Perfect Foresight Equilibrium Selection in Signaling Games

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December 1, 2009

Abstract

We study equilibrium selection in signaling games by perfect foresight dynamics. We consider a signaling game of the following properties. The informed player is of high or low productivity, which is her private information. She chooses whether or not to send a costly signal. She receives a wage equal to her expected productivity. If the informed player with high productivity prefers the separating equilibrium to the pooling equilibrium, the separating equilibrium is locally and globally stable. If the preference is reversed and the pooling equilibrium satisfy a condition which is analogous to risk dominance, the pooling equilibrium is locally and globally stable. Finally the Pareto inefficient pooling equilibrium is locally unstable.

JEL classification numbers: C72, C73, D82.

1 Introduction

Signaling games often have multiple equilibria, and the issue of equilibrium selection has attracted much attention (e.g., Cho and Kreps [3], Banks and Sobel [2], Mailath, Okuno-Fujiwara and Postlewaite [7]). The basic idea in the equilibrium selection is to put some restrictions on beliefs off the equilibrium path.

More recent literature has studied this issue in the context of evolutionary dynamics and given support to the equilibrium selection in the preceding literature (e.g., Nöldeke and Samuelson [9], Amaya [1], Jacobsen, Jensen and

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Sloth [5]). In the dynamics considered in these papers, a signaling game is played by a population of players. The players behave in a boundedly rational way, or myopically, in the sense that when they are given an opportunity to choose an action they take a best response to the current state of all players' behavior.

In the present paper, we study the perfect foresight dynamics with Poisson revision opportunities, originally proposed by Matsui and Matsuyama [8]. We consider a signaling game played between player I (*informed player*, or *sender*) and player II (*uninformed player*, or *receiver*). Our model has a large population of player I and another large population of player II, who are repeatedly and randomly matched to play the signaling game. Player I is endowed with a type, which is her private information. The type of each individual in the role of player I is fixed over time. Player II can switch actions at any moment and is assumed to take a best response to the current action distribution of player I. On the other hand, each player in the role of player I must make a commitment to a particular action for a random time interval. Opportunities to revise actions follow Poisson processes, which are independent across players. They maximize their expected discounted payoffs. This dynamic game with frictions generates nontrivial equilibrium paths of behavior patterns, whose stationary states correspond to the sequential equilibria of the signaling game.

We consider a simple signaling game with two types and two signals. In analogy with Spence [11], player I and II may be interpreted as a worker and the market. The worker's private information is her productivity, which may be high (type 1) or low (type 0). She chooses whether or not to send a costly signal such as education. The market pays a wage equal to the worker's expected productivity given the observed signal. A pure strategy sequential equilbrium belongs to one of the following classes: (i) A separating equilibrium where only the productive worker sends a costly signal and the worker receives a wage equal to her own productivity, (ii) a Pareto efficient pooling equilibrium where the worker does not send a costly signal regardless of her type and receives a wage equal to the average productivity in the society, (iii) a Pareto inefficient pooling equilibrium where the worker sends a costly signal regardless of her type and receives a wage equal to the average productivity in the society,

We examine the stability of these equilibria with respect to the perfect foresight dynamics. We say a stationary state is *absorbing* if all perfect foresight equilibrium with an initial state in the neighborhood of it converges to it. A stationary state is*globally accessible* if for any initial states, there exists an equilibrium path converging to it. If an absorbing state is globally accessible, it is the unique absorbing state.

We follow the view by Matsui and Matsuyama [8] that these two stability properties, taken together, make the unique absorbing state a natural choice among sequential equilibria.

Our result is threefold. First, if a separating equilibrium exists and player I with high type receives a higher payoff in it than in a Pareto efficient pooling equilibrium, it is absorbing and globally accessible.

Second, a Pareto efficient pooling equilibrium is absorbing and globally accessible for sufficiently small degree of friction. if it is *type-1 half dominant*. Roughly speaking, the type-1 half dominance condition means that if at least half of high type (type 1) workers do not send a costly signal and the market plays in a consistent fashion, a high type worker is better off by not sending a costly signal. If a Pareto efficient pooling equilibrium is type-1 half dominant, it is preferred by a high type (type 1) worker over the separating equilibrium (if any).

The intuition for this result is as follows. For simplicity consider the case where there exists a separating equilibrium. In this case, not sending a costly signal is a dominant strategy for low type workers. By assuming the low type workers choose not send a costly signal at every revision opportunity and the market always take a best response, we can reduce the analysis to a game played by a single population of player I. Roughly speaking, type-1 half dominance means that not sending a signal is the risk dominant strategy (Harsanyi and Selten [4]) of this reduced form game. Here works a similar argument to Matsui and Matsuyama [8], who showed that in a single population perfect foresight dynamic the risk dominant equilibrium is absorbing and globally accessible for sufficiently small degree of friction.

Third, a Pareto inefficient pooling equilibrium is not absorbing.

The paper is organized as follows. In Section 2 we describe the signaling game. In Section 3 we characterize the equilibria of the signaling game. In Section 4 we review the existing literature on equilibrium selection in signaling games and examine how these ideas apply to our model. In Section 5 we introduce the model of perfect foresight dynamics and describe the stability concepts. In Section 6 we present the stability results.

2 Signaling game

The signaling game G is described as follows. There are two players, I and II. Player I is endowed with a type $\theta \in \Theta \equiv \{0, 1\}$. The prior probability distribution of types is common knowledge, with probability of $\theta = 0$ given by $1-p$ and probability of $\theta = 1$ by p. Player I, knowing this type chooses a message m from the set $M = \{m_0, m_1\}$. Player II seeing m, responds with a choice r from the set $R = [0, 1]$. The payoff for type θ of player I for the pair of moves $(m, r) \in M \times R$ is given by $u(m, r, \theta)$. The payoff to player II when I is of type θ and (m, r) is chosen is given by $v(m, r, \theta)$.

