpha P(\alpha, \alpha^*)|\alpha\rangle\langle\alpha|,$$

- the phase space representation for $s = -1$ is thus the P-function,

$$f^{(-1)}(\alpha, \alpha^*) = P(\alpha, \alpha^*),$$

- equivalent, one can define

$$\begin{aligned}
P(\alpha, \alpha^*) &= \mathrm{Tr}[\hat{\rho}\delta(\alpha^* - \hat{a}^\dagger)\delta(\alpha - \hat{a})], \\
&= \mathrm{Tr}[\int \mathrm{d}^2\beta P(\beta, \beta^*)|\beta\rangle\langle\beta|\delta(\alpha^* - \hat{a}^\dagger)\delta(\alpha - \hat{a})], \\
&= \int \mathrm{d}^2\alpha \int \mathrm{d}^2\beta P(\beta, \beta^*)\langle\alpha|\beta\rangle\langle\beta|\delta(\alpha^* - \hat{a}^\dagger)\delta(\alpha - \hat{a})|\alpha\rangle,
\end{aligned}$$

note it is normally ordering in the trace,

$$\delta(\alpha^* - \hat{a}^\dagger)\delta(\alpha - \hat{a}),$$

P-representation, normally ordering

- the function $P(\alpha, \alpha^*)$ can be used to evaluate the expectation values of any normal ordered function of $\hat{a}$ and $\hat{a}^\dagger$ using the methods of classical statistical mechanics,

$$\begin{aligned}\langle \hat{A}_N \rangle &= \text{Tr}(\hat{A}_N) = \frac{1}{\pi}\int \text{d}^2\xi G(\xi, \xi^*)\text{exp}(\frac{-1}{2}|\xi|^2)\text{Tr}\{\hat{A}\text{exp}[i(\xi\hat{a} + \xi^*\hat{a}^\dagger)]\},\\ &= \pi \int \text{d}^2\alpha f^{(-1)}(\alpha, \alpha^*)A^{(1)}(\alpha, \alpha^*),\\ &= \int \text{d}^2\alpha P(\alpha, \alpha^*)A_N(\alpha, \alpha^*),\end{aligned}$$

- since $\text{Tr}(\hat{\rho}) = 1$,

$$\int \text{d}^2\alpha P(\alpha, \alpha^*) = 1,$$

- the function $P(\alpha, \alpha^*)$ is referred to as the P-representation or the coherent state representation,

$$\hat{\rho} = \int \text{d}^2\alpha P(\alpha, \alpha^*)|\alpha\rangle\langle\alpha|,$$

- the function $P(\alpha, \alpha^*)$ forms a connection between the classical and quantum coherence theory,

P-representation, normally ordering

- let $|\beta\rangle$ and $|-\beta\rangle$ be the coherent states, then

$$
\begin{aligned}
\langle -\beta|\hat{\rho}|\beta\rangle &= \int \mathrm{d}^2\alpha P(\alpha,\alpha^*)\langle -\beta|\alpha\rangle\langle\alpha|\beta\rangle, \\
&= e^{-|\beta|^2}\int \mathrm{d}^2\alpha P(\alpha,\alpha^*) e^{-|\alpha|^2} e^{\beta\alpha^*-\beta^*\alpha}, \\
&= e^{-|\beta|^2}\int \mathrm{d}x_\alpha \int \mathrm{d}y_\alpha P(x_\alpha,y_\alpha) e^{-(x_\alpha^2+y_\alpha^2)} e^{2i(y_\beta x_\alpha - x_\beta y_\alpha)},
\end{aligned}
$$

with

$$
\langle\alpha|\beta\rangle = \exp(-\frac{1}{2}|\alpha|^2+\alpha^*\beta-\frac{1}{2}|\beta|^2) = \exp(-\frac{1}{2}|\alpha-\beta|^2),
$$

where $\alpha = x_\alpha + iy_\alpha$ and $\beta = x_\beta + iy_\beta$ and this is the two-dimensional Fourier transform,

$$
\begin{aligned}
P(\alpha,\alpha^*) &= \frac{e^{x_\alpha^2+y_\alpha^2}}{\pi^2}\int \mathrm{d}x_\beta \int \mathrm{d}y_\beta \langle -\beta|\hat{\rho}|\beta\rangle e^{(x_\beta^2+y_\beta^2)} e^{-2i(y_\beta x_\alpha - x_\beta y_\alpha)}, \\
&= \frac{e^{|\alpha|^2}}{\pi^2}\int \mathrm{d}^2\beta \langle -\beta|\hat{\rho}|\beta\rangle e^{|\beta|^2} e^{-\beta\alpha^*+\beta^*\alpha},
\end{aligned}
$$

