One-to-Many Negotiation Between a Seller and Asymmetric Buyers

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September 3, 2009

Abstract

A seller with an indivisible object negotiates with two asymmetric buyers to determine who gets the object and at what price. The seller repeatedly submits take-it-or-leave-it offers to the two buyers until one of them accepts. Unlike a Dutch auction, the seller has the discretion to offer two different prices to the two buyers. We show that when committing to some price paths is possible, the optimal outcome for the seller stated by Myerson (1981) is achievable. When commitment is impossible, the optimal outcome is no longer attainable. Instead, there exists an equilibrium in which the seller's equilibrium payoff is the same as that in a second-price auction, which implies that the seller's payoff might be lower than in a Dutch auction. The result thus illustrates the value of a simple institution like a Dutch auction, which seems to restrict a player's freedom but actually benefits the player by providing a commitment tool. Our analysis also sheds light on the procurement literature and gives insights into the performance of atypical auctions conducted at Priceline.com.

1 Introduction

The Dutch auction is considered strategically equivalent to the first-price sealed-bid auction. In a Dutch auction, the auctioneer, who is usually the seller, plays a passive role and nonstrategically lowers the price until the object is sold. However, if the seller is allowed to submit different prices to different buyers at the

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same time, will the seller be able to attain a higher profit than when such discretion is not permitted?

In this paper, we consider a two-buyer auction where the buyers are ex ante asymmetric: the buyers' private values of the object are drawn independently from different distributions. When buyers are asymmetric, the seller's payoffs in a first-price and a second-price auction are not optimal. An optimal auction stated by Myerson (1981) requires the seller to treat buyers asymmetrically,[1] that is, the seller might not sell the object to the buyer with the highest value. Therefore, one may imagine that if the seller does not have to submit the same price to the buyers as in a Dutch auction, the seller might be able to do better. We show that if the seller is allowed to submit different prices to different buyers, when commitment to some price paths is possible, Myerson's optimal outcome for the seller is achievable; however, when commitment is impossible, the optimal outcome is no longer attainable, but instead, there exists an equilibrium such that the seller's equilibrium payoff is the same as that in a second-price auction. As the auction literature (Vickrey (1961) and Maskin and Riley (2000)) shows, with asymmetric bidders, the seller's payoff in a first-price auction (or a Dutch auction) might be greater than that in a second-price auction. Therefore, when allowed to submit different prices, the seller's payoff might be lower than the payoff when he must submit the same price. The result thus suggests that the benefit brought by the discretion to determine the price paths might be outweighed by the loss caused by not being able to commit.

An example of the trading process we consider is the Name-Your-Own-Price (NYOP) auctions conducted at Priceline.com. An NYOP auction is a procurement auction (also called reverse auction) where the roles of a buyer and a seller are reversed from an ordinary auction. In an NYOP auction, when a customer tries to get a travel-related item such as an air ticket or a hotel room, he can submit different prices to different groups of sellers. For instance, when bidding for a hotel room, a buyer can specify the area to stay and the rating of hotels, and submit different bids to hotels with different characteristics. The customer adjusts his bids over time until one of the sellers accepts. The analysis in this paper

[1] In a second-price auction, buyers bid their true value, so it is always the case that the buyer with the highest value gets the good, and hence it is obviously not optimal. In a first-price auction, a buyer whose value distribution is first-order stochastically dominated will bid more aggressively, but still, the result is not optimal for the seller.

helps us get a better understanding of the buyer's payoff and bidding behavior under the NYOP mechanism.

This paper is also related to the procurement literature that compares the performances of two different trading institutions—auctions versus negotiations. Manelli and Vincent (1995) model negotiations as sequential offers and find that under certain conditions, negotiations outperform auctions from the buyer's perspective. The sequential-offer process they consider for negotiation is such that the order of sellers that the buyer gets to negotiate and the prices offered are determined in advance, and the buyer only gets at most one chance to negotiate with each seller. In this paper, we generalize the negotiation process, considering the case when the number of chances to negotiate is not limited. We show that if the seller can commit to certain price paths in advance, the optimal outcome stated by Myerson (1981) can always be achieved, so negotiations always perform better. We further consider the case when the price paths are determined as the process goes along, which is closer to the real-world negotiation process. In this case, whether negotiations or auctions do better depends on the environment and is not conclusive.

This paper builds a bridge between the auction and the bargaining literature. The model differs from a Dutch auction environment by allowing the auctioneer to set different prices for different buyers, and differs from a bargaining setting by considering a one-to-many negotiation process in which an individual party chooses his partner from a group of candidates.[2]

The paper is organized as follows. Section 2 presents an example illustrating the main points of this paper. Section 3 describes the model. Section 4 characterizes the optimal outcome that the seller can achieve when commitment is possible. Section 5 characterizes the equilibrium without commitment, and Section 6 concludes the paper.

2 An Example

In this section, we use a simple example to illustrate some of the main points of this paper. Consider the case when a seller tries to sell a single object to

[2] The existing one-to-many negotiation literature considers the case where the individual party has to work with every member in the other party instead of choosing one partner from all of them.

two buyers. The buyers have private values for the object. Buyer 1's value is drawn from the set $\{0, 3\}$ with respective probabilities $\left\{\frac{1}{2}, \frac{1}{2}\right\}$; buyer 2's value is drawn from the set $\{1, 3.5, 6\}$ with respective probabilities $\left\{\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right\}$. The seller maximizes the expected amount of money he collects from the buyers. A buyer's payoff if he gets the object is the difference between his value and the payment. The seller makes offers to the two buyers until one of them accepts. If the two buyers accept at the same time, each of them gets the object with probability $\frac{1}{2}$. For simplicity, we assume that the prices offered must be integers. We consider three different sales mechanisms: (i) the seller offers the same price sequence $\{5, 4, 3, 2, 1\}$ to the two buyers, (ii) the seller determines two price paths in advance and makes offers according to the paths, and (iii) the seller makes offers without making commitment in advance.

2.1 Offer price sequence $\{5, 4, 3, 2, 1\}$

In this case, the mechanism works like a Dutch auction. The seller lowers the price gradually until a buyer accepts. Given the price path, the following table summarizes the prices accepted by different types of buyers: conditional on that the object is not sold in earlier periods, buyer 2 with value 6 accepts \$3,[3] buyer 1 with value 3 accepts \$2, and buyer 2 with value 1 and 3.5 accepts \$1. The seller's expected payoff is $\frac{9}{4}$.

	Price pairs offered to the two buyers				
	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
Buyer 1	5	4	3	2	1
Buyer 2	5	4	3	2	1
accepted by			B2 with 6	B1 with 3	B2 with 1, 3.5

Note that buyer 1 is stronger than buyer 2 in the sense that the distribution function of buyer 1's value, F_1, stochastically dominates the distribution function of buyer 2's value, F_2. That is, $F_1(x) \leq F_2(x)$. We can observe from the table

[3] To see why buyer 2 with value 6 accepts at \$3, note that if he waits and accepts \$2, with probability $\frac{1}{2}$, buyer 1 also accepts, and each of them gets the object with probability $\frac{1}{2}$, so the probability that he gets the object is $\frac{3}{4}$. Since the buyer's expected payoff if accepting at \$2 is $\frac{3}{4}(6-2) = 3$, the same as his payoff if he accepts at \$3, he is willing to accept at \$3. Similarly, buyers with other values will find that accepting at the price specified in the table is the optimal strategy.

that buyer 1 with value 3 accepts at a higher price than buyer 2 with value 3.5. This is analogous to the well-known fact that in a first-price auction (which is strategically equivalent to a Dutch auction), the weaker bidder bids more aggressively than the strong bidder.

2.2 Commit to certain price paths determined in advance

Now we allow the seller to determine the price paths in advance, and the prices offered to the two buyers can be different. Suppose the seller commits to offer the following price sequences:

	Price pairs offered to the two buyers			
	$t = 1$	$t = 2$	$t = 3$	$t = 4$
Buyer 1	2	1	1	0
Buyer 2	4	4	3	3
accepted by	B2 with 6	B1 with 3	B2 with 3.5	B1 with 0

The last row shows that in equilibrium, conditional on that the object is not sold in earlier periods, at $t = 1$, buyer 2 with value 6 accepts \$4, at $t = 2$, buyer 1 with value 3 accepts \$1, at $t = 3$, buyer 2 with value 3.5 accepts \$3, and at $t = 4$, buyer 1 with value 0 accepts \$0. The seller's expected payoff is $\frac{21}{8}$, greater than the payoff in the first mechanism.

Notice that given the paths, buyer 2 with value 1 never gets the object. Therefore, by committing to the price paths, the seller loses the chance to sell the object to a buyer with positive value. Furthermore, in each period, the price offered to buyer 1, the weaker buyer, is lower than the price offered to buyer 2. This design raises competition between the buyers, so the seller is able to extract more surplus from buyer 2.

2.3 Make offers without commitment

If the seller makes offers without committing to certain paths in advance, the price paths characterized in the first two mechanisms cannot be implemented. To see this, first consider the paths implemented in the second mechanism. In the last period $t = 4$, given that the two buyers have rejected all the offers, the seller believes that buyer 1 has value 0 and buyer 2 has value 1. Given the belief, the seller can offer \$1 to buyer 2 and a price higher than 0 to buyer 1, and get a higher payoff than if he follows the paths in the second mechanism at $t = 4$. Therefore, the paths in the second mechanism cannot be implemented without commitment.

Next consider the paths implemented in the first mechanism. After $t = 3$, if no buyer accepts, the seller believes that buyer 1's value is either 0 or 3, and buyer 2's value is either 1 or 3.5. Given the belief, the optimal paths for the seller in the continuation game are the following:

	Price pairs offered to the two buyers		
	$t = 4$	$t = 5$	$t = 6$
Buyer 1	3	2	2
Buyer 2	2	2	1
accepted by	B2 with 3.5	B1 with 3	B2 with 1

With the paths, buyer 2 with value 3.5 accepts \$2 at $t = 4$ and buyer 1 with value 3 accepts \$2 at $t = 5$, and buyer 1 with value 1 accepts \$1 at $t = 6$, so the seller gets a higher payoff than if he follows the paths in the first mechanism. Given the optimal paths in the continuation game, in equilibrium, buyer 2 with value 6 would not accept a price higher than 2. If in equilibrium, the seller believes that buyer 2 with value 6 accepts \$3, then after offering buyer 2 \$3 and getting rejected, in the next period, he follows the optimal paths specified in the table and offers the price pair $(3, 2)$. Given the pair $(3, 2)$, buyer 1 will not accept, so buyer 2 with value 6 gets a higher payoff if he rejects price \$3 and waits for one more period to accept \$2. The following table characterizes the equilibrium price paths and the types of buyers accepting in each period in the third mechanism. The seller's expected payoff is $\frac{15}{8}$.

	Price pairs offered to the two buyers		
	$t = 1$	$t = 2$	$t = 3$
Buyer 1	3	2	2
Buyer 2	2	2	1
accepted by	B2 with 3.5, 6	B1 with 3	B2 with 1

2.4 Comparison

From the above discussion, we can compare the seller's expected payoffs and the equilibrium allocations in the three mechanisms.

1. The seller can do better in the second mechanism than in the first mechanism. In both mechanisms, the price paths are determined in advance, but in the second mechanism, the seller has the freedom to offer different prices to different buyers, and hence, he can get a higher payoff. However, the fact that the seller's payoff in the third mechanism is less than the payoff in the first mechanism shows that if the seller cannot commit to certain price paths in advance, even with the freedom to offer different prices to different buyers, the seller might do worse.

2. In the first and the second mechanisms, the final allocation might be inefficient: if buyer 1 and buyer 2 have values 3 and 3.5 respectively, buyer 1 gets the object although his value is lower than buyer 2's. On the other hand, in the third mechanism where the seller cannot commit, the seller tends to induce the buyer with the higher value to buy first in every period. Therefore, the allocation is efficient — the buyer with the higher value always gets the object.

In the remaining sections, we analyze a model in which buyers' values are drawn from continuous distribution functions and the seller adjusts prices continuously. We show that the conclusions made above for the simple example hold in the general model.