The payoff functions considered here take the following form;

$$u(m, r, \theta) = \begin{cases} rv & \text{if} \quad m = m_0 \\ rv - c_\theta & \text{if} \quad m = m_1 \end{cases} \tag{1}$$

$$v(m, r, \theta) = -(r - \theta)^2, \tag{2}$$

where v, c_0 and c_1 are positive constants with $c_0 > c_1$.

This payoff specification follows the formulation of job market signaling by Spence [11]. In Spence's interpretation, player I is a worker seeking for a job and player II is a reduced representation of the market. Worker (player I) of type 0 (low type) has marginal productivity 0 and type 1 has marginal productivity v. Worker receives wage rv from the market (player II). Worker incurs a cost for sending higher signal (m_1) such as higher education, while lower signal (m_0) is costless. The cost of sending higher signal is higher for low type than high type. Since the wage for a worker should be equal to his expected marginal productivityunder Bertrand competition, player II's objective is assumed to be minimazation of the difference between them. Player II's payoff function follows from the fact that worker's productivity is θv and wage is rv.

Player II will never randomize in any equilibria from concavity of v, while player I may randomize. The set of all probability distributions on a set X is denoted Δ_X. Player I's mixed strategy is denoted $\mu : \Theta \to \Delta_M$. Player I of type θ sends message m with probability $\mu^\theta(m)$. Player II's pure strategy is denoted $\rho : M \to R$.

We write $\beta : M \to [0, 1]$ for II's belief upon observing m, so that $\beta(m)$ is II's conditional belief that player I is of type 1 when he observes $m \in M$. Finally, we let $BR : M \times \Delta_T \to R$ denote the best response to m given $\beta(m)$ (which is unique from the concavity of v), i.e., $BR(m, \beta(m)) = \arg\max_{r \in R} \sum_{\theta \in \Theta} v(m, r, \theta)\beta(\theta|m)$.

Definition 1 *$\sigma = (\mu, \rho, \beta)$ is a sequential equilibrium if*

1. *For all $\theta \in \Theta$ and $m \in M$, $\mu^\theta(m) > 0$ implies $m \in \arg\max_{m' \in M} u(m', \rho(m'), \theta)$.*

2. *For all $m \in M$, $\rho(m) = BR(m, \beta(m))$;*

3. *For all* $m \in M$,

$$\beta(m) = \frac{p\mu^1(m)}{(1-p)\mu^0(m) + p\mu^1(m)}$$

if the denominator is positive.

We denote the set of sequential equilibria for game G by $PSE(G)$. With an abuse of notation we write $u(\sigma, \theta)$ for type θ's payoff associated with an equilibrium σ.

3 Equilibria

This section characterizes all equilibria of game G.

Obviously, in any equilibrium $\rho(m) = \beta(m)$ for all m.

Any equilibrium should belong to one of the following five categories.

1. *Separating* (σ^S): Player I of type 0 sends m_0 and type 1 sends m_1.
2. *Pooling on* m_0 (σ^{P0}): Player I of both types sends m_0.
3. *Pooling on* m_1 (σ^{P1}): Player I of both types sends m_1.
4. *Semi-pooling on* m_0 (σ^{SP0}): Player I of type 0 sends m_0 and type 1 mixes between m_0 and m_1.
5. *Semi-pooling on* m_1 (σ^{SP1}): Player I of type 0 mixes between m_0 and m_1 and type 1 sends m_1.

For each class of equilibrium, the conditions that payoff parameters must satisfy for existence of equilibrium and the equilibrium payoffs for each type are as follows.

Separating

In a separating equilibrium, $\rho(m_0) = \beta(m_0) = 0$ and $\rho(m_1) = \beta(m_1) = 1$. The incentive conditions for player I of each type are $0 \geq v - c_0$ for type 0 and $v - c_1 \geq 0$ for type 1. Therefore, a separating equilibrium exists if and only if $\frac{c_1}{v} \leq 1 \leq \frac{c_0}{v}$. The equilibrium payoffs are $u(\sigma^S, 0) = 0$ and $u(\sigma^S, 1) = v - c_1$.

Pooling on m_0

When player I of both types sends m_0, $\rho(m_0) = \beta(m_0) = p$. The incentive condition for player I of type θ is $pv \geq \beta(m_1)v - c_\theta$. This condition is satisfied for a belief $\beta(m_1)$ with $0 < \beta(m_1) < p + \frac{c_1}{v}$, and such a belief is consistent because m_1 is off the equilibrium path. Therefore, a pooling equilibrium on m_0 always exsists. The equilibrium payoff is $u(\sigma^{P0}, \theta) = pv$ for $\theta \in \{0, 1\}$.

Pooling on m_1

When player I of both types sends m_1, $\rho(m_1) = \beta(m_1) = p$. The incentive condition for player I of type θ is $pv - c_\theta \geq \beta(m_0)v$. If LHS is nonnegative, the condition holds for sufficiently small $\beta(m_0)$ and such a belief is consistent because m_0 if off the equilibrium path. On the other hand if LHS is negative, there is no $\beta(m_0) \in [0, 1]$ that satisfy the condition. The LHS is nonnegative for both types if $pv - c_0 \geq 0$. Therefore, a pooling equilibrium on m_0 exists if and only if $\frac{c_0}{v} \leq p$. The equilibrium payoffs are $u(\sigma^{P1}, 0) = pv - c_0$ and $u(\sigma^{P1}, 1) = pv - c_1$.