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Thermal field expanded in Fock states

- expansion in the photon number distribution, $\hat{\rho} = \sum_n P_n |n\rangle\langle n|$,
- expansion in P-representation of a density operator, $\hat{\rho} = \int \mathrm{d}^2\alpha P(\alpha, \alpha^*)|\alpha\rangle\langle\alpha|$,
- for the thermal field,

$$\hat{\rho} = \frac{\exp(-\hat{H}/k_B T)}{\mathrm{Tr}[\exp(-\hat{H}/k_B T)]},$$

where k_B is the Boltzman constant and $\hat{H}$ is the free-field Hamiltonian,
$\hat{H} = \hbar\omega(\hat{a}^\dagger + \hat{a} + 1/2)$,

$$\hat{\rho} = \sum_n [1 - \exp(\frac{-\hbar\omega}{k_B T})]\exp(-\frac{n\hbar\omega}{k_B T})|n\rangle\langle n|,$$

- the expectation value of the photon number, $\langle\bar{n}\rangle = \mathrm{Tr}(\hat{a}^\dagger\hat{a}\hat{\rho}) = \frac{1}{\exp(\hbar\omega/k_B T)-1}$,
- the photon distribution in a thermal field,

$$\hat{\rho} = \sum_n \frac{\langle n\rangle^n}{(1+\langle n\rangle)^{n+1}}|n\rangle\langle n|,$$

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which is the Bose-Einstein distribution,

Thermal field expanded in P-representation

- expansion in P-representation of a density operator, $\hat{\rho} = \int \mathrm{d}^2\alpha P(\alpha, \alpha^*)|\alpha\rangle\langle\alpha|$,
- for the thermal field, $\hat{\rho} = \sum_n \frac{\langle n\rangle^n}{(1+\langle n\rangle)^{n+1}}|n\rangle\langle n|$, then

$$\langle -\beta|\hat{\rho}|\beta\rangle = \sum_n \frac{\langle n\rangle^n}{(1+\langle n\rangle)^{n+1}}\langle -\beta|n\rangle\langle n|\beta\rangle = \frac{e^{-|\beta|^2}}{1+\langle n\rangle}\mathrm{exp}[\frac{-|\beta|^2}{1+\frac{1}{\langle n\rangle}}],$$

with $|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$,

- the P-representation of the thermal field is

$$\begin{aligned} P(\alpha, \alpha^*) &= \frac{e^{|\alpha|^2}}{\pi^2}\int \mathrm{d}^2\beta\langle -\beta|\hat{\rho}|\beta\rangle e^{|\beta|^2}e^{-\beta\alpha^*+\beta^*\alpha}, \\ &= \frac{e^{|\alpha|^2}}{\pi^2(1+\frac{1}{\langle n\rangle})}\int \mathrm{d}^2\beta\mathrm{exp}[\frac{-|\beta|^2}{1+\frac{1}{\langle n\rangle}}]e^{-\beta\alpha^*+\beta^*\alpha}, \\ &= \frac{1}{\pi\langle n\rangle}e^{-|\alpha|^2/\langle n\rangle}, \end{aligned}$$

which is a Gaussian distribution with the width of $\langle n\rangle$ in phase space,

Coherent State expanded in P-representation

- for the coherent field, $\hat{\rho} = |\alpha_0\rangle\langle\alpha_0|$, then
- the P-representation of the coherent field is

$$
\begin{aligned}
P(\alpha, \alpha^*) &= \frac{1}{\pi^2} e^{|\alpha|^2 - |\alpha_0|^2} \int \mathrm{d}^2\beta \exp[-\beta(\alpha^* - \alpha_0^*) + \beta^*(\alpha - \alpha_0)], \\
&= \delta^{(2)}(\alpha - \alpha_0),
\end{aligned}
$$

which is a two-dimensional delta function in phase space, i.e.