3 The Model

A seller has one indivisible object for sale and faces two risk-neutral buyers. The value of the object to the seller is 0, which is publicly known. Buyers have private values for the object, and the values are independently distributed. Buyer i's value X^i is distributed over the interval $[\underline{w}_i, \overline{w}_i], \underline{w}_i \geq 0$, according to distribution function F_i with associated density function f_i. For $i = 1, 2$, $\psi_i(x) \equiv x - \frac{1-F_i(x)}{f_i(x)}$ strictly increases in x and $f_i(x) > 0, \forall x \in [\underline{w}_i, \overline{w}_i]$. The seller maximizes the expected amount of money he collects from the buyers. Buyer i's utility is $z_i x_i -$

M, where M is his payment to the seller, and $z_i = 1$ if buyer i gets the object, $z_i = 0$ otherwise. The setting is common knowledge to everyone in the market.

The seller is allowed to conduct a negotiation process. From the beginning, the seller makes simultaneous offers to the two buyers and adjusts the prices continuously until some buyer indicates his interest. Without loss of generality, we assume that the prices offered have to be weakly decreasing. The two simultaneous offers to the two buyers can be different and are observable by both buyers. Once a buyer accepts, the object is sold to the buyer at the price offered. There is no discounting.

4 Optimal Outcome with Commitment

In this section, we consider the case when the seller can commit to some price paths before the negotiation begins. Recall that when deriving the optimal mechanism, we compare the virtual valuation of buyers. The virtual valuation of a buyer with value x_i is defined as

$$\psi_i(x_i) \equiv x_i - \frac{1 - F_i(x_i)}{f_i(x_i)},$$

which is assumed strictly increasing in our model. The optimal allocation rule characterized by Myerson (1981) is that the good goes to the buyers whose virtual valuations are the greatest and positive. Therefore, the seller should design paths such that, in the following negotiation process, a buyer whose virtual valuation is higher will accept earlier.

Without loss of generality, assume that $\psi_1(\underline{w}_1) \geq \psi_2(\underline{w}_2)$, where $\underline{w}_i$ is the lowest possible value of buyer i. If $\psi_1(\underline{w}_1) < 0$, let $\underline{x}_1 = \psi_1^{-1}(0)$; otherwise, $\underline{x}_1 = \underline{w}_1$. Let $\widehat{x}_2(x_1) = \psi_2^{-1}(\psi_1(x_1))$ so that $\psi_1(x_1) = \psi_2(\widehat{x}_2(x_1))$, and let $\widehat{x}_1(x) \equiv \widehat{x}_2^{-1}(x)$. Furthermore, let

$$b_1(x_1) = x_1 - \frac{\int_{\underline{x}_1}^{x_1} F_2(\widehat{x}_2(x))\, dx}{F_2(\widehat{x}_2(x_1))},$$

and

$$b_2(x_2) = x_2 - \frac{\int_{\widehat{x}_2(\underline{x}_1)}^{x_2} F_1(\widehat{x}_1(x))\, dx}{F_1(\widehat{x}_1(x_2))}.$$

Note that both $b_1(x)$ and $b_2(x)$ are strictly increasing.

Theorem 1 *If the seller commits to a path on which the price to buyer* 1*,* p_1*, and the price to buyer* 2*,* p_2*, have the relation* $p_2(p_1) = b_2\left(\widehat{x}_2\left(b_1^{-1}(p_1)\right)\right)$ *and stop at* $\underline{x}_1$ *and* $\widehat{x}_2(\underline{x}_1)$ *respectively, then there exists an equilibrium such that buyer* 1 *with value* x_1 *accepts at price* $b_1(x_1)$ *and buyer* 2 *with value* x_2 *accepts at price* $b_2(x_2)$*. The seller's payoff is the same as that in an optimal mechanism.*

Proof. Given that buyer 2 with value $x_2 > \widehat{x}_2(\underline{x}_1)$ accepts at $b_2(x_2)$, we show that accepting at $b_1(x_1)$ is the best strategy for buyer 1 with value x_1. Suppose buyer 1 has value x_1. If the current price for buyer 1 is $p_1 > b_1(x_1)$, accepting now gives buyer 1 payoff

$$x_1 - p_1 = x_1 - b_1\left(x_1'\right), \text{ where } x_1' = b_1^{-1}(p_1) > x_1.$$

Accepting later at $b_1(x_1)$ gives expected payoff

$$\frac{F_2(\widehat{x}_2(x_1))}{F_2(\widehat{x}_2(x_1'))}(x_1 - b_1(x_1)).$$

$$\begin{aligned} F_2\left(\widehat{x}_2\left(x_1'\right)\right)\left(x_1 - b_1\left(x_1'\right)\right) &= \left(x_1 - x_1'\right)F_2\left(\widehat{x}_2\left(x_1'\right)\right) + \int_{\underline{x}_1}^{x_1'} F_2(\widehat{x}_2(x))\,dx \\ &< \int_{\underline{x}_1}^{x_1} F_2(\widehat{x}_2(x))\,dx = F_2(\widehat{x}_2(x_1))(x_1 - b_1(x_1)). \end{aligned}$$

Therefore waiting and accepting later is better for buyer 1. If the current price is $b_1(x_1)$, accepting now gives payoff $x_1 - b_1(x_1)$. Accepting later at $b_1\left(x_1''\right), x_1'' < x_1$, gives expected payoff $\frac{F_2\left(\widehat{x}_2\left(x_1''\right)\right)}{F_2(\widehat{x}_2(x_1))}(x_1 - b_1(x_1))$.

$$\begin{aligned} F_2\left(\widehat{x}_2\left(x_1''\right)\right)\left(x_1 - b_1\left(x_1''\right)\right) &= \left(x_1 - x_1''\right)F_2\left(\widehat{x}_2\left(x_1''\right)\right) + \int_{\underline{x}_1}^{x_1''} F_2(\widehat{x}_2(x))\,dx \\ &< \int_{\underline{x}_1}^{x_1} F_2(\widehat{x}_2(x))\,dx = F_2(\widehat{x}_2(x_1))(x_1 - b_1(x_1)). \end{aligned}$$

Therefore, accepting now at $b_1(x_1)$ is better. The same argument applies to buyer 2. Therefore, that buyer 1 with value x_1 accepts at price $b_1(x_1)$ and buyer 2 with value x_2 accepts at price $b_2(x_2)$ is an equilibrium.

To see that the seller gets the optimal payoff, notice that buyer i gets the object if and only if his virtual valuation is positive and at least as high as buyer j, i.e. $\psi_i(x_i) \geq \psi_j(x_j)$, so the allocation is the same as the optimal allocation rule. In addition, buyer 1 with value $\underline{x}_1$ and buyer 2 with value $\widehat{x}_2(\underline{x}_1)$ get zero utility. Thus, by the revenue equivalence principle, the seller gets the optimal payoff. ■

Manelli and Vincent (1995) compare the performance of negotiations and auctions in procurement and conclude that under certain conditions, negotiations are outperformed by auctions from the buyer's perspective (seller in our model). The sequential-offer process they consider for negotiation requires that the order of sellers (buyers in our model) the buyer gets to negotiate and the prices offered are both determined in advance, and the buyer only gets at most one chance to negotiate with each seller. However, we show that if the number of chances to negotiate is not limited and the price paths are determined in advance, Myerson's optimal outcome can always be achieved, so negotiations always outperform auctions. Therefore, Manelli and Vincent's result is contingent on the limited number of chances to negotiate.

5 Equilibrium without Commitment

In this section, we consider the case when the seller is not able to commit to any price paths. First we describe the equilibrium concept. Next we restrict our attention to a particular class of equilibria and show that without commitment, Myerson's optimal outcome, which can be attained when commitment is possible, is no longer achievable.

5.1 Equilibrium Concept

Let $p_i(\tau)$ be the price submitted by the seller to buyer i at time τ. At time t, $t \geq 0$, denote by $h^t = \{(p_1(\tau), p_2(\tau)); 0 \leq \tau \leq t\}$ the price history submitted by the seller to the two buyers until time t. Without loss of generality, let the prices offered to the two buyers start at $\overline{w}_1$ and $\overline{w}_2$, that is, $p_1(0) = \overline{w}_1$ and $p_2(0) = \overline{w}_2$. Then at time t, given history h^t and the fact that no buyer accepts, the seller determines the prices for the next instant. Therefore, the seller's strategy can be characterized by $\left(\frac{dp_1(t;h^t)}{dt}, \frac{dp_2(t;h^t)}{dt}\right)$, which determines the increments of p_1 and p_2. Because the prices are weakly decreasing, $\frac{dp_1(t;h^t)}{dt} \leq 0$ and $\frac{dp_2(t;h^t)}{dt} \leq 0$. Since there is no discounting, speeding up or slowing down the price paths should not affect the players' payoffs. What really matters is how the two prices are paired along the path. Therefore, we restrict the seller's strategy space to $\mathcal{S} = \left\{\left(\frac{dp_1(t;h^t)}{dt}, \frac{dp_2(t;h^t)}{dt}\right) \mid \frac{dp_1}{dt} + \frac{dp_2}{dt} = -1, \frac{dp_1}{dt} \leq 0, \text{ and } \frac{dp_2}{dt} \leq 0\right\}$, and denote the seller's strategy by $\mu(t;h^t) = \left(\frac{dp_1(t;h^t)}{dt}, \frac{dp_2(t;h^t)}{dt}\right)$.

At time t, given the current price (p_1, p_2) offered by the seller and the price history until t, h^t, buyer i determines whether to accept p_i. We use $P_1(x_1;h^t)$ and $P_2(x_2;h^t)$ to characterize buyer 1's and buyer 2's strategies: buyer i with value x_i accepts the current price $p_i(t)$ if and only if $p_i(t) \leq P_i(x_i;h^t)$. Given history h^t and an equilibrium in which $P_i(x_i;h^t)$ is monotone in x_i, the seller's and buyer j's belief about buyer i's value is summarized by function $y_i(h^t)$, which specifies the lowest upper bound of buyer i's value believed by the seller and buyer j given history h^t and the fact that no buyer accepts.

Denote by $v_0(\mu, P_1, P_2 \mid h^t, y_1(h^t), y_2(h^t))$ the seller's expected utility given h^t and belief y, and $v_i(\mu, P_1, P_2 \mid h^t, \theta_i, y_j(h^t))$ buyer i's expected utility, given h^t, the realization θ_i of buyer i's value, and belief $y_j(h^t), j \neq i$.

Definition 1 *A pure strategy perfect Bayesian equilibrium is a* $(\mu, y_1, y_2, P_1, P_2)$ *that satisfies*

(a) $y_i(h^t) = \inf\{x \mid P_i(x;h^\tau) \geq p_i(\tau), \textit{ for any } \tau \in [0,t]\}, \forall t, h^t.$

(b) $v_0(\mu, P_1, P_2 \mid h^t, y_1(h^t), y_2(h^t)) \geq v_0(\mu', P_1, P_2 \mid h^t, y_1(h^t), y_2(h^t))$ *and* $v_i(\mu, P_i, P_j \mid h^t, \theta_i, y_j(h^t)) \geq v_i(\mu, P_i', P_j \mid h^t, \theta_i, y_j(h^t))$, *for* $i = 1, 2$ *and* $\forall t, \theta_i, \mu', P_i', h^t.$

Condition (a) implies that players' belief about the lowest upper bound of buyer i's value at time t is the same as the infimum of buyer i's values with which buyer i would have accepted a price occurring on the historical price path. Condition (b) implies that players cannot do better by deviating from the equilibrium strategy.

5.2 The class of equilibria considered

We focus on pure strategy perfect Bayesian equilibria with the property that the seller's strategy depends on history only through his belief about the buyers' values, $y_1(h^t)$ and $y_2(h^t)$, and the current prices, $p_1(t)$ and $p_2(t)$; and buyer i's strategy depends on history only through his belief about buyer j's value, $y_j(h^t)$, if his value x_i is less than or equal to $\lim_{\tau \uparrow t} y_i(h^\tau)$, the lowest upper bound of his value believed by other players given that he did not accept any price before t. Note that if $x_i > \lim_{\tau \uparrow t} y_i(h^\tau)$, this implies that buyer i has deviated from the equilibrium strategy — he should have accepted some price occurring on the history path. Therefore, when there is no deviation, $P_i(x_i; h^t)$ can be expressed as $P_i(x_i, x_j)$ where $x_j = y_j(h^t)$, and $\mu(t; h^t) = \left(\frac{dp_1(t;h^t)}{dt}, \frac{dp_2(t;h^t)}{dt}\right)$ can be expressed as $\mu(t; p_1, p_2, x_1, x_2)$ where $p_i = p_i(t)$ and $x_i = y_i(h^t)$. We restrict our attention to the equilibria in which $\frac{\partial P_i(x_i,x_j)}{\partial x_i} \geq 0$, $\frac{\partial P_i(x_i,x_j)}{\partial x_j} \geq 0$, and $\frac{\frac{\partial P_1}{\partial x_1}}{\frac{\partial P_1}{\partial x_2}} \neq \frac{\frac{\partial P_2}{\partial x_1}}{\frac{\partial P_2}{\partial x_2}}$. $\frac{\partial P_i(x_i,x_j)}{\partial x_i} \geq 0$ is natural because given the other conditions the same, a buyer with higher value should be willing to accept a higher price.

