The Proof of Lemma 3.8

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We recall Lemma 3.8 in following and divide seven sections to prove it.

Lemma 3.8. Consider (1.1). For $\varepsilon_2 \leq \varepsilon < \varepsilon_5$, the following assertion (i)–(vii) hold:

(i) If $0 \leq \alpha \leq \frac{21\sigma}{50\varepsilon}$, then $H_1(u,\alpha)$ is a strictly decreasing function of u on $\left[0, \frac{21\sigma}{50\varepsilon}\right]$.

(ii) If $0 \leq u \leq \frac{21\sigma}{50\varepsilon}$, then $H_1(u,\alpha)$ is a strictly decreasing function of α on $\left[\frac{\sigma}{3\varepsilon}, \frac{21\sigma}{50\varepsilon}\right]$.

(iii) If $0 \leq \alpha \leq \frac{21\sigma}{50\varepsilon}$, then $I_1(u,\alpha)$ is a strictly increasing function of u on $(0, \hat{u}(\alpha))$ and is strictly decreasing function of u on $(\hat{u}(\alpha), \infty)$, where

$$\hat{u}(\alpha) \equiv \frac{1}{45\varepsilon}\left[14\sigma - 39\varepsilon\alpha + \sqrt{14\left(-171\varepsilon^2\alpha^2 + 57\varepsilon\sigma\alpha + 14\sigma^2\right)}\right] > 0.$$

Furthermore, $\hat{u}(\alpha)$ is a strictly decreasing function of α on $(\frac{\sigma}{6\varepsilon}, \frac{21\sigma}{50\varepsilon}]$, and $\hat{u}(\gamma) = \gamma$.

(iv) If $u \geq 0$, then $I_1(u,\alpha)$ is a strictly decreasing function of α on $\left[\frac{39\sigma}{100\varepsilon}, \frac{21\sigma}{50\varepsilon}\right]$.

(v) If $\frac{39\sigma}{100\varepsilon} \leq \alpha \leq \frac{21\sigma}{50\varepsilon}$, then $H_3(u,\alpha)$ is a strictly decreasing function of u on $[0, \frac{27\sigma}{100\varepsilon}]$.

(vi) If $0 \leq u \leq \frac{27\sigma}{100\varepsilon}$, then $H_3(u,\alpha)$ is a strictly increasing function of α on $\left[\frac{39\sigma}{100\varepsilon}, \frac{21\sigma}{50\varepsilon}\right]$.

(vii) $I_2(0,\alpha)$ is a negative and strictly decreasing functions of α on $\left[\frac{39\sigma}{100\varepsilon}, \frac{21\sigma}{50\varepsilon}\right]$.

where

$$H_1(u,\alpha) \equiv \frac{(\alpha - u)^{3/2}}{6\left[F(\alpha) - F(u)\right]^{3/2}}, \quad H_3(u,\alpha) \equiv \frac{u\,(p_2 - u)^{3/2}}{\left[F(p_2) - F(u)\right]^{3/2}},$$

$$
\begin{aligned}
I_1(u,\alpha) \quad &\equiv \quad \frac{2}{35}\left[-15\varepsilon u^3 - (39\varepsilon\alpha - 14\sigma)\,u^2 - \left(87\varepsilon\alpha^2 - 42\sigma\alpha\right)u - 279\varepsilon\alpha^3 \right.\\
&\quad\quad \left. +154\alpha^2\sigma - 210\rho\right],
\end{aligned}
$$

$$
I_2(u,\alpha) \equiv \frac{2}{15}\left[6au^3 + (12a\alpha - 5b)u^2 + (48a\alpha^2 - 20b\alpha)u - 96a\alpha^3 + 40b\alpha^2 - 15d\right].
$$

1. Proof of assertion (i)

We compute that

$$
\frac{\partial H_1(u,\alpha)}{\partial u} = \frac{\sqrt{3}}{144\left[F(\alpha) - F(\alpha)\right]^{5/2}}\bar{H}_1(u,\alpha), \tag{1.1}
$$