$$
\begin{aligned}
f^{(s)}(\alpha, \alpha^*) &= \frac{1}{\pi^2} \int \mathrm{d}^2\xi G(\xi, \xi^*) \exp[i(\alpha\xi + \alpha^*\xi^*)], \\
G(\xi, \xi^*) &= \int \mathrm{d}^2\alpha f^{(s)}(\alpha, \alpha^*) \exp[-i(\alpha\xi + \alpha^*\xi^*)]
\end{aligned}
$$

where $\mathrm{Tr}\{\exp[-i(\xi\hat{a} + \xi^*\hat{a}^\dagger)]\hat{\rho}\} \equiv G(\xi, \xi^*)\exp(\frac{s}{2}|\xi|^2)$,

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Number State expanded in P-representation

- the two-dimensional Fourier transform

$$
\begin{aligned}
f^{(s)}(\alpha, \alpha^*) &= \frac{1}{\pi^2} \int \mathrm{d}^2\xi G(\xi, \xi^*) \mathrm{exp}[i(\alpha\xi + \alpha^*\xi^*)], \\
G(\xi, \xi^*) &= \int \mathrm{d}^2\alpha f^{(s)}(\alpha, \alpha^*) \mathrm{exp}[-i(\alpha\xi + \alpha^*\xi^*)]
\end{aligned}
$$

  where $\mathrm{Tr}\{\mathrm{exp}[-i(\xi\hat{a} + \xi^*\hat{a}^\dagger)]\hat{\rho}\} \equiv G(\xi, \xi^*)\mathrm{exp}(\frac{s}{2}|\xi|^2)$,

- for thermal field, its P-representation is a Gaussian function in phase space,
- for coherent state, its P-representation is a 2D delta function in phase space,
- for a number state, $\hat{\rho} = |n\rangle\langle n|$, then

$$
\langle -\beta|\hat{\rho}|\beta\rangle = \langle -\beta|n\rangle\langle n|\beta\rangle = \mathrm{exp}(-|\beta|^2)\frac{(-1)^n|\beta|^{2n}}{n!},
$$

  and the corresponding P-representation is, $P(\alpha, \alpha^*) = \frac{e^{|\alpha|^2}}{n!}\frac{\partial^{2n}}{\partial\alpha^n\partial\alpha^{*n}}\delta^{(2)}(\alpha)$, which is not a *non-negative* definite function for $n > 0$,

- whenever the photon distribution ρ_{nn} is narrower than the Poisson distribution, $P(\alpha, \alpha^*)$ becomes badly behaved.

Number State expanded in P-representation

- the two-dimensional Fourier transform

$$f^{(s)}(\alpha,\alpha^*) = \frac{1}{\pi^2}\int \mathrm{d}^2\xi G(\xi,\xi^*)\mathrm{exp}[i(\alpha\xi+\alpha^*\xi^*)],$$
$$G(\xi,\xi^*) = \int \mathrm{d}^2\alpha f^{(s)}(\alpha,\alpha^*)\mathrm{exp}[-i(\alpha\xi+\alpha^*\xi^*)]$$

  where $\mathrm{Tr}\{\mathrm{exp}[-i(\xi\hat{a}+\xi^*\hat{a}^\dagger)]\hat{\rho}\} \equiv G(\xi,\xi^*)\mathrm{exp}(\frac{s}{2}|\xi|^2)$,

- for thermal field, its P-representation is a Gaussian function in phase space,
- for coherent state, its P-representation is a 2D delta function in phase space,
- for a number state, $\hat{\rho} = |n\rangle\langle n|$, then

$$\langle -\beta|\hat{\rho}|\beta\rangle = \langle -\beta|n\rangle\langle n|\beta\rangle = \mathrm{exp}(-|\beta|^2)\frac{(-1)^n|\beta|^{2n}}{n!},$$

  and the corresponding P-representation is, $P(\alpha,\alpha^*) = \frac{e^{|\alpha|^2}}{n!}\frac{\partial^{2n}}{\partial\alpha^n\partial\alpha^{*n}}\delta^{(2)}(\alpha)$, which is not a *non-negative* definite function for $n>0$,

- whenever the photon distribution ρ_{nn} is narrower than the Poisson distribution, $P(\alpha,\alpha^*)$ becomes badly behaved.