Condition 1 *We consider equilibria with the property that* $\frac{\partial P_i(x_i,x_j)}{\partial x_i} \geq 0$, $\frac{\partial P_i(x_i,x_j)}{\partial x_j} \geq 0$, *and* $\frac{\frac{\partial P_1}{\partial x_1}}{\frac{\partial P_1}{\partial x_2}} \neq \frac{\frac{\partial P_2}{\partial x_1}}{\frac{\partial P_2}{\partial x_2}}$ *for all* $x_i \in [\underline{w}_i, \overline{w}_i], x_j \in [\underline{w}_j, \overline{w}_j]$.

5.3 Seller's strategy

In this section, we characterize the seller's equilibrium strategies. First consider a continuation game at time t after the buyers have rejected prices $p_1(t)$ and $p_2(t)$. The seller then determines the prices for the next instant. Let the beliefs about the lowest upper bounds of the buyers' values be x_1 and x_2. The first condition in Definition 1 implies that $p_1(t) \geq P_1(x_1, x_2)$ and $p_2(t) \geq P_2(x_1, x_2)$.

5.3.1 Equilibrium strategy when $p_i(t) > P_i(x_1, x_2)$ for some i

In this section, we first characterize the set of price paths that result in the same expected payoff of the seller in Lemma 1 and Lemma 2. Then by the two lemmas, we conclude the seller's strategy when $p_i(t) > P_i(x_1, x_2)$ for some i in Proposition 1.

Lemma 1 *Given belief* (x_1, x_2)*, for any price path* $\{(p_1(\tau), p_2(\tau)) \mid t \leq \tau \leq t'\}$ *such that* $p_1(\tau) \geq P_1(x_1, x_2)$ *and* $p_2(\tau) \geq P_2(x_1, x_2)$ *for all* $\tau \in [t, t']$*, the seller's expected payoff received along the path is* 0.

Proof. Given the belief that buyer 1's and buyer 2's values are below x_1 and x_2 respectively, any price that is higher than or equal to $P_i(x_1, x_2)$ will not be accepted by buyer i with a value below x_i. Therefore, the seller's expected payoff along the path is 0. ■

Lemma 2 *Given belief* (x_1, x_2)*, consider two price paths*

1. $\{(p_1(\tau), p_2(\tau)) \mid t \leq \tau \leq t'\}$ *such that for all* $\tau \in [t, t']$*,* $p_i(\tau) = P_i(x, x_j)$ *and* $p_j(\tau) \geq P_j(x, x_j)$ *for some* $x \leq x_i$*;*

2. $\{(\bar{p}_1(\tau), \bar{p}_2(\tau)) \mid t \leq \tau \leq t''\}$ *such that for all* $\tau \in [t, t'']$*,* $\bar{p}_i(\tau) = P_i(x, x_j)$ *and* $\bar{p}_j(\tau) = P_j(x, x_j)$ *for some* $x \leq x_i$*,*

such that $p_i(t) = \bar{p}_i(t)$ *and* $p_i(t') = \bar{p}_i(t'')$*. The seller's expected payoffs received along the two price paths are the same.*

Proof. Given the belief that the lowest upper bounds of the buyers' values are x_1 and x_2 at t, along the first and the second paths, $p_i(\tau)$ is accepted by buyer i with value x such that $p_i(\tau) = P_i(x, x_j)$, $\bar{p}_i(\tau)$ is accepted by buyer i with value y such that $\bar{p}_i(\tau) = P_i(y, x_j)$, and $p_j(\tau)$ and $\bar{p}_j(\tau)$ are not accepted by buyer j with a value lower than x_j. Therefore, the seller receives no expected payoff from buyer j along the two paths, and the belief about the lowest upper bound of buyer j's value stays at x_j. Given that the belief about the lowest upper bound of buyer j's value stays at x_j on both paths, the prices accepted by buyer i with some value x are the same on both paths. Since both paths start at $p_i(t)$ and end at $p_i(t')$, the expected payoffs received from buyer i are the same along the two paths. ■

Proposition 1 *At* t*, given belief* $(x_1, x_2) = \left(y_1\left(h^t\right), y_2\left(h^t\right)\right)$ *and the current prices* $(p_1(t), p_2(t))$*, if* $p_i(t) > P_i(x_1, x_2)$ *and* $p_j(t) = P_j(x_1, x_2)$*, the seller's strategy is* $\frac{dp_i}{dt} = 1$ *and* $\frac{dp_j}{dt} = 0$*; if* $p_i(t) > P_i(x_1, x_2)$ *for both* $i = 1, 2$*, then any* $\left(\frac{dp_1}{dt}, \frac{dp_2}{dt}\right) \in \mathcal{S}$ *can be the seller's strategy.*

Proof. Lemmas 1 and 2 imply that given belief (x_1, x_2), for any path $\mathcal{P}$ starting with prices (p_1, p_2) such that $p_1 \geq P_1(x_1, x_2)$ and $p_2 \geq P_2(x_1, x_2)$, there exists a path starting with prices $P_1(x_1,x_2)$ and $P_2(x_1,x_2)$ that yields the same seller's payoff as $\mathcal{P}$, that is, a path starting with prices $p_1 \geq P_1(x_1,x_2)$ and $p_2 \geq P_2(x_1,x_2)$ is weakly dominated by some path starting with prices $P_1(x_1,x_2)$ and $P_2(x_1,x_2)$. Therefore, at t, if $p_i(t) > P_i(x_1,x_2)$ and $p_j(t) = P_j(x_1,x_2)$, it does not hurt the seller to first lower p_i to $P_i(x_1,x_2)$; and if $p_i(t) > P_i(x_1,x_2)$ for both $i = 1,2$, it does not hurt the seller to first lower p_i and p_j to $P_i(x_1,x_2)$ and $P_j(x_1,x_2)$ respectively. (Note that all the paths that lower p_i and p_j to $P_i(x_1,x_2)$ and $P_j(x_1,x_2)$ yield the same seller's payoff, 0.) ■

5.3.2 Equilibrium strategy when $p_i(t) = P_i(x_1,x_2)$ for both i

Proposition 1 characterizes the seller's strategy when $p_i(t) > P_i(x_1,x_2)$ for some i. In this section, we consider the case when $p_1(t) = P_1(x_1,x_2)$ and $p_2(t) = P_2(x_1,x_2)$. We show that without loss of generality, the seller's strategy of choosing $\left(\frac{dp_1}{dt},\frac{dp_2}{dt}\right)$ is equivalent to choosing $\left(\frac{dx_1}{dt},\frac{dx_2}{dt}\right)$, where x_i is the infimum of buyer i's values with which buyer i would have accepted a price occurring on the history price path.

If the seller chooses strategy $\left(\frac{dp_1}{dt},\frac{dp_2}{dt}\right) \in \mathcal{S}$, we can derive how the beliefs x_1 and x_2 change accordingly by solving

$$\begin{cases} \frac{dp_1}{dt} = \frac{\partial P_1}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial P_1}{\partial x_2}\frac{dx_2}{dt} \\ \frac{dp_2}{dt} = \frac{\partial P_2}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial P_2}{\partial x_2}\frac{dx_2}{dt} \end{cases}, \tag{1}$$

where $\frac{dp_1}{dt} + \frac{dp_2}{dt} = -1$. Note that since x_i is the belief about the lowest upper bound of a buyer's value, x_i can never go up after the belief is updated, so $\frac{dx_1}{dt} \leq 0$ and $\frac{dx_2}{dt} \leq 0$. Because $\frac{\partial P_i(x_i,x_j)}{\partial x_i} \geq 0$, $\frac{\partial P_i(x_i,x_j)}{\partial x_j} \geq 0$, $\frac{dp_1}{dt} \leq 0$, and $\frac{dp_2}{dt} \leq 0$, given (x_1,x_2), there exist $a_{x_1x_2}, b_{x_1x_2} \in [-1,0]$ such that if $\frac{dp_1}{dt} \in [a_{x_1x_2}, b_{x_1x_2}]$ and $\frac{dp_2}{dt} = 1 - \frac{dp_1}{dt}$, then $\frac{dx_1}{dt} \leq 0$ and $\frac{dx_2}{dt} \leq 0$.[4] Hence, when $\frac{dp_1}{dt} \in [a_{x_1x_2}, b_{x_1x_2}]$, choosing $\left(\frac{dp_1}{dt},\frac{dp_2}{dt}\right)$ is equivalent to choosing $\left(\frac{dx_1}{dt},\frac{dx_2}{dt}\right)$ derived from (1). Moreover, Lemma 2 implies that without loss of generality, for each (x_1,x_2), we can focus on a smaller strategy space, $\hat{\mathcal{S}}_{x_1x_2} = \left\{\left(\frac{dp_1}{dt},\frac{dp_2}{dt}\right) \in \mathcal{S} \mid \frac{dp_1}{dt} \in [a_{x_1x_2}, b_{x_1x_2}], \frac{dp_2}{dt} = -1 - dp_1\right\}$, because for any path derived from strategies in $\mathcal{S}$, there exists a path derived from strategies in $\left\{\hat{\mathcal{S}}_{x_1x_2}\right\}_{x_1,x_2}$ that yields the same seller's expected payoff.

[4] $a_{x_1x_2} = \min\left\{\frac{\frac{\partial P_1}{\partial x_1}}{\frac{\partial P_1}{\partial x_1}+\frac{\partial P_2}{\partial x_1}}, \frac{\frac{\partial P_1}{\partial x_2}}{\frac{\partial P_1}{\partial x_2}+\frac{\partial P_2}{\partial x_2}}\right\}$ and $b_{x_1x_2} = \max\left\{\frac{\frac{\partial P_1}{\partial x_1}}{\frac{\partial P_1}{\partial x_1}+\frac{\partial P_2}{\partial x_1}}, \frac{\frac{\partial P_1}{\partial x_2}}{\frac{\partial P_1}{\partial x_2}+\frac{\partial P_2}{\partial x_2}}\right\}$.

Therefore, when $p_1(t) = P_1(x_1, x_2)$ and $p_2(t) = P_2(x_1, x_2)$, we can focus on the smaller strategy space $\hat{\mathcal{S}}_{x_1x_2}$ and redefine the seller's strategy as $u(t; x_1, x_2) = \left(\frac{dx_1(t;x_1,x_2)}{dt}, \frac{dx_2(t;x_1,x_2)}{dt}\right)$. As we discussed before, since there is no discounting, what really matters is how the two paths $x_1(t)$ and $x_2(t)$ are paired, not the exact values of $\frac{dx_1}{dt}$ and $\frac{dx_2}{dt}$. Therefore, we let $\frac{dx_1}{dt} = 1$, and the seller determines $u(x_1, x_2) \equiv \frac{dx_2(x_1;x_1,x_2)}{dx_1} = \frac{dx_2(t;x_1,x_2)}{dt} \in [0, \infty]$. Note that for each $\frac{dx_2(x_1;x_1,x_2)}{dx_1} \in [0, \infty]$, there is a corresponding strategy $\left(\frac{dp_1}{dt}, \frac{dp_2}{dt}\right) \in \hat{\mathcal{S}}_{x_1x_2}$. After the transformation, the seller's strategy $\frac{dx_2(x_1;x_1,x_2)}{dx_1}$ also depends on history only through the belief about the buyers' values, $x_1 = y_1(h^t)$ and $x_2 = y_2(h^t)$.

