The Proof of Lemma 3.9(ii)

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Lemma 3.9(ii). Consider (1.1). If \( \varepsilon = \varepsilon_3 = \sqrt{\frac{31\sigma^3}{10000\varepsilon^3}} \), then \( T_{\varepsilon_3}'(\gamma) > 0 \) and \( T_{\varepsilon_3}'(p_2) < 0 \).

The proof of Lemma 3.9(ii) is similar the proof of Lemma 3.9(i). We recall the functions

\[
H_1(u, \alpha) \equiv \frac{(\alpha - u)^{3/2}}{6 [F(\alpha) - F(u)]^{3/2}}, \quad H_2(u, \alpha) \equiv \frac{6 [\theta(\alpha) - \theta(u)]}{(\alpha - u)^{3/2}},
\]

\[
I_1(u, \alpha) \equiv \frac{2}{35} \left[ -15 \varepsilon u^3 - (39 \varepsilon \alpha - 14 \sigma) u^2 - (87 \varepsilon \alpha^2 - 42 \sigma \alpha) u - 279 \varepsilon \alpha^3 + 154 \alpha^2 \sigma - 210 \rho \right].
\]

Next, we prove that \( T_{\varepsilon_3}'(\gamma) > 0 \) and \( T_{\varepsilon_3}'(p_2) < 0 \) in Sections 1 and 2, respectively.

We remark that most of the computation in this paper has been checked using the symbolic manipulator Maple 16.

1. Proof of \( T_{\varepsilon_3}'(\gamma) > 0 \)

Assume that \( \varepsilon = \varepsilon_3 \). By (3.34) in [1], we have that

\[
T_{\varepsilon_3}'(\gamma) \geq \frac{1}{2 \sqrt{2} \gamma} \sum_{i=0}^{5} \left[ H_1(\alpha_{i+1}, \beta_6) \left( \sqrt{\beta_6 - \alpha_{i+1} I_1(\alpha_{i+1}, \beta_6)} - \sqrt{\beta_6 - \alpha_i I_1(\alpha_i, \beta_6)} \right) 
+ H_1(\beta_i, \beta_6) \left( \sqrt{\beta_6 - \beta_{i+1} I_1(\beta_{i+1}, \beta_6)} - \sqrt{\beta_6 - \beta_i I_1(\beta_i, \beta_6)} \right) \right],
\]

(1.1)

where \( \alpha_i = i \tilde{\gamma}/6 \) and \( \beta_i = \tilde{\gamma} + i (\gamma - \tilde{\gamma})/6 \) for \( i = 0, 1, ..., 6 \). Let

\[
\alpha_i^* \equiv \frac{i \tilde{\gamma}}{6}, \quad \beta_i^* \equiv \frac{(6 - i) \tilde{\gamma} + i \gamma}{6}, \quad \text{and} \quad \beta_i^* \equiv \frac{(6 - i) \tilde{\gamma} + i \gamma}{6}
\]

for \( i = 0, 1, ..., 6 \), where

\[
\tilde{\gamma} \equiv \frac{1799 \sigma}{10000 \varepsilon_3} \quad \text{and} \quad \tilde{\gamma}^* \equiv \frac{9 \sigma}{50 \varepsilon_3}.
\]
By Lemma 3.3 in [1], we compute and observe that

\[
\hat{\gamma}_* < \hat{\gamma} = \frac{1 - 4 \cos(\frac{y_3}{2} + \frac{\pi}{3})}{9} \bigg|_{\varepsilon = \varepsilon_3} \frac{\sigma}{\varepsilon_3} \left( \approx \frac{17996\sigma}{10^5\varepsilon_3} \right) < \hat{\gamma}^* < \frac{21\sigma}{50\varepsilon_3},
\]

(1.2)

from which it follows that

\[
\alpha_i < \alpha_i < \alpha_i^* \quad \text{and} \quad \beta_i < \beta_i < \beta_i^* \quad \text{for} \quad i = 0, 1, \ldots, 6.
\]

(1.3)

By Lemma 3.8(i) in [1], \( H_1(u, \gamma) \) is a strictly decreasing function of \( u \) on \((0, \gamma)\). Also, by Lemma 3.1 in [1], \( H_2(u, \gamma) > 0 \) for \( 0 < u < \bar{\gamma} \) and \( H_2(u, \gamma) < 0 \) for \( \bar{\gamma} < u < \gamma \). Since

\[
\int H_2(u, \alpha) du = \sqrt{\alpha} - u I_1(u, \alpha),
\]

we observe that

\[
\sqrt{\beta_6 - \alpha_{i+1} I_1(\alpha_{i+1}, \beta_6)} - \sqrt{\beta_6 - \alpha_i I_1(\alpha_i, \beta_6)} = \int_{\alpha_i}^{\alpha_{i+1}} H_2(u, \alpha) du > 0
\]

and

\[
\sqrt{\beta_6 - \beta_{i+1} I_1(\beta_{i+1}, \beta_6)} - \sqrt{\beta_6 - \beta_i I_1(\beta_i, \beta_6)} = \int_{\beta_i}^{\beta_{i+1}} H_2(u, \alpha) du < 0.
\]

So by (1.1),

\[
T'_{\varepsilon_3}(\gamma) \geq \frac{1}{2\sqrt{2}\gamma} \sum_{i=0}^{5} \left[ H_1(\alpha_i^*, \gamma) \left( \sqrt{\gamma - \alpha_{i+1} I_1(\alpha_{i+1}, \gamma)} - \sqrt{\gamma - \alpha_i I_1(\alpha_i, \gamma)} \right) + H_1(\beta_i, \gamma) \left( \sqrt{\gamma - \beta_{i+1} I_1(\beta_{i+1}, \gamma)} - \sqrt{\gamma - \beta_i I_1(\beta_i, \gamma)} \right) \right].
\]