Properties of P-representation

- We may put a function $f(\hat{a}, \hat{a}^\dagger)$ into normal ordering by means of,

$$f^{(n)}(\hat{a}, \hat{a}^\dagger) = \langle\alpha|f(\hat{a}, \hat{a}^\dagger)|\alpha\rangle = f(\alpha + \frac{\partial}{\partial\alpha^*}, \alpha^*),$$

- for example

$$\langle\alpha|\hat{a}\hat{a}^\dagger|\alpha\rangle = (\alpha + \frac{\partial}{\partial\alpha^*})\alpha^* = \alpha\alpha^* + 1,$$

- for a coherent state $|\alpha\rangle$, then

$$\begin{aligned} |\alpha\rangle\langle\alpha|\hat{a} &= (e^{-\alpha^*\alpha}e^{\alpha\hat{a}^\dagger}|0\rangle\langle 0|e^{\alpha^*\hat{a}})\hat{a}, \\ &= e^{-\alpha^*\alpha}\frac{\partial}{\partial\alpha^*}\{e^{\alpha\hat{a}^\dagger}|0\rangle\langle 0|e^{\alpha^*\hat{a}}\}, \\ &= (\frac{\partial}{\partial\alpha^*} + \alpha)|\alpha\rangle\langle\alpha|, \end{aligned}$$

- by repeat $|\alpha\rangle\langle\alpha|\hat{a}^l = (\frac{\partial}{\partial\alpha^*} + \alpha)^l|\alpha\rangle\langle\alpha|$,
- its adjoint $\hat{a}^\dagger|\alpha\rangle\langle\alpha| = (\frac{\partial}{\partial\alpha} + \alpha^*)|\alpha\rangle\langle\alpha|$,

Glauber-Sudarshan P-representation

- consider a single electromagnetic field mode in a cavity with finite leakage rate, the time evolution of the field density is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}_f(t) = \frac{-1}{2}[R_e(\hat{a}\hat{a}^\dagger\hat{\rho}_f - \hat{a}^\dagger\hat{\rho}_f\hat{a}) + R_g(\hat{a}^\dagger\hat{a}\hat{\rho}_f - \hat{a}\hat{\rho}_f\hat{a}^\dagger)] + \text{adjoint},$$

where R_e and R_g are the photon emission and absorption rate coefficients,

- with the P-representation for the density operator, $\hat{\rho} = \int \mathrm{d}^2\alpha P(\alpha, \alpha^*)|\alpha\rangle\langle\alpha|$, then we have

$$\begin{aligned}\int \mathrm{d}^2\alpha \dot{P}|\alpha\rangle\langle\alpha| &= \frac{-1}{2}\int \mathrm{d}^2\alpha P[R_e(\hat{a}\hat{a}^\dagger|\alpha\rangle\langle\alpha| - \hat{a}^\dagger|\alpha\rangle\langle\alpha|\hat{a}) + R_g(\hat{a}^\dagger\hat{a}|\alpha\rangle\langle\alpha| - \hat{a}|\alpha\rangle\langle\\ &+ \text{adjoint},\end{aligned}$$

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Fokker-Planck equation

- with

$$|\alpha\rangle\langle\alpha|\hat{a} = (\frac{\partial}{\partial\alpha^*} + \alpha)|\alpha\rangle\langle\alpha|,$$
$$\hat{a}^\dagger|\alpha\rangle\langle\alpha| = (\frac{\partial}{\partial\alpha} + \alpha^*)|\alpha\rangle\langle\alpha|,$$

- we have the Fokker-Planck equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}P(\alpha,\alpha^*) = \frac{-1}{2}(R_e - R_g)\{\frac{\partial}{\partial\alpha}[\alpha P(\alpha,\alpha^*)] + \frac{\partial}{\partial\alpha^*}[\alpha^* P(\alpha,\alpha^*)]\} + R_e\frac{\partial^2}{\partial\alpha\partial\alpha^*}P(\alpha,\alpha^*),$$

compared with,

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}_f(t) = \frac{-1}{2}[R_e(\hat{a}\hat{a}^\dagger\hat{\rho}_f - \hat{a}^\dagger\hat{\rho}_f\hat{a}) + R_g(\hat{a}^\dagger\hat{a}\hat{\rho}_f - \hat{a}\hat{\rho}_f\hat{a}^\dagger)] + \text{adjoint},$$