Semi-pooling on m_0

In a semi-pooling equilibrium on m_0, player I of type 1 is indifferent between sending m_0 and m_1, while type 0 is strictly better off by sending m_0. Type 1's payoff is $v - c_1$ when sending m_1 because $\beta(m_1) = 1$. The indifference condition for type 1 is $v - c_1 = \beta(m_0)v$, or equivalently, $1 - \frac{c_1}{v} = \beta(m_0)$. Since

$$\beta(m_0) = \frac{p\mu^1(m_0)}{(1-p) + p\mu^1(m_0)},$$

$0 < \beta(m_0) < p$ should hold. Therefore, a semi-pooling equilibrium on m_0 exists if and only if $0 < 1 - \frac{c_1}{v} < p$, i.e., $1 - p < \frac{c_1}{v} < 1$. The equilibrium payoff is $u(\sigma^{SP0}, \theta) = \beta(0)v = v - c_1$ for $\theta \in \{0, 1\}$.

Semi-pooling on m_1

In a semi-pooling equilibrium on m_1, player I of type 0 is indifferent between sending m_0 and m_1, while type 1 is strictly better off by sending m_1. Type 0's payoff is 0 when sending $m = 0$ because $\beta(m_0) = 0$. The indifference condition for type 0 is $v\beta(m_1) - c_0 = 0$, or equivalently, $\frac{c_0}{v} = \beta(m_1)$. Since

$$\beta(m_1) = \frac{p}{(1-p)\mu^0(m_1) + p},$$

$p < \beta(m_1) < 1$ should hold. Therefore, a semi-pooling equilibrium on m_0 exists if and only if $p < \frac{c_0}{v} < 1$. The equilibrium payoffs are $u(\sigma^{SP1}, 0) = 0$ and $u(\sigma^{SP1}, 1) = \beta(m_1)v - c_1 = c_0 - c_1$.

Now we consider which equilibrium is preferred by player I of each type.

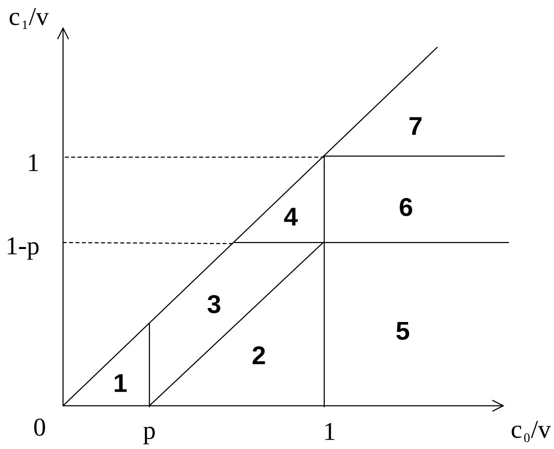

Figure 1: $p \leq \frac{1}{2}$

First consider the case of $p \leq \frac{1}{2}$. See Figure 1.

In Region 1, σ^{P0} and σ^{P1} exist, $u(\sigma^{P0}, 0) > u(\sigma^{P1}, 0)$ and $u(\sigma^{P0}, 1) > u(\sigma^{P1}, 1)$.

In Region 2, σ^{P0} and σ^{SP1} exist, $u(\sigma^{P0}, 0) > u(\sigma^{SP1}, 0)$ and $u(\sigma^{SP1}, 1) > u(\sigma^{P0}, 1)$.

In Region 3, σ^{P0} and σ^{SP1} exist, $u(\sigma^{P0}, 0) > u(\sigma^{SP1}, 0)$ and $u(\sigma^{P0}, 1) > u(\sigma^{SP1}, 1)$.

In Region 4, σ^{P0}, σ^{SP0} and σ^{SP1} exist, $u(\sigma^{P0}, 0) > u(\sigma^{SP0}, 0) > u(\sigma^{SP1}, 0)$ and $u(\sigma^{P0}, 1) > u(\sigma^{SP0}, 1) > u(\sigma^{SP1}, 1)$.

In Region 5, σ^{P0} and σ^{S} exist, $u(\sigma^{P0}, 0) > u(\sigma^{S}, 0)$ and $u(\sigma^{S}, 1) > u(\sigma^{P0}, 1)$.

In Region 6, σ^{P0}, σ^{SP0} and σ^{S} exist, $u(\sigma^{P0}, 0) > u(\sigma^{SP0}, 0) > u(\sigma^{S}, 0)$ and $u(\sigma^{P0}, 1) > u(\sigma^{SP0}, 1) = u(\sigma^{S}, 1)$.

In Region 7, only σ^{P0} exists.

For the case of $p > \frac{1}{2}$, see Figure 2. The same argument applies as in the case of $p \leq \frac{1}{2}$ for Region 1 through 7. In Region 8, σ^{P0}, σ^{P1} and σ^{SP0} exist, $u(\sigma^{P0}, 0) > u(\sigma^{SP0}, 0) > u(\sigma^{P1}, 0)$ and $u(\sigma^{P0}, 1) > u(\sigma^{SP0}, 1) > u(\sigma^{P1}, 1)$.

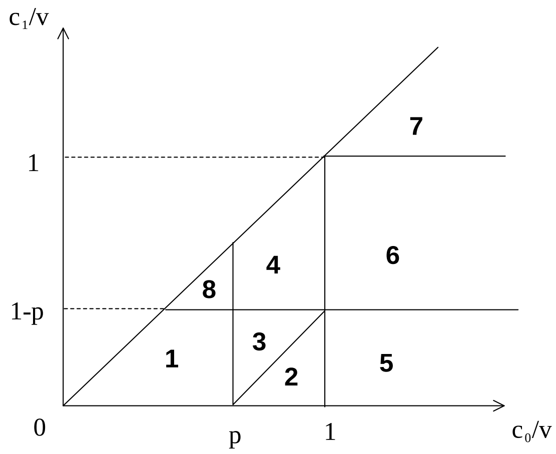

Figure 2: $p > \frac{1}{2}$

4 Equilibrium selection

This section reviews the existing literature on equilibrium selection and how the concepts and the ideas there work in our model.