The seller's strategy profile is thus $\left\{u(x_1, x_2) \equiv \frac{dx_2(x_1;x_1,x_2)}{dx_1} \mid x_1 \in [\underline{w}_1, \overline{w}_1], x_2 \in [\underline{w}_2, \overline{w}_2]\right\}$. In a continuation game starting with belief (w_1, w_2), we can derive the equilibrium path $x_{2,w_1w_2}(x)$ for $x \in [\underline{w}_1, w_1]$ from the strategy profile $\{u(x_1, x_2)\}$, and that path maximizes the seller's expected payoff in the continuation game given the buyers' strategies $P_1(x_1, x_2)$ and $P_2(x_1, x_2)$. Therefore, for any (w_1, w_2), the seller's strategy $u(x_1, x_2)$ along the equilibrium path $x_{2,w_1w_2}(x)$ is such that

$$
\begin{aligned}
& u(x_1, x_2, _{w_1w_2}(x_1)) \qquad\qquad \text{(P1)} \\
\in\ & \arg\max_{u(x_1, x_{2,w_1w_2}(x_1))} \int_{\underline{w}_1}^{w_1} \left\{ \begin{array}{c} P_1(x_1, x_2) F_2(x_2) f_1(x_1) \\ +P_2(x_1, x_2) F_1(x_1) f_2(x_2) u \end{array} \right\} dx_1 \\
& +P_2(x_1, x_{2,w_1w_2}(x_1)) [F_2(w_2) - F_2(x_{2,w_1w_2}(x_1))] \\
s.t.\quad & x_2(w_1) \leq w_2, \\
& \frac{dx_2}{dx_1} = u, \\
& 0 \leq u < \infty.
\end{aligned}
$$

The first part of the integrand represents the seller's payoff increment from buyer 1 when he decreases x_1 by dx_1. The good is sold to buyer 1 at price $P_1(x_1, x_2)$ if buyer 2's value is below x_2 and buyer 1's value is x_1, which occurs with probability $F_2(x_2) f_1(x_1) dx_1$. The second part represents the payoff increment from buyer 2. When $x_{2,w_1w_2}(w_1) \neq w_2$, it implies $u(w_1, w_2) = \infty$ for $x \in (x_{2,w_1w_2}(w_1), w_2]$, and the second line is the seller's payoff from buyer 2 with values between $x_2(w_1)$ and w_2, who would accept $P_2(w_1, x_{2,w_1w_2}(w_1))$. To apply standard dynamic programming techniques, we allow the seller to have jumps on $x_{2,w_1w_2}(x_1)$ path only at the beginning when he chooses the initial value $x_2(w_1)$. With the restriction on the seller's strategy, we derive an equilibrium, and will show that even jumps are allowed along the path, the seller would not deviate.

By considering program (P1) for all (w_1, w_2) in $[\underline{w}_1, \overline{w}_1] \times [\underline{w}_2, \overline{w}_2]$, we derive the seller's strategy profile $\{u(x_1, x_2) \mid x_1 \in [\underline{w}_1, \overline{w}_1], x_2 \in [\underline{w}_2, \overline{w}_2]\}$. Note that the principal of optimality ensures that if $x_2(x_1)$ is the optimal path for a game starting with (w_1, w_2), then for a continuation game starting with $(x_1', x_2(x_1'))$ where $x_1' \in (\underline{w}_1, w_1)$, $x_2(x_1)$ for $x_1 \leq x_1'$ is also the optimal path. Therefore, $u(x_1, x_2)$ derived from (P1) with different (w_1, w_2) must be consistent. By Lemma 2 and the discussion in this section, we have the following proposition.

Proposition 2 *At t, given belief $(w_1, w_2) = \left(y_1\left(h^t\right), y_2\left(h^t\right)\right)$ and the current prices $(p_1(t), p_2(t))$, if $p_1(t) = P_1(w_1, w_2)$ and $p_2(t) = P_2(w_1, w_2)$, the seller's strategy is $\left(\frac{dp_1(t;b_1,b_2,w_1,w_2)}{dt}, \frac{dp_2(t;b_1,b_2,w_1,w_2)}{dt}\right) \in \mathcal{S}$ such that*

$$\frac{\frac{dp_2}{dt}}{\frac{dp_1}{dt}} = \left\{ \begin{array}{ll} \frac{dP_2\left(x_1, x_{2,w_1w_2}(x_1)\right)}{dx_1} / \frac{dP_1\left(x_1, x_{2,w_1w_2}(x_1)\right)}{dx_1} \,|_{x_1=w_1} & \text{if } x_{2,w_1w_2}(w_1) = w_2 \\ \frac{\partial P_2(x_1,x_2)}{\partial x_2} / \frac{\partial P_1(x_1,x_2)}{\partial x_2} \,|_{x_1=w_1, x_2=w_2} & \text{if } x_{2,w_1w_2}(w_1) < w_2 \end{array} \right\},$$

where $x_{2,w_1w_2}(w_1)$ is derived from (P1).

5.3.3 Differences between the commitment and non-commitment cases

When commitment is possible, the buyers form their beliefs and strategies after seeing the pre-committed price paths. The seller's optimal strategy is to choose the path with the induced buyers' beliefs and strategies that maximizes his payoff. Without commitment, the buyers form their beliefs and strategies based on the price history. The seller determines the prices at every instant based on his expectation of how the buyers will react. In short, with commitment, the buyer's strategies are formed corresponding to the pre-committed price paths; and without commitment, the buyers' strategies are formed corresponding to each realized history. Therefore, the seller's payoff-maximizing problems are different under the two circumstances.

In addition, buyers' reactions to prices are different in the two situations. Consider the case when the optimal mechanism requires setting reserve prices $r_1 > \underline{w}_1$ and $r_2 > \underline{w}_2$ for buyers 1 and 2 respectively. If the buyers expect the seller to submit the path derived in Theorem 1 and stop at r_1 and r_2, then buyer 1 with value r_1 and buyer 2 with value r_2 will accept at prices r_1 and r_2 respectively, that is, $P_1(r_1, r_2) = r_1$ and $P_2(r_1, r_2) = r_2$. When there is no commitment, since

$P_1(x_1, x_2) < r_1$ and $P_2(x_1, x_2) < r_2$ for all $x_1 < r_1$ and $x_2 < r_2$, once the seller submits prices (r_1, r_2) and no buyer accepts, the seller believes that the buyers' values are below r_1 and r_2 respectively and will lower the prices further. Hence, $P_1(r_1, r_2) = r_1$ and $P_2(r_1, r_2) = r_2$ cannot be sustained in equilibrium.[5] The differences in buyers' reactions to prices prevent the seller from achieving the optimal outcome even though he maximizes his payoff in response to the buyers' strategies.

5.4 Buyers' strategy

$P_1(x_1, x_2)$ and $P_2(x_1, x_2)$ characterize the buyers' strategies, and they are the prices that the seller needs to offer in order to induce buyer 1 and buyer 2 whose values are above x_1 and x_2 respectively to accept. We can represent $P_1(x_1, x_2)$ and $P_2(x_1, x_2)$ in the following forms

$$\begin{aligned} P_1(x_1, x_2) &= x_1 - \frac{C_1(x_1, x_2)}{F_2(x_2)} \text{ if } x_2 > \underline{w}_2, \\ P_2(x_1, x_2) &= x_2 - \frac{C_2(x_1, x_2)}{F_1(x_1)} \text{ if } x_2 > \underline{w}_1. \end{aligned} \tag{B1}$$

$C_i(x_1, x_2) = F_j(x_j)(x_i - P_i(x_1, x_2))$ can be regarded as the information rent asked by buyer i with value x_i when he believes that if buyer j does not accept the current price, the lowest upper bound of buyer j's value is x_2. Buyer i with value x_i accepts a price less than or equal to $P_i(x_1, x_2)$, and he gets the good if buyer j's value is below x_j. How much a buyer asks for as information rent depends on his expectation of the prices that the seller will offer later and his belief about how the other buyer behaves. Therefore, $C_i(x_1, x_2)$ also characterizes buyer i's strategy, and must be the best response to the other players' strategies. Given the current beliefs (w_1, w_2), if the seller's strategy is to implement a path $x_{2,w_1w_2}(x_1)$ in the future, then to satisfy incentive compatibility so that buyer 1 and buyer 2 will not deviate, we must have

$$\begin{aligned} C_1(w_1, w_2) &= \int_{\underline{w}_1}^{w_1} F_2(x_{2,w_1w_2}(x))\, dx, \\ C_2(w_1, w_2) &= \int_{\underline{w}_2}^{w_2} F_1\left(x_{2,w_1w_2}^{-1}(x)\right) dx \end{aligned} \tag{B2}$$

[5] In equilibrium, P_1 and P_2 satisfy (B1) and (B2) defined in the next section.

in equilibrium, where

$$x_{2,w_1w_2}^{-1}(x_2) \equiv \sup\{x_1 \mid x_{2,w_1w_2}(x_1) \leq x_2\} \tag{2}$$

which shows the same path as $x_{2,w_1w_2}(x_1)$ but represents x_1 as a function of x_2.

Proposition 3 formalizes the arguments above and characterizes the buyers' strategies.

Proposition 3 *After the seller submits $p_1(t)$ and $p_2(t)$ at time t, with the belief that the lowest upper bound of buyer j's value is $x_j = y_j(h^t)$ provided that buyer j rejects all the prices on the history path until t, if buyer i's value $x_i \leq y_i \equiv \lim_{\tau\uparrow t} y_i(h^\tau)$, then the maximum price he accepts is $P_i(x_i, x_j)$ defined in (B1) and (B2); if $x_i > y_i$, which implies buyer i has deviated, then the maximum price he accepts is $P_i(y_i, x_j)$.*

Proof. We show that buyer i maximizes his payoff by following the strategy specified in the proposition. Let $p_1(\tau)$ and $p_2(\tau)$ be the price paths submitted by the seller. Consider a continuation game starting at t with belief $(y_1, y_2) = (\lim_{\tau\uparrow t} y_1(h^\tau), \lim_{\tau\uparrow t} y_2(h^\tau))$ and prices $(p_1(t), p_2(t))$.

First consider the case when there exist $w_1 \leq y_1$ and $w_2 \leq y_2$ such that $p_1(t) = P_1(w_1, w_2)$ and $p_2(t) = P_2(w_1, w_2)$. Given the seller's strategy profile characterized in Proposition 2, we can derive the price paths after t as well as the other two paths $x_1(\tau)$ and $x_2(\tau)$ by solving $\begin{cases} p_1(\tau) = P_1(x_1(\tau), x_2(\tau)) \\ p_2(\tau) = P_2(x_1(\tau), x_2(\tau)) \end{cases}$. $x_i(\tau)$ is the seller's and buyer j's beliefs about the greatest lower bound of buyer i's values with which buyer i would have accepted $p_i(\tau)$. In addition, $(x_1(\tau), x_2(\tau))$, $\tau \geq t$, constitute the graph of $x_{2,w_1w_2}(x_1)$, the solution path to program (P1). Note that if $x_{2,w_1w_2}(w_1) < w_2$, then there exists $s > t$ such that $x_1(\tau) = w_1$, and $x_2(\tau)$ decreases from w_2 to $x_{2,w_1w_2}(w_1)$ for $\tau \in [t, s]$. Because $\frac{\partial P_i(x_i,x_j)}{\partial x_i} \geq 0$, to prove the proposition, it is enough to show that given the price paths $(p_1(\tau), p_2(t))$ derived from the seller's strategy profile, at any time $t' \geq t$, buyer i with a value higher than or equal to $x_i(t')$ should accept $p_i(t')$, and buyer i with a value lower than $x_i(t')$ should reject. At t', buyer 1 with value x_1 gets payoff

$$x_1 - p_1(t') = (x_1 - x_1(t')) + \int_{\underline{w}_1}^{x_1(t')} F_2(x_{2,w_1w_2}(x))\,dx \text{ [6]}$$

if he accepts now, and gets expected payoff

$$\frac{F_2(x_2(t''))}{F_2(x_2(t'))}(x_1 - p_1(t'')) = \frac{F_2(x_2(t''))}{F_2(x_2(t'))}\left((x_1 - x_1(t'')) + \int_{\underline{w}_1}^{x_1(t'')} F_2(x_{2,w_1w_2}(x))\,dx\right)$$

if he accepts later at t''. Because $x_2(t'') \le x_2(t')$, similar to the argument in Theorem 1, buyer 1 with $x_1 \ge x_1(t')$ gets weakly higher payoff by accepting at t' and buyer 1 with $x_1 = x_1(t'')$ gets weakly higher payoff by accepting at $t'' > t'$. The same argument applies to buyer 2. Therefore, buyer i with a value higher than or equal to $x_i(t')$ should accept $p_i(t')$, and buyer i with a value lower than $x_i(t')$ should reject.