Positive-P-representation

- The advantage of the Fokker-Planck equation is that it significantly simplifies the calculation process for the fields that are approximately coherent states,
- when the fields become nonclassical, the P-representation is no longer well-behaved, such as the squeezed and photon number states,
- in order to map an arbitrary nonclassical state into a classical probability density, the dimension of the phase space must at least be doubled,
- one may use off-diagonal or *positive-P-representation* for nonclassical states,

"Quantum noise," by C. W. Gardiner

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Q-representation, normally ordering

- for $s=1$, the density matrix in the coherent state representation is,

$$\begin{aligned} \langle\alpha|\hat{\rho}|\alpha\rangle &= \frac{1}{\pi}\int \mathrm{d}^2\xi G(\xi,\xi^*)\mathrm{exp}[i(\alpha\xi+\alpha^*\xi^*))], \\ &= \pi f^{(1)}(\alpha,\alpha^*) \equiv Q(\alpha,\alpha^*), \end{aligned}$$

- $f^{(1)}(\alpha,\alpha^*)$ is simply the matrix element of the operator in the coherent states representation, known as the Q-function,
- the expectation value, $\mathrm{Tr}[\hat{A}\hat{\rho}] = \frac{1}{\pi}\int \mathrm{d}^2\alpha f^{(s)}(\alpha,\alpha^*)A^{(-s)}(\alpha,\alpha^*)$,
- if the density operator is represented by P-function, then

$$\langle\hat{a}^{\dagger m}\hat{a}^n\rangle = \int \mathrm{d}^2\alpha P(\alpha,\alpha^*)\alpha^{*m}\alpha^m,$$

- if the density operator is represented by P-function, then

$$\langle\hat{a}^n\hat{a}^{\dagger m}\rangle = \int \mathrm{d}^2\alpha Q(\alpha,\alpha^*)\alpha^{*m}\alpha^m,$$

Q-representation, normally ordering

- Q-representation defineds as the antinormally ordering in the trace,

$$
\begin{aligned}
Q(\alpha,\alpha^*) &= \mathrm{Tr}[\hat{\rho}\delta(\alpha-\hat{a})\delta(\alpha^*-\hat{a}^\dagger)], \\
&= \frac{1}{\pi}\mathrm{Tr}\int \mathrm{d}^2\beta[\hat{\rho}\delta(\alpha-\hat{a})|\beta\rangle\langle\beta|\delta(\alpha^*-\hat{a}^\dagger)], \\
&= \frac{1}{\pi}\mathrm{Tr}[\hat{\rho}|\alpha\rangle\langle\alpha|], \\
&= \frac{1}{\pi}\langle\alpha|\hat{\rho}|\alpha\rangle,
\end{aligned}
$$

i.e. $Q(\alpha,\alpha^*)$ is proportional to the diagonal element of the density operator in the coherent state representation,

- unlike P-representation, $Q(\alpha,\alpha^*)$ isis non-negative definite and bounded, i.e.

$$Q(\alpha,\alpha^*) = \frac{1}{\pi}\sum_\psi P_\psi|\langle\psi|\alpha\rangle|^2,$$

since $|\langle\psi|\alpha\rangle|^2 \leq 1$, we have

$$Q(\alpha,\alpha^*) \leq \frac{1}{\pi},$$

Q-representation, normally ordering

- Q-representation may be related to the P-representation as,

$$Q(\alpha, \alpha^*) = \frac{1}{\pi} \int \mathrm{d}^2\beta P(\beta, \beta^*) e^{-|\alpha-\beta|^2},$$

- for a number state $|n\rangle$, its Q-representation is,

$$Q(\alpha, \alpha^*) = \frac{1}{\pi} |\langle n|\alpha\rangle|^2 = \frac{e^{-|\alpha|^2} |\alpha|^{2n}}{\pi n!},$$