4.1 Intuitive criterion and divine equilibrium

Intuitive criterion (Cho and Kreps [3]) puts restrictions on beliefs off the equilibrium path. Suppose message m is not used in an equilibrium. If for some types the highest possible payoff that player I can expect by sending m given that player II is taking a best response for some belief (not necessarily the equilibrium belief), then m should be interpreted as being sent by such types.

Divine equilibrium (Banks and Sobel [2]) puts further restrictions, in addition to intuitive criterion. If the set of beliefs for which type θ can receive a payoff higher than the equilibrium by sending m given player II is taking a best response is a proper subset of that for θ', then the belief given m should put probability zero on type θ.

Since these concepts is concerned with off-path beliefs, they say nothing if an equilibrium has no off-path information set. Therefore, all separating equilibria and semi-pooing equilibria in our model survive these equilibrium selection.

A pooling equilibrim on m_0 fails intuitive criterion if $v - c_0 < pv < v - c_1$, or equivalently, $\frac{c_1}{v} < 1 - p < \frac{c_0}{v}$. It fails to be divine if $pv < v - c_1$, or equivalently, $\frac{c_1}{v} < 1 - p$. Therefore, a pooling equilibrium fails to be divine if and only if player I of type 1 receives higher payoff in a separating equilibrium. Even when this is the case, a pooling equilibrium may or may not satisfy intuitive criterion.

A pooling equilibrium on m_1 meets intuitive criterion and is divine. Player I of both types is better off by sending m_0 if player II's belief $\beta(m_0)$ is higher than some threshold value. Therefore, intuitive criterion puts no restriction on $\beta(m_0)$. Since the threshold value of belief is lower for type 0, divine equilibrium requires $\beta(m_0) = 0$. This belief indeed constitutes a pooling equilibrium on $m = 1$.

4.2 Undefeated equilibrium

Mailath, Okuno-Fujiwara and Postlewaite [7] offers an alternative concept, called *undefeated equilibrium*, of restricting off-path beliefs. Their idea is that if player II observes an off-path message, it should be interpreted as player I's attempt to move to another equilibrium. Equilibrium σ is called to *defeat* equilibrium σ' if there is a message m which is used in σ but not in σ' and all types who sends m in σ receives higher payoffs in σ than in σ'. An equilibrium is said to be undefeated if there is no other equilibrium that defeats it.

Again, since these concepts is concerned with off-path beliefs, all separating equilibria and semi-pooing equilibria in our model is undefeated.

A pooling equilibrim on m_0 fails to be undefeated if $pv < v - c_1$. In this case, a separating equilibrium exists and gives higher payoff to player I of type 1. Since only type 1 sends m_1 in a separating equilbrium, a pooling equilibrium on m_0 is defeated by a separating equilibrium.

A pooling equilibirum on m_1 is defeated by a pooling equilibrium on m_0 becase player I of both types receives a higher payoff in the latter equilibirum and messege m_0 is used only in the latter equilibrium.

4.3 Evolutionary equilibrium selection

There are several works, such as Nöldeke and Samuelson [9], Amaya [1] and Jacobsen, Jensen and Sloth [5], which analyze evolutionary dynamics to study equilibrium selection in signaling games. All these papers consider a discrete model with generic payoffs to do away with semi-pooling equilibria.

Nöldeke and Samuelson [9] work on an evolutionary dynamic of Kandori, Mailath and Rob [6], where there are many agents in the role of player I and they meet a single agent in the role of player II. Nöldeke and Samuelson [9] show that a pooling equilibrium is stable if it is undefeated and divine, while a separating equilibrium is stable if it is undefeated.

Amaya [1] works on Kandori-Mailath-Rob dynamic, in which there are many agents in the role of both players and they are randomly matched. Jacobsen, Jensen and Sloth [5] work on an evolutionary dynamic of Young [12]. These papers show that Riley equilibrium (Pareto efficient separating equilibrium) is stable[1].

5 Perfect foresight dynamics

We consider *perfect foresight dynamics* proposed by Matsui and Matsuyama [8].

5.1 Dynamics

There is a unit mass of agents in the role of player I and that of agents in the role of player II. The type of each agent in the role of player I is fixed over time. Proportion $1-p$ of player I agents has type 0 and proportion p has type 1. Time is continuous. At each point in time, player I and player II are randomly matched and play the signaling game G. We assume that player I cannot switch messages at every point in time. Instead, every agent in the role of player I must make a comminment to a particular message for a random time interval. Time instants at which each player can switch actions follow a Poisson process with the mean arrival rate λ. The processes are independent across players.

On the other hand, we assume that player II can revise beliefs and switch actions at any point in time. At any time t, each agent in the role of player II forms belief $\beta(t) : M \to [0,1]$ that is consistent with player I's actual behavior, and play strategy $\rho(t) : M \to [0,1]$, which is optimal given $\beta(t)$. We assume all agents of player II share the same belief $\beta(t)$. The optimality of $\rho(t)$ implies $\rho(t)(m) = \beta(t)(m)$ for all t and m.

The message distribution among type θ agents at time t is denoted by

$$x^\theta(t) = (x^\theta(t)(m_0), x^\theta(t)(m_1)),$$

[1]The three papers mentioned here impose some assumptions on the structure of the game and thus the results mentioned here do not necessarily apply directly to our model

and let $x(t) = (x^0(t), x^1(t))$.