Next consider the situation when there does not exist $w_1 \le y_1$ and $w_2 \le y_2$ such that $p_1(t) = P_1(w_1, w_2)$ and $p_2(t) = P_2(w_1, w_2)$. This implies either (i) $p_i(t) > P_i(y_i, y_j)$ for both $i = 1, 2$, or (ii) there exists $w_j \le y_j$ such that $p_i(t) > P_i(y_i, w_j)$ and $p_j(t) = P_j(y_i, w_j)$ for some $i \ne j$. Given the seller's strategy in Proposition 2 and buyer j's strategy $P_j(x_1, x_2)$, in case (i), buyer i expects that buyer j would not accept before p_i falls to $P_i(y_i, y_j)$ so buyer i rejects all the prices $p_i > P_i(y_i, y_j)$; and in case (ii), buyer i expects that buyer j would not accept before p_i falls to $P_i(y_i, w_j)$ so buyer i rejects all the prices $p_i > P_i(y_i, w_j)$.

■

Substituting $P_1(x_1, x_2)$ and $P_2(x_1, x_2)$ in (B1) into the seller's problem (P1), we get

$$\max_{u(x_1, x_{2,w_1w_2}(x_1))} \int_{\underline{w}_1}^{w_1} \left\{ \begin{array}{c} [x_1F_2(x_2) - C_1(x_1, x_2)]\, f_1(x_1) \\ +[x_2F_1(x_1) - C_2(x_1, x_2)]\, f_2(x_2)\, u \end{array} \right\} dx_1 \qquad \text{(P2)}$$
$$+ [x_{2,w_1w_2}(w_1) F_1(w_1) - C_2(w_1, x_{2,w_1w_2}(w_1))] [F_2(w_2) - F_2(x_{2,w_1w_2}(w_1))]$$

$$\begin{aligned} s.t. \quad & x_{2,w_1w_2}(w_1) \le w_2, \\ & \frac{dx_2}{dx_1} = u, \\ & 0 \le u < \infty. \end{aligned}$$

[6] $x_{2,w_1w_2}(x) = x_{2,x_1(t')x_2(t')}(x)$ for $x \in [\underline{w}_1, x_1(t')]$.

5.5 An equilibrium

In this section, we first characterize the conditions for an equilibrium in Proposition 4. Then we show that there exists an equilibrium in which the equilibrium allocation and payoffs of all the players are the same as those in a second-price auction in Theorem 2.

Proposition 4 *Suppose there exist* $x_{2,w_1w_2}(x_1), C_1(w_1,w_2), C_2(w_1,w_2)$ *such that for any* $(w_1,w_2) \in [\underline{w}_1, \overline{w}_1] \times [\underline{w}_2, \overline{w}_2]$,

1. $x_{2,w_1w_2}(x_1)$, *for* $x_1 \in [\underline{w}_1, w_1]$, *is the path derived from (P2);*
2. $C_1(w_1,w_2) = \int_{\underline{w}_1}^{w_1} F_2(x_{2,w_1w_2}(x))\,dx;$
3. $C_2(w_1,w_2) = \int_{\underline{w}_2}^{w_2} F_1\left(x_{2,w_1w_2}^{-1}(x)\right)dx.$[7]

If the vector-valued function $P(w_1,w_2) = (P_1(w_1,w_2), P_2(w_1,w_2))$ *defined in (B1) satisfies Condition 1, then there exists a perfect Bayesian equilibrium as follows:*
Let $X_{w_1w_2}(p_1,p_2) = (X_{1,w_1w_2}(p_1,p_2), X_{2,w_1w_2}(p_1,p_2)) \equiv P^{-1}(p_1,p_2)$, *for* $(p_1,p_2) \in \mathcal{B}_{w_1w_2}$, *where* $\mathcal{B}_{w_1w_2} = \{P(y_1,y_2) \mid 0 < y_1 \le w_1, 0 < y_2 \le w_2\}$.

- *The belief about the lowest upper bound of buyer i's value,* $y_i(h^t)$, *is formed as follows:*
  At t, let $(w_1,w_2) = \begin{cases} (\overline{w}_1, \overline{w}_2), & \text{if } t = 0 \\ (\lim_{\tau\uparrow t} y_1(h^\tau), \lim_{\tau\uparrow t} y_2(h^\tau)), & \text{if } t > 0 \end{cases}$,
  - *If the prices* (p_1,p_2) *offered by the seller is in* $\mathcal{B}_{w_1w_2}$, $(y_1(h^t), y_2(h^t)) = (X_{1,w_1w_2}(p_1,p_2), X_{2,w_1w_2}(p_1,p_2))$.
  - *If* $p_1 \le P_1(w_1,w_2)$ *and* (p_1,p_2) *is above* $\mathcal{B}_{w_1w_2}$, $y_1(h^t) = x_1$ *such that* $P_1(x_1,w_2) = p_1$, *and* $y_2(h^t) = w_2$.
  - *If* $p_2 \le P_2(w_1,w_2)$ *and* (p_1,p_2) *is on the right of* $\mathcal{B}_{w_1w_2}$, $y_1(h^t) = w_1$, *and* $y_2(h^t) = x_2$ *such that* $P_1(w_1,x_2) = p_2$.
  - *If* $p_1 > P_1(w_1,w_2)$ *and* $p_2 > P_2(w_1,w_2)$, $(y_1(h^t), y_2(h^t)) = (w_1,w_2)$.

[7] $x_{2,w_1w_2}^{-1}(x)$ is defined in (2).

- *At t, given belief $(w_1, w_2) = \left(y_1\left(h^t\right), y_2\left(h^t\right)\right)$ and the current prices (b_1, b_2) (note that $P_i(w_1, w_2) \leq b_i$ for $i = 1, 2$[8]), if $b_1 = P_1(w_1, w_2)$ and $b_2 = P_2(w_1, w_2)$, the seller's strategy is $\left(\frac{dp_1(t; b_1, b_2, w_1, w_2)}{dt}, \frac{dp_2(t; b_1, b_2, w_1, w_2)}{dt}\right) \in \mathcal{S}$ such that*
$$\frac{\frac{dp_2}{dt}}{\frac{dp_1}{dt}} = \left\{ \begin{array}{ll} \frac{dP_2\left(x_1, x_{2,w_1w_2}(x_1)\right)}{dx_1} / \frac{dP_1\left(x_1, x_{2,w_1w_2}(x_1)\right)}{dx_1} \,|_{x_1 = w_1} & \text{if } x_{2,w_1w_2}(w_1) = w_2 \\ \frac{\partial P_2(x_1, x_2)}{\partial x_2} / \frac{\partial P_1(x_1, x_2)}{\partial x_2} \,|_{x_1 = w_1, x_2 = w_2} & \text{if } x_{2,w_1w_2}(w_1) < w_2 \end{array} \right\};$$
if $b_i > P_i(x_1, x_2)$ and $b_j = P_j(x_1, x_2)$, the seller's strategy is $\frac{dp_i}{dt} = 1$ and $\frac{dp_j}{dt} = 0$; if $b_i > P_i(x_1, x_2)$ for both $i = 1, 2$, then any $\left(\frac{dp_1}{dt}, \frac{dp_2}{dt}\right) \in \mathcal{S}$ can be the seller's strategy.

- *At t, given history h^t, if buyer i's value $x_i \leq \lim_{\tau \uparrow t} y_i(h^\tau)$, the maximum price he accepts is $P_i\left(x_i, y_j\left(h^t\right)\right)$; if $x_i > \lim_{\tau \uparrow t} y_i(h^\tau)$, the maximum price he accepts is $P_i\left(\lim_{\tau \uparrow t} y_i(h^\tau), y_j\left(h^t\right)\right)$.*

Proof. The result comes directly from Propositions 1, 2, and 3. ■

Example 1 *When the two buyers' values are uniformly distributed on $[0, \overline{w}_1]$ and $[0, \overline{w}_2]$ respectively, there exists an equilibrium:*
$$x_{2,w_1w_2}(x_1) = \left\{ \begin{array}{l} x_1, \text{ for } 0 \leq x_1 \leq \min\{w_1, w_2\} \\ w_2, \text{ for } \min\{w_1, w_2\} < x_1 \leq w_1 \end{array} \right.$$
so that by Proposition 4 and (B1),
$$P_i(w_i, w_j) = \left\{ \begin{array}{l} w_i - \frac{w_i^2}{2w_j}, \text{ if } w_j \geq w_i \\ \frac{w_j}{2}, \text{ if } w_j < w_i \end{array} \right. .$$

- *The belief about the lowest upper bound of buyer i's value, $y_i\left(h^t\right)$, is formed as follows:*
  At t, let $(w_1, w_2) = \left\{ \begin{array}{l} (\overline{w}_1, \overline{w}_2), \text{ if } t = 0 \\ \left(\lim_{\tau \uparrow t} y_1(h^\tau), \lim_{\tau \uparrow t} y_2(h^\tau)\right), \text{ if } t > 0 \end{array} \right.$, and (p_1, p_2) be the prices submitted by the seller (without loss of generality, suppose $p_i \geq p_j$).

  - *If $2p_j \leq w_i$ and $\frac{2p_j^2}{2p_j - p_i} \leq w_j$, $y_i\left(h^t\right) = 2p_j$ and $y_j\left(h^t\right) = \frac{2p_j^2}{2p_j - p_i}$.*
  - *If $2p_j \leq w_i$ and $\frac{2p_j^2}{2p_j - p_i} > w_j$, $y_i\left(h^t\right) = w_i$ and $y_j\left(h^t\right) = \min\left\{w_j, w_i - \sqrt{w_i^2 - 2w_ip_j}\right\}$.*

[8] When the seller determines $\left(\frac{dp_1}{dt}, \frac{dp_2}{dt}\right)$, the buyers have rejected (b_1, b_2), so the belief $(w_1, w_2) = \left(y_1\left(h^t\right), y_2\left(h^t\right)\right)$, and $P_i(w_1, w_2) \leq b_i$ for $i = 1, 2$.

– *If* $2p_j > w_i$ *and* $\frac{2p_j^2}{2p_j - p_i} \le w_j$, $y_i\left(h^t\right) = \min\left\{w_i, w_j - \sqrt{w_j^2 - 2w_j p_i}\right\}$ *and* $y_j\left(h^t\right) = w_j$.

– *If* $2p_j > w_i$ *and* $\frac{2p_j^2}{2p_j - p_i} > w_j$, $y_i\left(h^t\right) = w_i$ *and* $y_j\left(h^t\right) = w_j$.

- *At time* $t \ge 0$*, suppose the current posted prices are* (p_1, p_2) *and beliefs are* $(w_1, w_2) = \left(y_1\left(h^t\right), y_2\left(h^t\right)\right)$*. Without loss of generality, suppose* $w_i \le w_j$*. (Note that* $p_i \ge w_i - \frac{w_i^2}{2w_j}$ *and* $p_j \ge \frac{w_i}{2}$*.*[9]*) The seller's strategy is*

$$\left(\frac{dp_i\left(t; p_i, p_j, w_i, w_j\right)}{dt}, \frac{dp_j\left(t; b_i, b_j, w_i, w_j\right)}{dt}\right)$$
$$= \begin{cases} \left(\frac{1}{2}, \frac{1}{2}\right), \text{ if } p_i = p_j = \frac{1}{2}w_i = \frac{1}{2}w_j \\ (0,1), \text{ if } p_i \ge w_i - \frac{w_i^2}{2w_j} \text{ and } p_j > \frac{w_i}{2} \\ (1,0), \text{ if } p_j = \frac{w_i}{2} \text{ and } p_i > \frac{w_i}{2} \end{cases}.$$

- *At* t*, given history* h^t*, if buyer* i*'s value* $x_i \le \lim_{\tau \uparrow t} y_i\left(h^\tau\right)$*, the maximum price he accepts is* $P_i\left(x_i, y_j\left(h^t\right)\right)$*; if* $x_i > \lim_{\tau \uparrow t} y_i\left(h^\tau\right)$*, the maximum price he accepts is* $P_i\left(\lim_{\tau \uparrow t} y_i\left(h^\tau\right), y_j\left(h^t\right)\right)$*.*

Figure 1 shows the equilibrium price path submitted by the seller and the corresponding $x_1 - x_2$ path in Example 1 when $\overline{w}_2 > \overline{w}_1$. The seller first keeps p_1 at $\overline{w}_1$ and lowers p_2 to $\frac{1}{2}\overline{w}_1$. Then he keeps p_2 at $\frac{1}{2}\overline{w}_1$ and lowers p_1 to $\frac{1}{2}\overline{w}_1$. Finally he lowers both p_1 and p_2 to 0 at the same speed. The buyers correctly expect the path, so buyer 2 will not accept any price along the vertical part of the path. When the seller starts lowering p_1 (the horizontal part), buyer 2 starts accepting $p_2 = \frac{1}{2}\overline{w}_1$. To be more precise, buyer 2 starts accepting after p_1 drops to $a = \overline{w}_1 - \frac{\overline{w}_1^2}{2\overline{w}_2}$. Buyer 1 then has to consider whether to accept a higher price now or to accept a lower price later at the risk of buyer 2 accepting before him. In the equilibrium characterized in Example 1, buyer 2 accepts p_2 at a speed such that buyer 1 is better to wait until p_1 drops to $\frac{1}{2}\overline{w}_1$. When p_1 reaches $\frac{1}{2}\overline{w}_1$, buyer 2 with a value higher than $\overline{w}_1$ would have accepted. Therefore, the dash-line part of the $p_1 - p_2$ path is mapped to the dash-line part of the $x_1 - x_2$ path.