- for a squeezed state $|\beta, \xi\rangle$, its Q-representation is,

$$\begin{aligned} Q(\alpha, \alpha^*) &= \frac{1}{\pi} |\langle \alpha|\beta, \xi\rangle|^2, \\ &= \frac{\mathrm{sech} r}{\pi} \exp\{-(|\alpha|^2 + |\beta|^2) + (\alpha^*\beta + \beta^*\alpha)\mathrm{sech} r \\ &- \frac{1}{2}[e^{i\theta}(\alpha^{*2} - \beta^{*2} + e^{-i\theta}(\alpha^2 - \beta^2)]\tanh r\}, \end{aligned}$$

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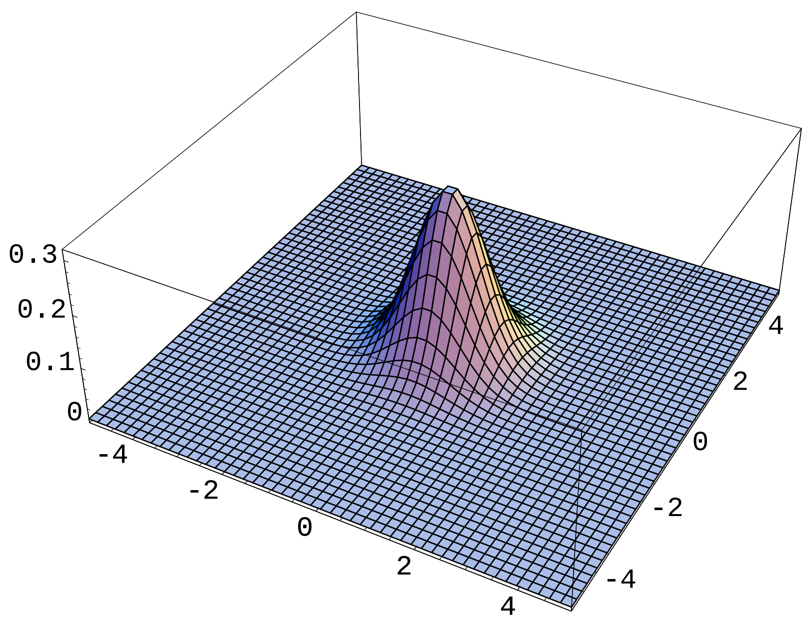
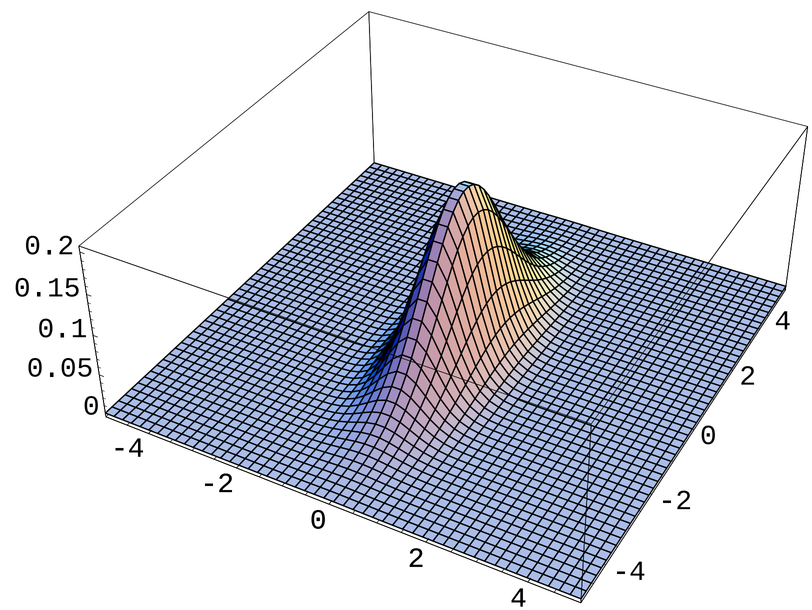
Q-representation

In the quarature phase-space, $X_1 = (\alpha + \alpha^*)/2$ and $X_1 = (\alpha - \alpha^*)/2i$,

$$\begin{aligned} Q(\alpha, \alpha^*) &= \frac{\text{sech}r}{\pi} \exp\{-(|\alpha|^2 + |\beta|^2) + (\alpha^* \beta + \beta^* \alpha)\text{sech}r \\ &- \frac{1}{2}[e^{i\theta}(\alpha^{*2} - \beta^{*2} + e^{-i\theta}(\alpha^2 - \beta^2)]\tanh r\}, \end{aligned}$$





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coherent state (vaccum), squeezd vaccum ($r = 1.0$).