Since player II forms a consistent belief at any time, the belief $\beta(t)$ satisfies

$$\beta(t)(m) = \frac{px^1(t)(m)}{(1-p)x^0(t)(m) + px^1(t)(m)}$$

if the denominator is positive.

We call $x(t)$ the *message state* at time t. The state of the society at time t is a pair $(x(t), \beta(t))$, and we consider the evolution of the state over time. Notice if a type θ agent sends m at time t and the belief is $\beta(t)$, he receives a payoff $u(m, \beta(t)(m), \theta)$, because $\rho(t)(m) = \beta(t)(m)$. A rational player anticipates the future evolution of state $(x(t), \beta(t))$, and, if given the opportunity to switch actions, commits to an action that maximizes the expected discounted payoff. Since the duration of the commitment has an exponential distribution with the mean $\frac{1}{\lambda}$, the expected discounted payoff for type θ of commiting to message m at time t with a given anticipated path $(x(t), \beta(t))$ is represented by

$$\begin{aligned} V^\theta(t)(m) &= (\lambda+\alpha)\int_0^\infty e^{-(\lambda+\alpha)s} u(m, \beta(t+s)(m), \theta) ds \\ &= (\lambda+\alpha)\int_0^\infty e^{-(\lambda+\alpha)s} \beta(t+s)(m) \cdot v ds - I_m c_\theta \end{aligned} \tag{3}$$

where $\alpha > 0$ is a common discount rate, and

$$I_m = \begin{cases} 0 & \text{if} \quad m = m_0 \\ 1 & \text{if} \quad m = m_1 \end{cases}.$$

Endowed with perfect foresight, players correctly anticipate the future evolution of $(x(t), \beta(t))$. Hence, the messege distribution path $x(t)$ with an initial state $(x(0), \beta(0))$ satisfies

$$\dot{x}^\theta(t)(m) \in \begin{cases} \{\lambda(1 - x^\theta(t)(m))\} & \text{if } V^\theta(t)(m) > V^\theta(t)(1-m) \\ \{-\lambda x^\theta(t)(m)\} & \text{if } V^\theta(t)(m) < V^\theta(t)(1-m) \\ [-\lambda x^\theta(t)(m), \lambda(1 - x^\theta(t)(m))] & \text{if } V^\theta(t)(m) = V^\theta(t)(1-m) \end{cases} \tag{4}$$

for all t where $x^\theta(t)(m)$ is differentiable.

Finally define the *degree of friction* by $\delta = \frac{\alpha}{\lambda} > 0$.

5.2 Stability concepts

We say a message state x^* is *stationary* if a path $(x(\cdot), \beta(\cdot))$ such that $x(t) = x^*$ for all t is a perfect foresight equilibrium path for some $\beta(\cdot)$. Obviously, s^* is stationary if and only if there exists a belief β consistent with $x(\cdot)$ such that player I's strategy represented by x^* and β compose a sequential equilibrium. Our analysis of equilibrium selection is to study the stability of a message state x.

We use the concepts of local and global stability proposed by Matsui and Matsuyama [8]. An *absorbing* state x is locally stable in the sense that all perfect foresight dynamics beginning from a nearby state converge to x. A state is *globally accessible* if for any state x' there exists a perfect foresight path from x' converging to x.

Definition 2 *A message state x is absorbing if there exists $\varepsilon > 0$ such that message state of any perfect foresight equilibrium path whose message state begins from $B_\varepsilon(x)$ converges to x. A sequential equilibrium σ of the signaling game G is absorbing under the perfect foresight dynamics if the corresponding message state is absorbing.*

Definition 3 *A message state x is globally accessible if for any x', there exists a perfect foresight equilibrium path whose message state starts from x' and converges to x. A sequential equilibrium σ of the signaling game G is globally accessible under the perfect foresight dynamics if the corresponding message state is globally accessible.*

It is also convenient to use the concept of *linear stability* proposed by Oyama [10]. We first define a *linear path* from a message state x' to x by

$$x(t) = x - (x - x')e^{-\lambda t}.$$

Definition 4 *A message state x is linearly stable if for any x', the linear path from x' to x is a perfect foresight equilibrium path from x' with some belief path $\beta(\cdot)$. A sequential equilibrium σ of the signaling game G is linearly stable under the perfect foresight dynamics if the corresponding message state is linearly stable.*

Clearly, if a message state x is linearly stable, it is globally accessible.

6 Results

6.1 Stability of Separating Equilibrium

Proposition 1 *If $\frac{c_0}{v} \geq 1$ and $\frac{c_1}{v} < 1 - p$, the separating equilibrium σ^S is globally accessible.*

The proof is by construction. The objective is to show that for any message state x there exists a perfect foresight path from x to the separating equilibrium message state. If the linear path from x to the separating equilibrium message state is a perfect foresight path, the work is done. In other cases, consider a path (which is not necessarily a perfect foresight equilibrium path) such that type 0 chooses m_0 at any revision opportunity, type 1 chooses m_0 before time τ and m_1 after τ. It is shown that if the "switching point" τ is large enough, with expectation of this path, player I of type 1's expected discounted payoff evaluated at τ from choosing m_1 is greater than equal to that from choosing m_0. Let τ^* be the minimum value of τ for which this condition is satisfied. We show that the path with "switching point" τ^* is a perfect foresight path.