[9] This is because in this example, if $w_j > w_i$, $P_i\left(w_i, w_j\right) = w_i - \frac{w_i^2}{2w_j}$, and $P_j\left(w_i, w_j\right) = \frac{w_i}{2}$. It must be that $P_i\left(w_i, w_j\right) \le p_i$ and $P_j\left(w_i, w_j\right) \le p_j$ so that the belief is updated consistently (see Proposition 4).

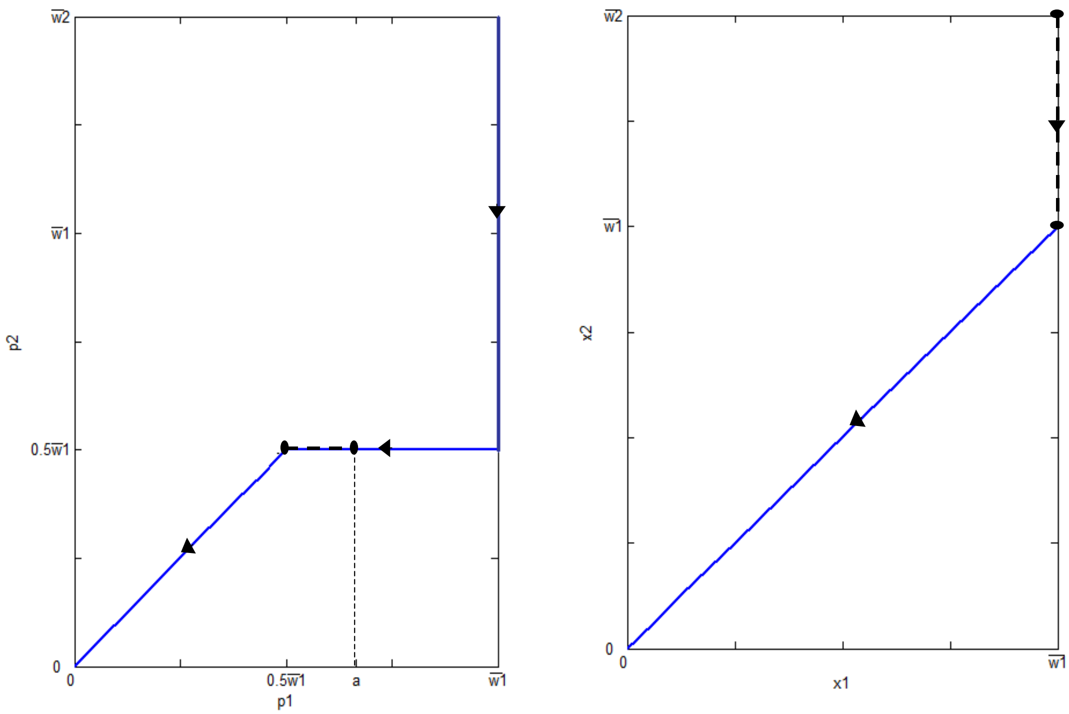

Figure 1: Equilibrium $p_1 - p_2$ and $x_1 - x_2$ paths

After that, the seller lowers p_1 and p_2 at the same speed, and buyer i with a value equal to $2p_i$ accepts p_i. The seller can choose other paths. However, given the buyers' reactions and expectations of the price path after deviation, which is consistent with the seller's strategy in the continuation game after deviation, the seller cannot do better than following the path in Example 1.

In this example, the seller's and the buyers' expected payoffs are the same as in a second-price auction, which is lower than the payoff in a Dutch auction. The lower payoff results from the seller's lack of commitment power. In equilibrium, the seller lowers the two prices simultaneously only when $p_1 = p_2 = \frac{1}{2}\overline{w}_1 = \frac{1}{2}\overline{w}_2$. Otherwise, he only lowers one of the prices. Given the seller's strategy, a buyer does not feel competition from the other buyer as much as in a Dutch auction, so he is reluctant to accept a higher price. And given the buyers' strategies, the seller can do no better by deviating. On the other hand, in a Dutch auction, the two prices are lowered simultaneously. This causes competition between the buyers, so the buyers are willing to accept higher prices. The example thus shows that the inability to make a commitment might result in an equilibrium unfavorable to the seller. The advantage of being able to determine the price paths at will is dominated by the loss caused by not being able to commit. In the next theorem, we show that existing an equilibrium in which the buyers' and the seller's expected payoffs are the same as in a second-price auction is generally true, not just for the uniform distribution case. And in Section 5.6, we further prove the uniqueness of the equilibrium in the uniform distribution case.

Theorem 2 $x_{2,w_1w_2}(x_1) = \begin{cases} \underline{w}_2, & \text{for } \underline{w}_1 \le x_1 < \max\{\underline{w}_1, \underline{w}_2\} \\ x_1, & \text{for } \max\{\underline{w}_1, \underline{w}_2\} \le x_1 \le \min\{w_1, w_2\} \\ w_2, & \text{for } \min\{w_1, w_2\} < x_1 \le w_1 \end{cases}$, $C_1(w_1, w_2) = \int_{\underline{w}_1}^{w_1} F_2(x_{2,w_1w_2}(x_1))\, dx_1$, *and* $C_2(w_1, w_2) = \int_{\underline{w}_2}^{w_2} F_1\left(x_{2,w_1w_2}^{-1}(x_2)\right) dx_2$[10] *are a set of functions satisfying the conditions in Proposition 4. Therefore, there exists an equilibrium such that the allocation rule, the buyers', and the seller's expected payoffs are the same as in a second-price auction.*

Proof. We show that given the forms of $C_1(w_1, w_2)$ and $C_2(w_1, w_2)$, any path gives the seller the same expected payoff, so the seller would not deviate. Let (w_1, w_2) be the initial belief. First consider the case when $\underline{w}_1 \ge \underline{w}_2$ and $w_2 > w_1$.

[10] If $\underline{w}_2 < \underline{w}_1$, $x_2^{-1}(x_2) = \underline{w}_1$ for $x_2 < \underline{w}_1$. If $w_2 > w_1$, $x_2^{-1}(x_2) = w_1$ for $x_2 > w_1$.

(Other cases can be proved in the same way.) The seller knows that $\underline{w}_1$ is buyer 1's lowest possible value, so the lowest price he will offer to the two buyers is $\underline{w}_1$, and buyer 2 with value lower than $\underline{w}_1$ has no chance to get the object. Given $C_1(x_1, x_2)$ and $C_2(x_1, x_2)$, if the seller chooses a path $x_2(x_1)$ such that $x_2(x_1) \geq x_1$ for all $x_1 \in [\underline{w}_1, w_1]$, i.e. $x_2(x_1)$ is above the forty-five degree line, then his expected payoff given initial belief (w_1, w_2) is

$$\begin{aligned}
& \int_{\underline{w}_1}^{w_1} \left[x_1 F_2(x_2(x_1)) - \int_{\underline{w}_1}^{x_1} F_2(x)\, dx \right] f_1(x_1)\, dx_1 \\
& + \int_{\underline{w}_1}^{w_1} \left[x_1 F_1(x_1) - \int_{\underline{w}_1}^{x_1} F_1(x)\, dx \right] f_2(x_2(x_1)) \frac{dx_2(x_1)}{dx_1} dx_1 \\
& + \left[w_1 F_1(w_1) - \int_{\underline{w}_1}^{w_1} F_1(x)\, dx \right] [F_2(w_2) - F_2(x_2(w_1))] \\
= & \int_{\underline{w}_1}^{w_1} x_1 [F_2(x_2(x_1)) - F_2(x_1)]\, dF_1(x_1) + \int_{\underline{w}_1}^{w_1} \int_{\underline{w}_1}^{x_1} x dF_2(x)\, dF_1(x_1) \\
& + \int_{\underline{w}_1}^{x_2(w_1)} \int_{\underline{w}_1}^{x_1} x dF_1(x)\, dF_2(x_2) \\
& + \int_{\underline{w}_1}^{w_1} x dF_1(x) [F_2(w_2) - F_2(x_2(w_1))] \\
= & \int_{\underline{w}_1}^{w_1} \int_{\underline{w}_1}^{x_1} x dF_2(x)\, dF_1(x_1) + \int_{\underline{w}_1}^{w_1} x_1 [F_2(x_2(x_1)) - F_2(x_1)]\, dF_1(x_1) \\
& + \int_{\underline{w}_1}^{w_1} x_1 [F_2(x_2(w_1)) - F_2(x_2(x_1))]\, dF_1(x_1) \\
& + \int_{\underline{w}_1}^{w_1} x_1 [F_2(w_2) - F_2(x_2(w_1))]\, dF_1(x_1) \\
= & \int_{\underline{w}_1}^{w_1} \int_{\underline{w}_1}^{x_1} x dF_2(x)\, dF_1(x_1) + \int_{\underline{w}_1}^{w_1} x_1 [F_2(w_2) - F_2(x_1)]\, dF_1(x_1),
\end{aligned}$$

which is independent of the path $x_2(x_1)$. Therefore, the seller gets the same payoff for all the paths $x_2(x_1)$ such that $x_2(x_1) \geq x_1$. Similarly, we can prove that the seller gets the same payoff for all the paths $x_2(x_1)$ such that $x_2(x_1) \leq x_1$, and can further show that the seller gets the same payoff for any path $x_2(x_1)$.

Next, one can check that $P(w_1, w_2) = (P_1(w_1, w_2), P_2(w_1, w_2))$ defined in (B1) satisfies Condition 1. Thus, there exists an equilibrium by Proposition 4. The allocation rule is that the buyer with the higher value gets the good, and a buyer whose value is smaller than or equal to $\max\{\underline{w}_1, \underline{w}_2\}$ gets zero payoff. Therefore, by the revenue equivalence principle, all the players' payoffs are the same as in the second-price auction. ■

A Dutch auction requires the seller to call out a single price for all the buyers even though they are asymmetric. In the previous section, we show that if different prices for different buyers are permitted and the seller is able to commit to a price path in advance, then the seller can achieve the optimal outcome. However, if different prices are allowed but the seller is not able to commit, there exists an equilibrium in which the seller's payoff is the same as in a second-price auction. Vickrey (1961) and Maskin and Riley (2000) show that, with asymmetric bidders, the seller's payoff in a first-price auction (or a Dutch auction) might be greater than that in a second-price auction. Our result together with the conclusion from the literature suggests that the benefit brought by the discretion to determine the price paths might be outweighed by the loss caused by not being able to commit.

In reality, an extremely sophisticated institution is difficult to implement, so the mechanisms adopted are usually simple. At times, people complain that simple mechanisms hinder their ability and freedom to do what is best for them. However, our result shows that even though people lose their discretion with a simple institution, that institution still retains its value because it helps people make commitment. In a Dutch auction, the seller is forced to commit to a price path and cannot charge different buyers different prices. It seems that the seller might get a better payoff without the restriction and benefit from having the discretion to design price paths. However, sometimes the advantage of having the discretion might be sabotaged by not being able to commit, and the seller's payoff turns out to be lower when he can negotiate with buyers at will.