÷where

$$
\bar{H}_1(u,\alpha) \equiv 9\varepsilon u^2 + (6\varepsilon\alpha - 8\sigma)u + 3\varepsilon\alpha^2 - 4\sigma\alpha - 6\tau.
$$

Since $\tau \geq 0$, we observe that, for $0 \leq u \leq \frac{21\sigma}{50\varepsilon}$ and $0 \leq \alpha \leq \frac{21\sigma}{50\varepsilon}$,

$$
\begin{aligned}
\bar{H}_1(u,\alpha) \quad &\leq \quad u\,(9\varepsilon u + 6\varepsilon\alpha - 8\sigma) + (3\varepsilon\alpha - 4\sigma)\,\alpha \\
&\leq \quad u\left[9\varepsilon\left(\frac{21\sigma}{50\varepsilon}\right) + 6\varepsilon\left(\frac{21\sigma}{50\varepsilon}\right) - 8\sigma\right] + \alpha\left[3\varepsilon\left(\frac{21\sigma}{50\varepsilon}\right) - 4\sigma\right] \\
&= \quad -\frac{17\sigma u}{10} - \frac{137\sigma\alpha}{50} < 0.
\end{aligned}
$$

So by (1.1), $\partial H_1(u,\alpha)/\partial u < 0$ for $0 \leq u \leq \frac{21\sigma}{50\varepsilon}$ and $0 \leq \alpha \leq \frac{21\sigma}{50\varepsilon}$. Hence, assertion (i) holds.

2. Proof of assertion (ii)

We compute that

$$
\frac{\partial H_1(u,\alpha)}{\partial \alpha} = \frac{(\alpha - u)^{3/2}}{48\left[F(\alpha) - F(u)\right]^{5/2}}\tilde{H}_1(u,\alpha),
$$

where

$$
\tilde{H}_1(u,\alpha) \equiv 3\varepsilon u^2 + (6\varepsilon\alpha - 4\sigma)u + 9\varepsilon\alpha^2 - 8\sigma\alpha - 6\tau.
$$

Since $\tau \geq 0$, we observe that, for $\frac{\sigma}{3\varepsilon} \leq \alpha \leq \frac{21\sigma}{50\varepsilon}$,

$$\begin{aligned}
\tilde{H}_1(\frac{21\sigma}{50\varepsilon}, \alpha) &= \frac{1}{48\varepsilon}\left(432\varepsilon^2\alpha^2 - 264\varepsilon\sigma\alpha - 55\sigma^2\right) - 6\tau \\
&< \frac{1}{48\varepsilon}\left[432\varepsilon^2\left(\frac{21\sigma}{50\varepsilon}\right)^2 - 264\varepsilon\sigma\left(\frac{\sigma}{3\varepsilon}\right) - 55\sigma^2\right] \\
&= -\frac{41747\sigma^2}{30000\varepsilon} < 0, \qquad (2.1)
\end{aligned}$$

$$\begin{aligned}
\tilde{H}_1(0, \alpha) &= 9\varepsilon\alpha^2 - 8\sigma\alpha - 6\tau < 9\varepsilon\left(\frac{21\sigma}{50\varepsilon}\right)^2 - 8\sigma\left(\frac{39\sigma}{100\varepsilon}\right) - 6\tau \\
&= -\frac{1}{7500\varepsilon}\left(8093\sigma^2 + 45000\tau\varepsilon\right) < 0. \qquad (2.2)
\end{aligned}$$

Since $\tilde{H}_1(u, \alpha)$ is a quadratic polynomial of u with positive leading coefficient, and by (2.1) and (2.2), we see that $\tilde{H}_1(u, \alpha) < 0$ for $0 \leq u \leq \frac{21\sigma}{50\varepsilon}$ and $\frac{\sigma}{3\varepsilon} \leq \alpha \leq \frac{21\sigma}{50\varepsilon}$. Hence, assertion (ii) holds.