Proof:
Let the initial message state be

$$\begin{aligned} x(0) =& ((x^0(0)(m_0), x^0(0)(m_1)), (x^1(0)(m_0), x^1(0)(m_1))) \\ =& ((1 - x_0^0, x_0^0), (1 - x_0^1, x_0^1)). \end{aligned}$$

Consider a path beginning from $x(0)$ such that type 0 chooses m_0 at any revision opportunity, type 1 chooses m_0 before time τ and m_1 after τ. We call it a "switching path at τ".

Notice any switching path at τ converges to the message state corresponding to σ^S and switching path at $\tau = 0$ is a linear path to σ^S. We show that for any initial message state $x(0)$, there exists τ^* such that the switching path at τ^* is a perfect foresight path.

On the switching path at τ, the population state at time t is

$$\begin{aligned} x^\theta(t)(m_0) &= 1 - x_0^\theta e^{-\lambda t} \\ x^\theta(t)(m_1) &= x_0^\theta e^{-\lambda t} \end{aligned}$$

for $t \le \tau$ and $\theta \in \{0,1\}$. For $t = \tau + s$ with $s \ge 0$,

$$
\begin{aligned}
x^0(\tau+s)(m_0) &= 1 - x_0^\theta e^{-\lambda(\tau+s)} = 1 - x_0^\theta e^{-\lambda\tau} \cdot e^{-\lambda s} \\
x^0(\tau+s)(m_1) &= x_0^\theta e^{-\lambda(\tau+s)} = x_0^\theta e^{-\lambda\tau} \cdot e^{-\lambda s} \\
x^1(\tau+s)(m_0) &= (1 - x_0^1 e^{-\lambda\tau}) e^{-\lambda s} \\
x^1(\tau+s)(m_1) &= 1 - (1 - x_0^1 e^{-\lambda\tau}) e^{-\lambda s}.
\end{aligned}
$$

Thus, the belief consistent with the path is

$$
\begin{aligned}
\beta(t)(m_0) &= \frac{p(1 - x_0^1 e^{-\lambda t})}{(1-p)(1 - x_0^0 e^{-\lambda t}) + p(1 - x_0^1 e^{-\lambda t})} \\
\beta(t)(m_1) &= \frac{p x_0^1}{(1-p)x_0^0 + p x_0^1}
\end{aligned}
$$

for $t \le \tau$. For $t = \tau + s$ with $s \ge 0$,

$$
\begin{aligned}
\beta(\tau+s)(m_0) &= \frac{p(1 - x_0^1 e^{-\lambda\tau}) e^{-\lambda s}}{(1-p)(1 - x_0^0 e^{-\lambda\tau} \cdot e^{-\lambda s}) + p(1 - x_0^1 e^{-\lambda\tau}) e^{-\lambda s}} \\
\beta(\tau+s)(m_1) &= \frac{p(1 - (1 - x_0^1 e^{-\lambda\tau}) e^{\lambda s})}{(1-p)x_0^0 e^{-\lambda\tau} \cdot e^{-\lambda s} + p(1 - (1 - x_0^1 e^{-\lambda\tau}) e^{\lambda s})}.
\end{aligned}
$$

Consider the expected discounted payoff for type 1 evaluated at time τ.

$$
V^1(\tau)(m_0) = (\lambda+\alpha)v \int_0^\infty e^{-(\lambda+\alpha)s} \cdot \frac{p(1 - x_0^1 e^{-\lambda\tau}) e^{-\lambda s}}{(1-p)(1 - x_0^0 e^{-\lambda\tau} \cdot e^{-\lambda s}) + p(1 - x_0^1 e^{-\lambda\tau}) e^{-\lambda s}} ds.
$$

Thus,

$$
\begin{aligned}
\lim_{\tau\to\infty} V^1(\tau)(m_0) =& (\lambda+\alpha)v \int_0^\infty e^{-(\lambda+\alpha)s} \cdot \frac{p e^{-\lambda s}}{(1-p) + p e^{-\lambda s}} ds \\
<& (\lambda+\alpha)v \int_0^\infty e^{-(\lambda+\alpha)s} \cdot p ds \\
=& pv.
\end{aligned}
$$

Also,

$$
V^1(\tau)(m_0) = (\lambda+\alpha)v \int_0^\infty e^{-(\lambda+\alpha)s} \cdot \frac{p(1 - (1 - x_0^1 e^{-\lambda\tau}) e^{\lambda s})}{(1-p)x_0^0 e^{-\lambda\tau} \cdot e^{-\lambda s} + p(1 - (1 - x_0^1 e^{-\lambda\tau}) e^{\lambda s})} ds - c_1.
$$

Thus,

$$
\lim_{\tau\to\infty} V^1(\tau)(m_1) = v - c_1.
$$

Therefore, for sufficiently large τ, $V^1(\tau)(m_1) > V^1(\tau)(m_0)$ holds. Let $\Delta V^1(\tau) = V^1(\tau)(m_1) - V^1(\tau)(m_0)$. It is easily verified that $\Delta V^1(\tau)$ is continuous. Define τ^* by

$$\tau^* = \min\{\tau | \Delta V^1(\tau) \geq 0\}.$$

If $\tau^* = 0$, the linear path from $x(0)$ to the message state corresponding to the separating equilibrium is a perfect foresight dynamic.

For the case with $\tau^* > 0$, consider the switching path at τ. It can be verified that the expected discounted payoff for type 1 at time t with the anticipation of the the switching path at τ satisfies $V^1(t)(m_0) \geq V^1(t)(m_1)$ for $t \leq \tau$ and $V^1(t)(m_1) \geq V^1(t)(m_0)$ for $t \geq \tau$. Therefore, the switching path at τ is a perfect foresight dynamic. ∎

Proposition 2 *If $\frac{c_0}{v} \geq 1$ and $\frac{c_1}{v} < 1 - p$, the separating equilibrium σ^S is absorbing.*

Proof:

It suffices to show that for any perfect foresight equilibrium path whose initial message state is in $B_\varepsilon(x)$, $V^0(0)(m_0) > V^0(0)(m_1)$ and $V^1(0)(m_1) > V^1(0)(m_0)$. Since m_0 is the dominant strategy for type 0, the incentive condition for type 0 is obviously satisfied.