5.6 Uniqueness of the equilibrium

It might be the concern that there exist multiple equilibria for the negotiation game, so the comparison between the seller's payoffs in different institutions is not conclusive. In this section, we show that under some circumstances, there is actually a unique equilibrium, so our result is robust. We consider the case in Example 1 when the two buyers' values are uniformly distributed on $[0, \overline{w}_1]$

and $[0,\overline{w}_2]$ respectively. To see why the number of equilibria is limited, note that each off-equilibrium history h^t is reachable by deviations of the seller. Since the deviations are made by a player who does not have private information, the beliefs about the buyers' values cannot be arbitrary and must be consistent with the buyers' strategies at each h^t. In addition, the buyers' expectations of future paths are not arbitrary either. They must be consistent with the seller's strategy, which is the best response to the buyers' strategies.

The following analysis focuses on the equilibrium in which given any (w_1,w_2), the seller's strategy $u\left(x_1,x_{2,w_1w_2}\left(x_1\right)\right)$ (defined in Section 5.3 and derived from program (P2)) on the continuation equilibrium path starting with belief (w_1,w_2) is in $(0,\infty)$ for all $x_1\in(0,\overline{x}_{1,w_1w_2})$ for some $\overline{x}_{1,w_1w_2}\leq w_1$. With the restriction, program (P2) becomes

$$\begin{aligned} &\max_{\overline{x}_{1,w_1w_2},u\left(x_1,x_{2,w_1w_2}(x_1)\right)}\int_{\underline{w}_1}^{\overline{x}_1}\left\{\begin{array}{c}\left[x_1F_2\left(x_2\right)-C_1\left(x_1,x_2\right)\right]f_1\left(x_1\right)\\ +\left[x_2F_1\left(x_1\right)-C_2\left(x_1,x_2\right)\right]f_2\left(x_2\right)u\end{array}\right\}dx_1 \qquad \text{(P3)}\\ &+\mathbf{1}_{(\overline{x}_1<w_1)}\left[\overline{x}_1F_2\left(x_{2,w_1w_2}\left(\overline{x}_1\right)\right)-C_1\left(\overline{x}_1,x_{2,w_1w_2}\left(\overline{x}_1\right)\right)\right]\left[F_1\left(w_1\right)-F_1\left(\overline{x}_1\right)\right]\\ &+\mathbf{1}_{(x_2(\overline{x}_1)<w_2)}\left[x_{2,w_1w_2}\left(\overline{x}_1\right)F_1\left(\overline{x}_1\right)-C_2\left(\overline{x}_1,x_{2,w_1w_2}\left(\overline{x}_1\right)\right)\right]\left[F_2\left(w_2\right)-F_2\left(x_{2,w_1w_2}\left(\overline{x}_1\right)\right)\right]\\ s.t.\quad &\overline{x}_1=w_1 \text{ and } x_{2,w_1w_2}\left(w_1\right)\leq w_2, \text{ or } \overline{x}_1\leq w_1 \text{ and } x_{2,w_1w_2}\left(\overline{x}_1\right)=w_2,\\ &\frac{dx_2}{dx_1}=u,\\ &0<u<\infty,\end{aligned}$$

where $\mathbf{1}_{(\cdot)}$ is the indicator function. Note that we do not restrict the seller's strategy space. The seller can still adopt a strategy such that $u\left(x_1,x_2\right)=0$ or ∞, but he will find that doing so does not make him better.

In addition, we require $C_1\left(x_1,x_2\right)$ and $C_2\left(x_1,x_2\right)$, the functions that characterize buyers' information rent, to be smooth, i.e. $\frac{\partial C_i}{\partial x_j},i,j=1,2$, is continuous. If $\frac{\partial C_1}{\partial x_2}$ is not continuous at (x_1,x_2), it is implied that the equilibrium strategies and beliefs are quite different in the two continuation games with beliefs (x_1,x_2) and $(x_1,x_2-\epsilon)$, where ϵ is small. Moreover, we require $\frac{\partial C_i}{\partial x_j}\geq 0$.

Definition 2 *An equilibrium is smooth if the seller's equilibrium strategy* $u\left(x_1,x_2\left(x_1\right)\right)$ *on the continuation equilibrium path starting with any belief* (w_1,w_2) *is in* $(0,\infty)$ *for all* $x_1\in\left(0,\overline{x}_{1,w_1w_2}\right)$, *where* $u\left(x_1,x_2\left(x_1\right)\right)$ *and* $\overline{x}_{1,w_1w_2}$ *are derived from program (P3). Moreover,* $\frac{\partial C_i}{\partial x_j}$, $i,j=1,2$, *is continuous and* $\frac{\partial C_i}{\partial x_j}\geq 0$.

First, we use Pontryagin's maximum principle to derive necessary conditions for the seller's equilibrium strategy. Along with the condition implied by the revenue equivalence principle, we show that there is only one set of strategies and beliefs satisfying all the conditions and the requirements for a smooth equilibrium. Therefore, the equilibrium derived in Theorem 2 is the unique smooth equilibrium. The details of the proof can be found in Appendix A.

Proposition 5 *When the two buyers' values are uniformly distributed on* $[0, \overline{w}_1]$ *and* $[0, \overline{w}_2]$ *respectively, the equilibrium characterized in Example 1 is the unique smooth perfect Bayesian equilibrium.*

Proof. The proposition comes from lemma 10 in Appendix A. ■

6 Conclusion

Our paper studies the case when a seller with an indivisible object negotiates with two asymmetric buyers to determine who gets the object and at what price. The seller repeatedly submits take-it-or-leave-it offers to the two buyers until one of them accepts. Unlike a Dutch auction, the two prices offered to the two buyers do not have to be the same. We show that if the seller can commit to some price paths, the payoff realized in Myerson's optimal mechanism is achievable. However, if commitment is not possible, the seller's equilibrium payoff is the same as that in a second-price auction, which might be lower than the payoff in a Dutch auction. Therefore, although a simple institution, like a Dutch auction, restricts a player's freedom, it might actually benefit the player by providing a commitment tool. Our analysis also sheds light on the procurement literature and gives insights into the performance of atypic auctions conducted at Priceline.com.

The paper builds a bridge between the auction and the bargaining literature. The model differs from an auction environment by allowing the auctioneer to set different prices for different buyers, and differs from a bargaining setting by considering a one-to-many negotiation process in which one party chooses his partner from a group of people. Although we only consider a two-buyer case, the analysis and methodology can be applied to a more complex environment, and the conclusion can be generalized to a n-buyer case.

A Appendix

Consider the seller's optimal control problem in (P3). Define the initial value function as

$$I\left(\overline{x}_1, x_2\left(\overline{x}_1\right)\right) = \begin{cases} \left[\overline{x}_1 F_2\left(x_2\left(\overline{x}_1\right)\right) - C_1\left(\overline{x}_1, x_2\left(\overline{x}_1\right)\right)\right]\left[F_1\left(w_1\right) - F_1\left(\overline{x}_1\right)\right] & \text{if } \overline{x}_1 < w_1, x_2\left(\overline{x}_1\right) = w_2 \\ \left[x_2\left(\overline{x}_1\right) F_1\left(\overline{x}_1\right) - C_2\left(\overline{x}_1, x_2\left(\overline{x}_1\right)\right)\right]\left[F_2\left(w_2\right) - F_2\left(x_2\left(\overline{x}_1\right)\right)\right] & \text{if } \overline{x}_1 = w_1, x_2\left(\overline{x}_1\right) < w_2 \end{cases}.$$

Define the Hamiltonian function H as

$$H\left(x_1, x_2, u, \lambda\right) = G\left(x_1, x_2, u\right) + \lambda g\left(x_1, x_2, u\right),$$

where

$$\begin{aligned} G\left(x_1, x_2, u\right) &= \left[x_1 F_2\left(x_2\right) - C_1\left(x_1, x_2\right)\right] f_1\left(x_1\right) + \left[x_2 F_1\left(x_1\right) - C_2\left(x_1, x_2\right)\right] f_2\left(x_2\right) u, \\ g\left(x_1, x_2, u\right) &= u. \end{aligned}$$

The following theorem is a restatement of Pontryagin's maximum principle.

Theorem 3 *If a control $u(\cdot)$ with a corresponding state trajectory $x(\cdot)$ is optimal, there exists an absolutely continuous function $\lambda : [0, w_1] \mapsto \mathbb{R}$ such that the maximum condition*

$$H\left(x_1, x_2\left(x_1\right), u\left(x_1\right), \lambda\left(x_1\right)\right) = \max\left\{H\left(x_1, x_2\left(x_1\right), u, \lambda\left(x_1\right)\right) \mid 0 \leq u \leq \infty\right\},$$

the adjoint equation

$$\lambda'\left(x_1\right) = -\frac{\partial H\left(x_1, x_2\left(x_1\right), u\left(x_1\right), \lambda\left(x_1\right)\right)}{\partial x_2},$$

and the transversality conditions

$$\begin{aligned} \lambda(0) &= 0; \\ \text{if } \overline{x}_{1,w_1 w_2} &= w_1 \text{ and } x_2\left(w_1\right) < w_2,\ \lambda\left(w_1\right) = \frac{\partial I}{\partial x_2}; \qquad (3) \\ \text{if } \overline{x}_{1,w_1 w_2} &< w_1 \text{ and } x_2\left(\overline{x}_1\right) = w_2,\ H\left(\overline{x}_1\right) + \frac{\partial I}{\partial \overline{x}_1} = 0. \qquad (4) \end{aligned}$$

are satisfied.

For later convenience, let $c_1(x_1, x_2) = \overline{w}_2 C_1(x_1, x_2)$ and $c_2(x_1, x_2) = \overline{w}_1 C_2(x_1, x_2)$.

Lemma 3 $c_1(x_1, x_2) = x_1 x_2 - c_2(x_1, x_2)$.

Proof. Recall that $C_i(w_1, w_2)$ can be regarded as information rent to buyer i with value w_i while at some stage of the negotiation game, it is the belief that w_1 and w_2 are the lowest upper bounds of the buyers' values. Given w_1 and w_2, suppose $x_{2,w_1w_2}(x_1) : [0, \overline{x}_{1,w_1w_2}] \mapsto [0, w_2]$, is the equilibrium path of the continuation game. In equilibrium, incentive compatibility is satisfied, so the information rent to buyer 1 with value w_1 must be

$$C_1(w_1, w_2) = \int_0^{\overline{x}_{1,w_1w_2}} F_2(x_{2,w_1w_2}(x))\,dx + F_2(x_{2,w_1w_2}(\overline{x}_{1,w_1w_2}))[w_1 - \overline{x}_{1,w_1w_2}],$$

and the information rent to buyer 2 with value w_2 must be

$$C_2(w_1, w_2) = \int_0^{x_2(\overline{x}_{1,w_1w_2})} F_1\left(x_{2,w_1w_2}^{-1}(x)\right)dx + F_1(\overline{x}_{1,w_1w_2})[w_2 - x_{2,w_1w_2}(\overline{x}_{1,w_1w_2})].$$

Multiply both sides of the two equations by $\overline{w}_2$ and $\overline{w}_1$ respectively, we get

$$c_1(w_1, w_2) = \int_0^{\overline{x}_{1,w_1w_2}} x_{2,w_1w_2}(x)\,dx + x_{2,w_1w_2}(\overline{x}_{1,w_1w_2})[w_1 - \overline{x}_{1,w_1w_2}], \quad (5)$$

$$c_2(w_1, w_2) = \int_0^{x_{2,w_1w_2}(\overline{x}_{1,w_1w_2})} x_{2,w_1w_2}^{-1}(x)\,dx + \overline{x}_{1,w_1w_2}[w_2 - x_{2,w_1w_2}(\overline{x}_{1,w_1w_2})].$$

Note that either $\overline{x}_{1,w_1w_2} = w_1$ or $x_{2,w_1w_2}(\overline{x}_{1,w_1w_2}) = w_2$. Therefore, $c_1(w_1, w_2) + c_2(w_1, w_2) = w_1 w_2$. ■

Lemma 4 $c_i(0, x_2) = 0$ *and* $c_i(x_1, 0) = 0$.