3. Proof of assertion (iii)

We see that $\hat{u}(\alpha)$ is well-defined for $0 \leq \alpha \leq 21\sigma/(50\varepsilon)$ because

$$-171\varepsilon^2\alpha^2 + 57\varepsilon\sigma\alpha + 14\sigma^2\Big|_{\alpha=\frac{21\sigma}{50\varepsilon}} = \frac{19439}{2500}\sigma^2 > 0.$$

We compute and find that

$$\frac{\partial \hat{u}(\alpha)}{\partial \alpha} = \frac{266\varepsilon\left(\frac{\sigma}{6\varepsilon} - \alpha\right)}{5\sqrt{14\left(-171\varepsilon^2\alpha^2 + 57\varepsilon\sigma\alpha + 14\sigma^2\right)}} - \frac{13}{15} < 0 \quad \text{for } \frac{\sigma}{6\varepsilon} < \alpha < \frac{21\sigma}{50\varepsilon}.$$

Then $\hat{u}(\alpha)$ is a strictly decreasing function of α on $(\frac{\sigma}{6\varepsilon}, \frac{21\sigma}{50\varepsilon}]$. We further compute that

$$\frac{\partial I_1(u, \alpha)}{\partial u} = \frac{2}{35}\left[-45\varepsilon u^2 + (-78\varepsilon\alpha + 28\sigma)u - 87\varepsilon\alpha^2 + 42\sigma\alpha\right]. \qquad (3.1)$$

By (3.1), it is easy to see that

$$\left.\frac{\partial I_1(u, \alpha)}{\partial u}\right|_{u=\hat{u}(\alpha)} = 0, \qquad (3.2)$$

$$\left.\frac{\partial I_1(u,\alpha)}{\partial u}\right|_{u=0} = 87\varepsilon\left(\frac{42\sigma}{87\varepsilon}-\alpha\right)\alpha > 0 \quad \text{for } 0 \le \alpha \le \frac{21\sigma}{50\varepsilon}. \tag{3.3}$$

Since $\partial I_1(u,\alpha)/\partial u$ is a quadratic polynomial of u with negative leading coefficient, and by (3.2) and (3.3), we observe that

$$\frac{\partial I_1(u,\alpha)}{\partial u}\begin{cases} >0 & \text{for } 0<u<\hat{u}(\alpha),\\ =0 & \text{for } u=\hat{u}(\alpha),\\ <0 & \text{for } u>\hat{u}(\alpha).\end{cases}$$

Finally, we directly compute that $\hat{u}(\gamma)=\gamma$. Hence, the assertion (iii) holds.

4. Proof of assertion (iv)

We compute that

$$\frac{\partial I_1(u,\alpha)}{\partial \alpha} = \frac{2}{35}\left[-39\varepsilon u^2 + (-174\varepsilon\alpha+42\sigma)\,u - 837\varepsilon\alpha^2 + 308\sigma\alpha\right].$$

Let

$$\begin{aligned}\Delta(\alpha) &\equiv (-174\varepsilon\alpha+42\sigma)^2 - 4\,(-39\varepsilon)\left(-837\varepsilon\alpha^2+308\sigma\alpha\right)\\ &= -100296\varepsilon^2\alpha^2 + 33432\varepsilon\sigma\alpha + 1764\sigma^2.\end{aligned}$$

We note that $\Delta(\alpha)$ is a quadratic polynomial of α with negative leading coefficient,

$$\Delta(0) = 1764\sigma^2 > 0 \quad \text{and} \quad \Delta(\frac{39\sigma}{100\varepsilon}) = -\frac{565677}{1250}\sigma^2 < 0.$$

So $\Delta(\alpha)<0$ for $\frac{39\sigma}{100\varepsilon} \le \alpha \le \frac{21\sigma}{50\varepsilon}$. It follows that

$$\frac{\partial I_1(u,\alpha)}{\partial \alpha} < 0 \quad \text{for } u \ge 0 \text{ and } \frac{39\sigma}{100\varepsilon} \le \alpha \le \frac{21\sigma}{50\varepsilon}.$$

Hence, assertion (iv) holds.