Now we show the incentive condition for type 1. Let the initial messsage state be such that $x^0(0)(m_0) = 1 - \varepsilon_0$, $x^0(0)(m_1) = \varepsilon_0$, $x^1(0)(m_0) = \varepsilon_1$ and $x^1(0)(m_1) = 1 - \varepsilon_1$.

In any perfect foresight equilibrium path, type 0 switches to m_0 at any revision opportunity. The expected discounted payoff for type 1 of choosing m_0 with the expectation of perfect foresight path is at most that with the expectation that both types switches to m_0 at any revision opportunity. Therefore,

$$\begin{aligned}
V^1(0)(m_0) \leq &(\lambda+\alpha)\int_0^\infty e^{-(\lambda+\alpha)s}\frac{p(1-(1-\varepsilon_1)e^{-\lambda s})}{(1-p)(1-\varepsilon_0 e^{-\lambda s}) + p(1-(1-\varepsilon_1)e^{-\lambda s})} v ds \\
<&pv(\lambda+\alpha)\int_0^\infty e^{-(\lambda+\alpha)s}(1-(1-\varepsilon_1)e^{-\lambda s})ds \\
=&pv(\lambda+\alpha)\int_0^\infty e^{-(\lambda+\alpha)s} - (1-\varepsilon_1)e^{-(2\lambda+\alpha)s}ds \\
=&pv\big(1-(1-\varepsilon_1)\cdot\frac{1+\delta}{2+\delta}\big) \\
<&pv.
\end{aligned}$$

The expected discounted payoff for type 1 of choosing m_1 with the expectation of perfect foresight path is at least that with the expectation that both types switches to m_0 at any revision opportunity. Therefore,

$$V^1(0)(m_1) \geq \frac{p(1-\varepsilon_1)}{(1-p)\varepsilon_0 + p(1-\varepsilon_1)} \cdot v - c_1.$$

For all $\eta > 0$, there exist sufficiently small ε_0 and ε_1 such that

$$V^1(0)(m_1) \geq v - c_1 - \eta.$$

Therefore, for any perfect foresight equilibrium path whose initial message state is in $B_\varepsilon(x)$, $V^1(0)(m_1) > V^1(0)(m_0)$. ∎

6.2 Stability of Pooling Equilibrium on m_0

Definition 5 *Pooling equilibrium on m_0, σ^{P0}, is called type-1 half dominant if $\frac{1}{2}pv > v - c_1$.*

The condition of type-1 half dominance means that if at least half of player I is sending m_0 and player II is taking a best responce to a consistent belief, then player I of both types receives a higher payoff by sending m_0. To check this, notice if at least half of player I is sending m_0, player II's belief upon observing m_0 is at least $\frac{1}{2}p$, implying the payoff for both types of player I of sending m_0 is at least $\frac{1}{2}pv$. On the other hand, the payoff for type θ of player I of sending m_1 is at most $v - c_\theta$.

The condition can be rewritten as $\frac{c_1}{v} \geq 1 - \frac{1}{2}p$. For the case of $p < \frac{1}{2}$, σ^{P0} is type-1 half dominant in the shaded area in Figure 3. A similar picture can be drawn for the case of $p > \frac{1}{2}$.

Proposition 3 *If a pooling equilibrium on m_0, σ^{P0}, is type-1 half dominant, there exists $\bar{\delta}$ such that for all $\delta < \bar{\delta}$, σ^{P0} is linearly stable.*

Proof:
Let $x^* = ((1,0),(1,0))$ be the population state where all agents play the equilibrium strategy of σ^{P0} We show that if $\frac{1}{2}pv \geq v - c_1$, then for any initial condition $x(0)$, the linear path to x^*, associated with a consistent belief satisfies the equilibrium condition, that is, for $\theta \in \{1,2\}$, $V^\theta(t)(m_0) \geq V^\theta(t)(m_1)$ for all t along this linear path, given the belief.

Suppose, for any dynamic considered in this proof, the belief satisfies $\beta(t)(m) = 1 - p$ if it cannot be derived by Bayes' rule. Then, for any population state, the accompanying belief is uniquely determined. Thus, it

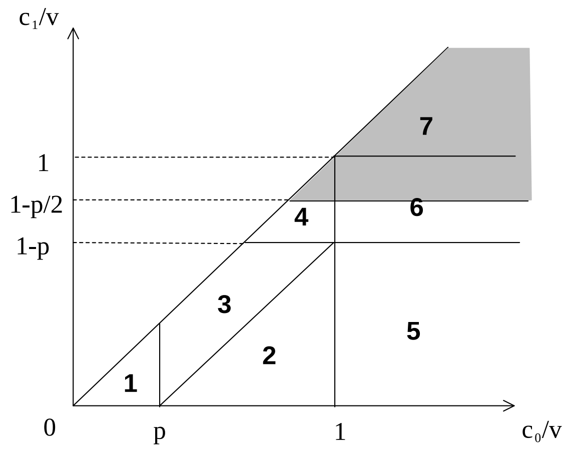

Figure 3: type-1 half dominance, $p \leq \frac{1}{2}$

suffices to show that the incentive condition is satisfied only at $t = 0$ for any initial condition $x(0)$.