W-representation, symmetric ordering

- the quantum analog of the classical phase space distribution function in the s-ordering is,

$$f^{(s)}(\alpha,\alpha^*) = \frac{1}{\pi^2}\int \mathrm{d}^2\xi \exp[i(\alpha\xi+\alpha^*\xi^*)]\exp(-\frac{s}{2}|\xi|^2)\mathrm{Tr}\{\exp[-i(\xi\hat{a}+\xi^*\hat{a}^\dagger)]\hat{\rho}\},$$

- for $s=-1$,

$$f^{(-1)}(\alpha,\alpha^*) = P(\alpha,\alpha^*) = \frac{1}{\pi^2}\int \mathrm{d}^2\xi \exp[i(\alpha\xi+\alpha^*\xi^*)]\mathrm{Tr}\{\exp(-i\xi^*\hat{a}^\dagger)\exp(-i\xi\hat{a})\hat{\rho}\},$$

- for $s=+1$,

$$f^{(+1)}(\alpha,\alpha^*) = Q(\alpha,\alpha^*) = \frac{1}{\pi^2}\int \mathrm{d}^2\xi \exp[i(\alpha\xi+\alpha^*\xi^*)]\mathrm{Tr}\{\exp(-i\xi\hat{a})\exp(-i\xi^*\hat{a}^\dagger)\hat{\rho}\},$$

- for $s=0$,

$$f^{(0)}(\alpha,\alpha^*) = W(\alpha,\alpha^*) = \frac{1}{\pi^2}\int \mathrm{d}^2\xi \exp[i(\alpha\xi+\alpha^*\xi^*)]\mathrm{Tr}\{\exp[-i(\xi\hat{a}+\xi^*\hat{a}^\dagger)]\hat{\rho}\},$$

W-representation, symmetric ordering

- for $s = 0$,

$$f^{(0)}(\alpha, \alpha^*) = W(\alpha, \alpha^*) = \frac{1}{\pi^2} \int \mathrm{d}^2\xi \exp[i(\alpha\xi + \alpha^*\xi^*)] \mathrm{Tr}\{\exp[-i(\xi\hat{a} + \xi^*\hat{a}^\dagger)]\hat{\rho}\},$$

- the Wigner-Weyl distibution function $W(\alpha, \alpha^*)$ is associated with symmetric ordering,
- for example

$$\frac{1}{2}\langle \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}\rangle = \int \mathrm{d}^2\alpha W(\alpha, \alpha^*)\alpha\alpha^a st,$$

- the Wigner function can be measured experimentally, including its negative values,

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Wigner function in terms of $\hat{q}$ and $\hat{p}$

- in terms of $\hat{q}$ and $\hat{p}$,

$$
\begin{aligned}
W(p,q) &= \frac{1}{(2\pi)^2}\int \mathrm{d}\sigma \int \mathrm{d}\tau \exp[i(\tau p+\sigma q)]\mathrm{Tr}\{\exp[-i(\tau\hat{p}+\sigma\hat{q})]\hat{\rho}\}, \\
&= \frac{1}{(2\pi)^2}\int \mathrm{d}\sigma \int \mathrm{d}\tau e^{[i(\tau p+\sigma q)]}\mathrm{Tr}\{e^{(-i\tau\hat{p}/2)}e^{(-i\sigma\hat{q})}\hat{\rho}e^{(-i\tau\hat{p}/2)}\}e^{(-i\sigma\tau/2)}, \\
&= \frac{1}{(2\pi)^2}\int \mathrm{d}\sigma \int \mathrm{d}\tau e^{[i(\tau p+\sigma q)]}\int \mathrm{d}q'\langle q'|e^{(-i\tau\hat{p}/2)}e^{(-i\sigma\hat{q})}\hat{\rho}e^{(-i\tau\hat{p}/2)}|q'\rangle e^{(-
\end{aligned}
$$