5. Proof of assertion (v)

Since $\tau \ge 0$, we compute and find that

$$\frac{\partial H_3(u,\alpha)}{\partial \alpha} = \frac{(\alpha-u)^{3/2}\,u}{8\left[F(\alpha)-F(u)\right]^{5/2}}\left[\bar{H}_3(u,\alpha)-6\tau\right]$$

$$\leq \frac{(\alpha-u)^{3/2}\,u}{8\left[F(\alpha)-F(u)\right]^{5/2}}\bar{H}_3(u,\alpha), \tag{5.1}$$

where

$$\bar{H}_3(u,\alpha) \equiv 3\varepsilon u^2 + (6\varepsilon\alpha - 4\sigma)\,u + 9\varepsilon\alpha^2 - 8\sigma\alpha,$$

Let $\alpha \in \left[\frac{39\sigma}{100\varepsilon}, \frac{21\sigma}{50\varepsilon}\right]$ be given. We compute and observe that, for $0 \leq u \leq \frac{27\sigma}{100\varepsilon}$,

$$\bar{H}_3(0,\alpha) \leq 9\varepsilon\alpha^2 - 8\sigma\alpha \leq 9\varepsilon\left(\frac{21\sigma}{50\varepsilon}\right)^2 - 8\sigma\left(\frac{39\sigma}{100\varepsilon}\right) = -\frac{3831}{2500}\frac{\sigma^2}{\varepsilon} < 0,$$

$$\begin{aligned}
\bar{H}_3(\frac{27\sigma}{100\varepsilon},\alpha) &= 3\varepsilon\left(\frac{27\sigma}{100\varepsilon}\right)^2 + 6\varepsilon\alpha\left(\frac{27\sigma}{100\varepsilon}\right) - 4\sigma\left(\frac{27\sigma}{100\varepsilon}\right) + 9\varepsilon\alpha^2 - 8\sigma\alpha \\
&\leq 3\varepsilon\left(\frac{27\sigma}{100\varepsilon}\right)^2 + 6\varepsilon\left(\frac{21\sigma}{50\varepsilon}\right)\left(\frac{27\sigma}{100\varepsilon}\right) - 4\sigma\left(\frac{27\sigma}{100\varepsilon}\right) \\
&\quad +9\varepsilon\left(\frac{21\sigma}{50\varepsilon}\right)^2 - 8\sigma\left(\frac{39\sigma}{100\varepsilon}\right) = -\frac{17133}{10000}\frac{\sigma^2}{\varepsilon} < 0.
\end{aligned}$$

It follows that $\bar{H}_3(u,\alpha) < 0$ for $0 \leq u \leq \frac{21\sigma}{50\varepsilon}$. So by (5.1), $H_3(u,\alpha)$ is a strictly decreasing function of α on $\left[\frac{39\sigma}{100\varepsilon}, \frac{21\sigma}{50\varepsilon}\right]$. Hence, assertion (v) holds.

6. Proof of assertion (vi)

We compute that

$$\frac{\partial H_3(u,\alpha)}{\partial u} = \frac{(\alpha-u)^{3/2}}{24\left[F(\alpha)-F(u)\right]^{5/2}}\left[\tilde{H}_3(u,\alpha) - 6\tau\,(u-2\alpha)\right], \tag{6.1}$$

where

$$\begin{aligned}
\tilde{H}_3(u,\alpha) \equiv{}& 21\varepsilon u^3 + (12\varepsilon\alpha - 16\sigma)u^2 + (3\varepsilon\alpha^2 - 4\alpha\sigma)u - 6\varepsilon\alpha^3 \\
&+8\alpha^2\sigma + 24\rho.
\end{aligned}$$

We assert that

$$\frac{\partial}{\partial\alpha}\tilde{H}_3(u,\alpha) > 0 \text{ for } u \geq 0 \text{ and } \frac{39\sigma}{100\varepsilon} \leq \alpha \leq \frac{21\sigma}{50\varepsilon}. \tag{6.2}$$

Indeed, we compute that

$$\frac{\partial}{\partial\alpha}\tilde{H}_3(u,\alpha) = 12\varepsilon u^2 + (6\varepsilon\alpha - 4\sigma)u - 18\varepsilon\alpha^2 + 16\alpha\sigma.$$