For a initial condition $x(0)$, define x_0^θ by $x_0^\theta = x^\theta(0)(m_1)$ for $\theta = 1, 2$. Then, on the linear path to x^*,

$$x^\theta(t)(m_0) = 1 - x_0^\theta e^{-\lambda t}$$
$$x^\theta(t)(m_1) = x_0^\theta e^{-\lambda t}$$

for all t, for all θ. We have

$$\beta(t)(m_0) = \frac{p(1 - x_0^1 e^{-\lambda t})}{(1-p)(1 - x_0^0 e^{-\lambda t}) + p(1 - x_0^1 e^{-\lambda t})}$$

for all t, with only exception at $t = 0$ in the case of $x_0^0 = x_0^1 = 1$. Notice $\beta(t)(m_0) \geq p(1 - e^{-\lambda t})$ for all t.

The expected discounted payoff for type θ to message m_0 at $t = 0$ is given by

$$
\begin{aligned}
V^{\theta}(0)(m_0) &= (\lambda+\alpha)\int_0^{\infty} e^{-(\lambda+\alpha)s}\beta(s)(m_0)\cdot vds \\
&\geq (\lambda+\alpha)\int_0^{\infty} e^{-(\lambda+\alpha)s}p(1-e^{-\lambda s})\cdot vds \\
&= (\lambda+\alpha)pv\int_0^{\infty} e^{-(\lambda+\alpha)s} - e^{-(2\lambda+\alpha)s}\cdot vds \\
&= (\lambda+\alpha)pv\left(\frac{1}{\lambda+\alpha} - \frac{1}{2\lambda+\alpha}\right) \\
&= pv\left(1-\frac{1+\delta}{2+\delta}\right).
\end{aligned}
$$

The expected discounted payoff for type θ to message m_1 at $t=0$ is at most $v-c_\theta$, or equivalently,

$$V^{\theta}(0)(m_1) \leq v - c_\theta$$

The condition of type-1 half dominance, $\frac{1}{2}pv > v - c_1$, implies that for sufficiently small δ,

$$V^{\theta}(0)(m_0) > V^{\theta}(0)(m_1).$$

∎

Proposition 4 *If a pooling equilibrium on m_0, σ^{P0}, is type-1 half dominant, σ^{P0} is absorbing for all δ.*

Proof:
It suffices to show that for any perfect foresight equilibrium path whose initial message state is in $B_\varepsilon(x)$, $V^{\theta}(0)(m_0) > V^{\theta}(0)(m_1)$ for $\theta \in \{0,1\}$.

Let the initial messege state $x(0)$ be such that $x^{\theta}(0)(m_0) = 1-\varepsilon_\theta$ and $x^{\theta}(0)(m_1) = \varepsilon_\theta$ for $\theta \in \{0,1\}$.

First, for $\theta \in \{0,1\}$,

$$V^{\theta}(0)(m_1) \leq v - c_\theta \leq v - c_1. \tag{5}$$

Next, for type $\theta \in \{0,1\}$, the expected discounted payoff of commiting to message m_0 at time $t=0$ with anticipation of any path $(x(t),\beta(t))$ is at least the discounted payoff with anticipation that at every revision opportunity type 0 switches to m_0 and type 1 switches to m_1.

Therefore, for $\theta \in \{0,1\}$,

$$\begin{aligned}
V^\theta(0)(m_0) \geq & (\lambda+\alpha)\int_0^\infty e^{-(\lambda+\alpha)s}\beta(s)(m_0)v ds \\
= & (\lambda+\alpha)pv\int_0^\infty e^{-(\lambda+\alpha)s}\frac{(1-\varepsilon_1)e^{-\lambda s}}{(1-p)(1-\varepsilon_0 e^{-\lambda s})+p(1-\varepsilon_1)e^{-\lambda s}}v ds \\
= & (\lambda+\alpha)pv\int_0^\infty \frac{(1-\varepsilon_1)e^{-(2\lambda+\alpha)s}}{(1-p)(1-\varepsilon_0 e^{-\lambda s})+p(1-\varepsilon_1)e^{-\lambda s}} ds \\
\geq & (\lambda+\alpha)pv(1-\varepsilon_1)\int_0^\infty e^{-(2\lambda+\alpha)s} ds \\
= & pv(1-\varepsilon_1)(\frac{1+\delta}{2+\delta}) \\
> & \frac{1}{2}(1-\varepsilon_1)pv. \qquad (6)
\end{aligned}$$

From (5) and (6), if σ^{P0} is type-1 half dominant, then for sufficiently small ε_1, $V^\theta(0)(m_0) > V^\theta(0)(m_1)$ holds for $\theta \in \{0,1\}$. ∎

6.3 Instability of Pooling Equilibrium on m_1

Proposition 5 *The pooling equilibrium on m_1, σ^{P1} is not absorbing.*

Proof:
We show that the linear path from the message state corresponding to σ^{P1} to the message state corresponding to σ^{P0} is a perfect foresight equilibrium path.

Let the initial message state $x(0)$ be such that $x^\theta(0)(m_0) = 0$ and $x^\theta(0)(m_1) = 1$ for $\theta \in \{0,1\}$. In the linear path considered here, the evolution of the message state is described by

$$x^\theta(0)(m_1) = 1 - e^{-\lambda t} x^\theta(0)(m_1) = e^{-\lambda t}$$

for $\theta \in \{0,1\}$. Therefore, the consistent belief satisfies $\beta(t)(m_0) = \beta(t)(m_1) = p$ for all $t > 0$. Thus,

$$V^\theta(t)(m_0) = pv > pv - c_\theta = V^\theta(t)(m_1)$$

for all t and $\theta \in \{0,1\}$, implying the incentive condition for the linear path to be a perfect foresight equilibrium path is satisfied. ∎

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