since

$$\exp(-i\tau\hat{p})|q'\rangle = |q'-\hbar\tau/2\rangle,$$

we have

$$
\begin{aligned}
W(p,q) &= \frac{1}{(2\pi)^2}\int \mathrm{d}\sigma \int \mathrm{d}\tau e^{i\sigma(q-q')}\int \mathrm{d}q'\langle q'+\hbar\tau/2|\hat{\rho}|q'-\hbar/tau/2\rangle e^{i\tau p}, \\
&= \frac{1}{\pi\hbar}\int \mathrm{d}y e^{-2yp/\hbar)}\langle q'-y|\hat{\rho}|q'+y\rangle,
\end{aligned}
$$

where $y = -\hbar\tau/2$

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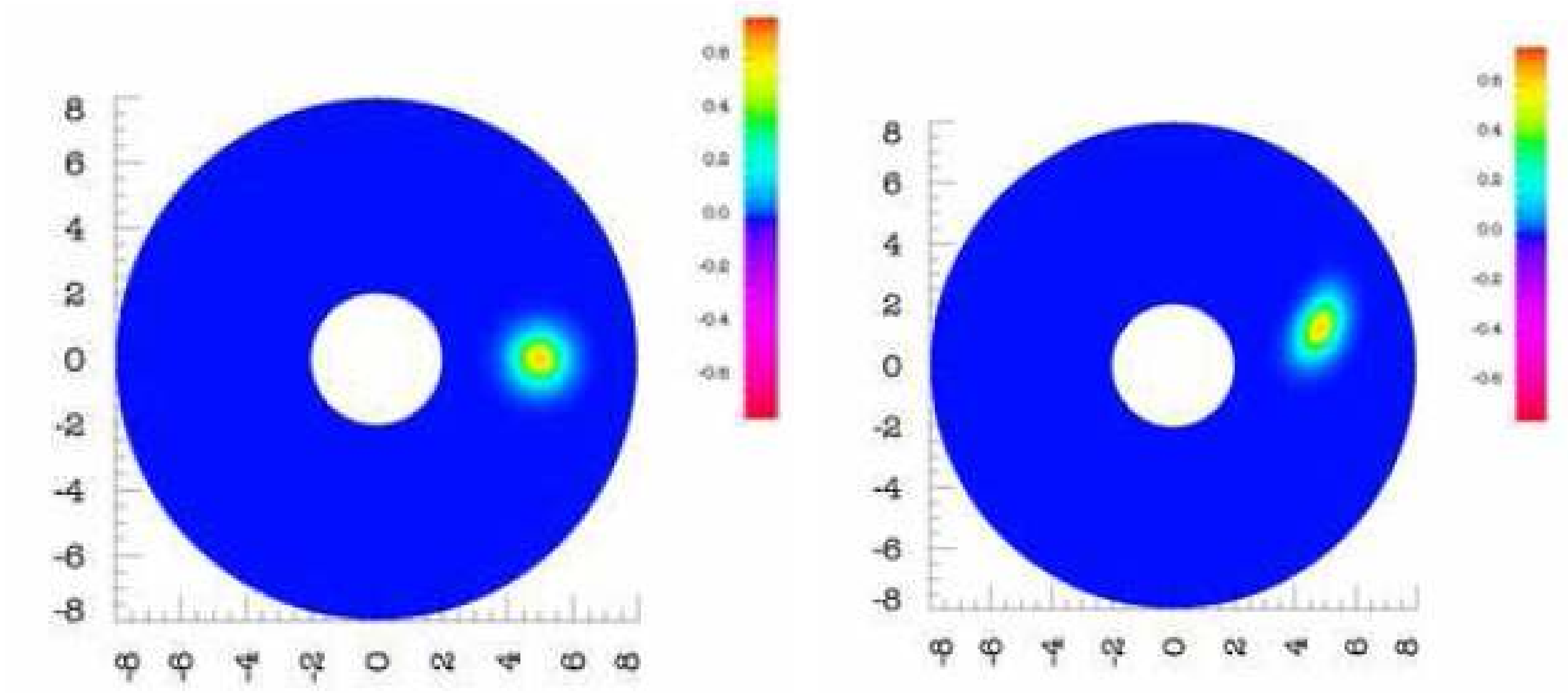
Wigner function for a Kerr state



M. Stobinska *et al.*, quant-ph/0605166