Then we observe that, for $\frac{39\sigma}{100\varepsilon} \le \alpha \le \frac{21\sigma}{50\varepsilon}$,

$$\begin{aligned} & (6\varepsilon\alpha - 4\sigma)^2 - 4\,(12\varepsilon)\left(-18\varepsilon\alpha^2 + 16\alpha\sigma\right) \\ = \;& 4\left(225\varepsilon^2\alpha^2 + 4\sigma^2 - 204\varepsilon\sigma\alpha\right) \\ < \;& 4\left[225\varepsilon^2\left(\frac{21\sigma}{50\varepsilon}\right)^2 + 4\sigma^2 - 204\varepsilon\sigma\left(\frac{39\sigma}{100\varepsilon}\right)\right] \\ = \;& -\frac{3587}{25}\sigma^2 < 0. \end{aligned} \tag{6.3}$$

Since $\partial\tilde{H}_3(u,\alpha)/\partial\alpha$ is a quadratic polynomial of u with positive leading coefficient, and by (6.3), we see that (6.3) holds. By (6.3), we obtain that

$$\tilde{H}_3(u,\alpha) \ge \tilde{H}_3(u, \frac{39\sigma}{100\varepsilon}) \quad \text{for } u \in \mathbf{R} \text{ and } \frac{39\sigma}{100\varepsilon} \le \alpha \le \frac{21\sigma}{50\varepsilon}. \tag{6.4}$$

We observe that

$$\frac{\partial\tilde{H}_3(u, \frac{39\sigma}{100\varepsilon})}{\partial u} = 63\varepsilon u^2 - \frac{566}{25}\sigma u - \frac{11037}{10000\varepsilon}\sigma^2 < 0 \quad \text{for } 0 \le u \le \frac{27\sigma}{100\varepsilon},$$

because

$$\left.\frac{\partial\tilde{H}_3(u, \frac{39\sigma}{100\varepsilon})}{\partial u}\right|_{u=0} = -\frac{11037}{10000\varepsilon}\sigma^2 < 0,$$

$$\left.\frac{\partial\tilde{H}_3(u, \frac{39\sigma}{100\varepsilon})}{\partial u}\right|_{u=\frac{27\sigma}{100\varepsilon}} = -\frac{13119}{5000}\frac{\sigma^2}{\varepsilon} < 0.$$

So by (6.4), we observe that, for $0 \le u \le \frac{27\sigma}{100\varepsilon}$ and $\frac{39\sigma}{100\varepsilon} \le \alpha \le \frac{21\sigma}{50\varepsilon}$,

$$\tilde{H}_3(u,\alpha) \ge \tilde{H}_3(\frac{261\sigma}{1000\varepsilon}, \frac{39\sigma}{100\varepsilon}) = \frac{3}{\varepsilon^2}\left(58353927 \times 10^{-9}\sigma^3 + 8\varepsilon^2\rho\right) > 0,$$

from which it follows that by (6.1), $H_3(u,\alpha)$ is a strictly increasing function of u on $[0, \frac{27\sigma}{100\varepsilon}]$. Hence, assertion (vi) holds.

7. Proof of assertion (vii)

We compute and observe that

$$\frac{\partial I_2(0,\alpha)}{\partial \alpha} = \frac{192\varepsilon\alpha}{5}(\frac{5\sigma}{18\varepsilon} - \alpha) < 0 \quad \text{for} \quad \frac{39\sigma}{100\varepsilon} \le \alpha \le \frac{21\sigma}{50\varepsilon},$$

which implies that $I_2(0,\alpha)$ is a strictly decreasing function of α on $\left[\frac{39\sigma}{100\varepsilon}, \frac{21\sigma}{50\varepsilon}\right]$. Moreover, for $\frac{39\sigma}{100\varepsilon} \le \alpha \le \frac{21\sigma}{50\varepsilon}$,

$$I_2(0,\alpha) \le I_2(0, \frac{39\sigma}{100\varepsilon}) = \frac{4056\sigma^3}{78125\varepsilon^2} - 2\rho < \frac{4056\sigma^3}{78125\varepsilon_2^2} - 2\rho = -\frac{401866\rho}{1953125} < 0.$$

Hence, assertion (vii) holds.