Proof. By lemma 3, $c_1(x_1, 0) + c_2(x_1, 0) = 0$. Since $c_i(\cdot) \geq 0$, $c_1(x_1, 0) = 0$ and $c_2(x_1, 0) = 0$. ■

Lemma 5 *Let $x_{2,w_1w_2}(x_1)$ be the equilibrium path of the continuation game starting with belief w_1, w_2. Then*

$$c_1(x_1, x_2{}_{,w_1w_2}(x_1)) - \int_0^{x_1} x - \frac{\partial c_1(x, x_{2,w_1w_2}(x))}{\partial x_2} + \frac{\partial c_1(x, x_{2,w_1w_2}(x))}{\partial x_2} \frac{dx_{2,w_1w_2}(x)}{dx_1} dx = 0 \quad (6)$$

for all $x_1 \in [0, \overline{x}_{1,w_1w_2}]$.

Proof. By the maximum condition,

$$u \in (0,\infty) \ \text{ if } \frac{\partial H}{\partial u} = \left[x_2 F_1(x_1) - C_2(x_1,x_2)\right] f_2(x_2) + \lambda = 0. \qquad (7)$$

By the adjoint equation,

$$\lambda'(x_1) = -\left\{\left[x_1 f_2\left(x_{2,w_1w_2}(x_1)\right) - \frac{\partial C_1}{\partial x_2}\right] f_1(x_1) + \left[F_1(x_1) - \frac{\partial C_2}{\partial x_2}\right] f_2\left(x_{2,w_1w_2}(x_1)\right) u\right\},$$

and $\lambda(0) = 0$ by the transversality condition, so $\lambda(x_1) = -\int_0^{x_1} \left[x f_2\left(x_{2,w_1w_2}(x)\right) - \frac{\partial C_1}{\partial x_2}\right] f_1(x) + \left[F_1(x) - \frac{\partial C_2}{\partial x_2}\right] f_2\left(x_{2,w_1w_2}(x)\right) u(x)\, dx$. Plugging into equation (7),

$$\begin{aligned} & \left[x_{2,w_1w_2}(x_1) F_1(x_1) - C_2(x_1,x_2)\right] f_2(x_2) \\ - & \int_0^{x_1} \left[x f_2\left(x_{2,w_1w_2}(x)\right) - \frac{\partial C_1}{\partial x_2}\right] f_1(x) + \left[F_1(x) - \frac{\partial C_2}{\partial x_2}\right] f_2\left(x_2(x)\right) u(x)\, dx = 0. \end{aligned}$$

Multiplying both sides by $\overline{w}_1 \overline{w}_2 = \frac{1}{f_1(x) f_2(y)}$,

$$\left[x_1 x_{2,w_1w_2}(x_1) - c_2\left(x_1, x_{2,w_1w_2}(x_1)\right)\right] - \int_0^{x_1} \left[x - \frac{\partial c_1}{\partial x_2}\right] + \left[x - \frac{\partial c_2}{\partial x_2}\right] u(x)\, dx = 0.$$

By lemma 3, we get $c_1\left(x_1, x_{2,w_1w_2}(x_1)\right) - \int_0^{x_1} x_1 - \frac{\partial c_1}{\partial x_2} + \frac{\partial c_1}{\partial x_2} \frac{dx_{2,w_1w_2}}{dx_1} dx_1 = 0$ for all $x_1 \in \left[0, \overline{x}_{1,w_1w_2}\right]$. ■

Lemma 6 *Letting $x_{2,w_1w_2}(x_1)$ be the equilibrium path of the continuation game starting with belief w_1, w_2,*

$$\frac{\partial c_1\left(x_1, x_{2,w_1w_2}(x_1)\right)}{\partial x_1} + \frac{\partial c_1\left(x_1, x_{2,w_1w_2}(x_1)\right)}{\partial x_2} = x_1 \ \textit{for all } x_1 \in \left[0, \overline{x}_{1,w_1w_2}\right]. \qquad (8)$$

Proof. Since equation (6) holds for all $x_1 \in [0, \overline{x}_{1,w_1w_2}]$, taking derivative with respect to x_1 on both sides, we get

$$\frac{\partial c_1}{\partial x_1} + \frac{\partial c_1}{\partial x_2} u = x_1 - \frac{\partial c_1}{\partial x_2} + \frac{\partial c_1}{\partial x_2} u.$$

Therefore, $\frac{\partial c_1\left(x_1, x_{2,w_1w_2}(x_1)\right)}{\partial x_1} + \frac{\partial c_1\left(x_1, x_{2,w_1w_2}(x_1)\right)}{\partial x_2} = x_1$. ■

Definition 3 *Let* $A = \left\{\left(x_1, x_{2,w_1w_2}(x_1)\right) \mid x_1 \in \left[0, \overline{x}_{1,w_1w_2}\right], 0 \le w_1 \le \overline{w}_1, 0 \le w_2 \le \overline{w}_2\right\}$, *and $\overline{A}$ is the closure of A.*

A contains all the points on the equilibrium paths of all the continuation games. Note that $(x, x) \in A$, where $x \in [0, \min\{\overline{w}_1, \overline{w}_2\}]$ by Theorem 2.

Lemma 7 *For all $(x_1, x_2) \in \overline{A}$, $c_1(x_1, x_2)$ is of the form $\frac{x_1^2}{2} + \phi(x_2 - x_1)$.*

Proof. If (x_1, x_2) is on the equilibrium path of a continuation game so that it is in A, then the differential equation (8) has to hold. The general solution to the differential equation is $c_1(x_1, x_2) = \frac{x_1^2}{2} + \phi(x_2 - x_1)$. Since $c_1(x_1, x_2)$ is totally differentiable on $[0, \overline{w}_1] \times [0, \overline{w}_2]$, $c_1(x_1, x_2) = \frac{x_1^2}{2} + \phi(x_2 - x_1)$ for all $(x_1, x_2) \in \overline{A}$. ■

Let $\overline{d} = \max\left\{x_2 - x_1 \mid (x_1, x_2) \in \overline{A}\right\}$ and $\underline{d} = \min\left\{x_2 - x_1 \mid (x_1, x_2) \in \overline{A}\right\}$.

Lemma 8 *$\phi(d) = 0$ if $d \in \left[0, \overline{d}\right]$.*

Proof. Let $UL(x_2) = \min\left\{x_1 \in \overline{A}\right\}$ for $x_2 \in [0, \overline{w}_2]$. UL describes the upper-left boundary of $\overline{A}$ and provides information about how $\phi(d)$ is like when $d \in \left[0, \overline{d}\right]$. Let $B_u = \{(x_1, x_2) \mid 0 \leq x_2 \leq \overline{w}_2, 0 \leq x_1 \leq UL(x_2)\}$ be the set above the upper-left boundary. If there exists (a_1, a_2) in the interior of B_u, it is implied that in a continuation game with belief (a_1, a_2), the continuation equilibrium path is $\left(x_1, UL^{-1}(x_1)\right), 0 \leq x_1 \leq a_1$, which means buyer 1 with value a_1 is paired with buyer 2 with value $UL^{-1}(a_1)$, so $c_1(a_1, a_2) = c_1\left(a_1, UL^{-1}(a_1)\right)$. Since we focus on the set of c_i such that $\frac{\partial c_i}{\partial x_j} \geq 0, i, j = 1, 2$, $c_1(a_1, x_2) = c_1(a_1, a_2)$ for all $x_2 \in \left[UL^{-1}(a_1), a_2\right]$, i.e. c_1 is independent of x_2. Besides, $c_1(0, x_2) = 0$, so $c_1(x_1, x_2)$ is of the form $\psi(x_1)$.

For any $d \in \left[0, \overline{d}\right]$, there exists (x_1, x_2) such that $x_2 - x_1 = d$, $(x_1, x_2) \in B_u \cap \overline{A}$, that is, (x_1, x_2) is on the upper-left boundary of $\overline{A}$. Therefore, $c_1(x_1, x_2)$ is of the form $\psi(x_1)$ as well as $\frac{x_1^2}{2} + \phi(x_2 - x_1)$. Since we require $\frac{\partial c_1}{\partial x_2}$ to be continuous, $\frac{\partial c_1(x_1, x_2)}{\partial x_2} = \phi'(x_2 - x_1) = \phi'(d) = \frac{d\psi(x_1)}{dx_2} = 0$. Since $c_1(0, 0) = 0$, $\phi(0) = 0$. So $\phi(d) = 0$ for all $d \in \left[0, \overline{d}\right]$, and $\psi(x_1) = \frac{x_1^2}{2}$ ■

Lemma 9 *$\phi(d) = -\frac{(x_2 - x_1)^2}{2}$ if $d \in [\underline{d}, 0]$.*

Proof. Let $LR(x_1) = \min\left\{x_2 \in \overline{A}\right\}$ for $x_1 \in [0, \overline{w}_1]$. LR describes the lower-right boundary of $\overline{A}$ and provides information about how $\phi(d)$ is like when $d \in [\underline{d}, 0]$. Let $B_l = \{(x_1, x_2) \mid 0 \leq x_1 \leq \overline{w}_1, 0 \leq x_2 \leq LR(x_1)\}$ be the set below the lower-right boundary. If there exists (a_1, a_2) in the interior of B_l, it is implied that in a continuation game with belief (a_1, a_2), the continuation equilibrium path is $\left(LR^{-1}(x_2), x_2\right), 0 \leq x_2 \leq a_2$, which means buyer 2 with value a_2 is paired with buyer 1 with value $LR^{-1}(a_2)$, so $c_2(a_1, a_2) = c_2\left(LR^{-1}(a_2), a_2\right)$. Since we focus on the set of c_i such that $\frac{\partial c_i}{\partial x_j} \geq 0, i, j = 1, 2$, $c_2(x_1, a_2) = c_2(a_1, a_2)$ for all $x_1 \in \left[LR^{-1}(a_2), a_1\right]$. Besides, $c_2(x_1, 0) = 0$, so $c_2(x_1, x_2)$ is of the form $\varphi(x_2)$.

For any $d \in [\underline{d}, 0]$, there exists (x_1, x_2) such that $x_2 - x_1 = d$, $(x_1, x_2) \in B_l$, and $(x_1, x_2) \in \overline{A}$. Therefore, $c_2(x_1, x_2)$ is of the form $\varphi(x_2)$ as well as $x_1 x_2 - \frac{x_1^2}{2} - \phi(x_2 - x_1)$. Since we require $\frac{\partial c_1}{\partial x_1}$ and $\frac{\partial c_1}{\partial x_2}$ to be continuous, $\frac{\partial c_2(x_1, x_2)}{\partial x_1} = x_2 - x_1 + \phi'(x_2 - x_1) = d + \phi'(d) = 0$. Since $c_2(0,0) = 0$, $\phi(0) = 0$. So $\phi(d) = -\frac{(x_2 - x_1)^2}{2}$ for all $d \in [\underline{d}, 0]$, and $\varphi(x_2) = \frac{x_2^2}{2}$. ■

Lemma 10 *In equilibrium,* $C_1(x_1, x_2) = \begin{cases} \frac{1}{\overline{w}_2} \frac{x_1^2}{2}, & \text{if } x_1 \leq x_2 \\ \frac{1}{\overline{w}_2}\left(x_1 x_2 - \frac{x_2^2}{2}\right), & \text{if } x_1 \geq x_2 \end{cases}$, $C_2(x_1, x_2) = \begin{cases} \frac{1}{\overline{w}_1}\left(x_1 x_2 - \frac{x_1^2}{2}\right), & \text{if } x_1 \leq x_2 \\ \frac{1}{\overline{w}_1} \frac{x_2^2}{2}, & \text{if } x_1 \geq x_2 \end{cases}$, *and* $A = \{(x_1, x_2) \mid x_1 = x_2, 0 \leq x_1 \leq \min\{\overline{w}_1, \overline{w}_2\}\}$.

Proof. From lemmas 8, 9 and the discussion in their proofs, we know that if the equilibrium is smooth, the only possible c_1 is that

$$c_1(x_1, x_2) = \begin{cases} \frac{x_1^2}{2}, & \text{if } x_1 \leq x_2 \\ \left(x_1 x_2 - \frac{x_2^2}{2}\right), & \text{if } x_1 \geq x_2 \end{cases},$$

which implies that the seller's strategy leads to the path $x_{2,w_1 w_2}(x_1) = \begin{cases} x_1, x_1 \in [0, \min\{w_1, w_2\}] \\ w_{2,} x_1 \in (\min\{w_1, w_2\}, w_1] \end{cases}$, given the belief (w_1, w_2) about the buyers' values at the beginning of a continuation game. ■

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