(1.4)

By Lemma 3.8(iii) in [1],

\[
I_1(u_2, \gamma) > I_1(u_1, \gamma) > I_1(0, \gamma) = \frac{964\rho}{1953} > 0 \quad \text{for} \quad 0 < u_1 < u_2 < \gamma.
\]

(1.5)

By Lemma 3.8(iii)(iv) in [1], (1.2)–(1.5), we compute and observe that

\[
2\sqrt{2}\gamma T'_{\varepsilon_3}(\gamma) \geq \sum_{i=0}^{5} H_1(\alpha_i^*, \gamma) \left[ \sqrt{\gamma - \alpha_{i+1} I_1(\alpha_{i+1}, \gamma)} - \sqrt{\gamma - \alpha_i I_1(\alpha_i, \gamma)} \right]
\]

2
\[
\sum_{i=0}^{4} H_1(\beta_{is}, \gamma) \left[ \sqrt{\gamma - \beta_i^s I_1(\beta_{is}, \gamma)} - \sqrt{\gamma - \beta_i^s I_1(\beta_i^s, \gamma)} \right] - H_1(\beta_{5s}, \gamma) \sqrt{\gamma - \beta_{5s} I_1(\beta_{5s}^s, \gamma)}.
\]

Let \( k = \tau / \sqrt{\sigma \rho} \). We compute and find that, for \( i = 0, 1, ..., 5 \) and \( j = 0, 1, ..., 4 \),

\[
H_1(\alpha_{i+1}^s, \gamma) \left[ \sqrt{\gamma - \alpha_{i+1}^s I_1(\alpha_{i+1}s, \gamma)} - \sqrt{\gamma - \alpha_i^s I_1(\alpha_i^s, \gamma)} \right] = \frac{c_i(k)}{(\sigma \rho)^{1/4}},
\]

\[
\sum_{i=0}^{4} H_1(\beta_{js}, \gamma) \left[ \sqrt{\gamma - \beta_j^s I_1(\beta_{js}, \gamma)} - \sqrt{\gamma - \beta_j^s I_1(\beta_j^s, \gamma)} \right] = \frac{d_j(k)}{(\sigma \rho)^{1/4}},
\]

and

\[
-H_1(\beta_{5s}, \gamma) \sqrt{\gamma - \beta_{5s} I_1(\beta_{5s}^s, \gamma)} = \frac{d_5(k)}{(\sigma \rho)^{1/4}}.
\]

So by (1.6),

\[
T''_{\zeta_3}(\gamma) \geq \frac{1}{2 \sqrt{2} \gamma \sigma \rho^{1/4}} \sum_{i=0}^{5} [c_i(k) + d_i(k)].
\]  \hspace{1cm} (1.7)

We note that functions \( c_i(k), d_i(k) \) for \( i = 0, 1, ..., 5 \) are all independent on \( \varepsilon, \sigma, \tau \) and \( \rho \). In fact, we see that the form of functions \( c_i(k), d_i(k) \) for \( i = 0, 1, ..., 5 \) are similar to the form of \( a_i(k) \) defined in the proof of Lemma 3.9(i) in [1] for \( i = 1, 2, 3 \). So we can prove the following assertion

\[
\sum_{i=0}^{5} [c_i(k) + d_i(k)] > 0 \text{ for } k \geq 0
\]  \hspace{1cm} (1.8)

by applying similar technique used to prove

\[
a_1(k) + a_2(k) + a_3(k) > 0 \text{ for } k \geq 0
\]

in the proof of Lemma 3.9(i) in [1]. See Figure 1.1. Since the proof of assertion (1.8) is easy and lengthy, we put the proof in Appendix A and omit it here. So by (1.7) and (1.8), \( T''_{\zeta_2}(\gamma) > 0 \).
Figure 1.1: The graph of $\sum_{i=0}^{5} [a_i(k) + b_i(k)]$ for $k \geq 0$.

2. Proof of $\mathcal{T}_{\varepsilon_3}'(p_2) < 0$

Assume that $\varepsilon = \varepsilon_3$. By (3.35) in [1], we have that

$$
\mathcal{T}_{\varepsilon_3}'(p_2) \leq \frac{1}{2\sqrt{2}p_2} \sum_{i=0}^{19} H_1(\alpha_i, \beta_{20}) \left[ \sqrt{\beta_{20} - \alpha_{i+1}} I_1(\alpha_{i+1}, \beta_{20}) - \sqrt{\beta_{20} - \alpha_i} I_1(\alpha_i, \beta_{20}) \right] 
+ H_1(\beta_{i+1}, \beta_{20}) \left[ \sqrt{\beta_{20} - \beta_{i+1}} I_1(\beta_{i+1}, \beta_{20}) - \sqrt{\beta_{20} - \beta_i} I_1(\beta_i, \beta_{20}) \right],
$$

(2.1)

where $\alpha_i = i\bar{p}_2/20$ and $\beta_i = \bar{p}_2 + i(p_2 - \bar{p}_2)/20$ for $i = 0, 1, ..., 20$. Let

$$
\alpha_i = \frac{i\bar{p}_2}{20}, \quad \alpha_i^* = \frac{ip_2^*}{20}, \quad \beta_i = \bar{p}_2 + \frac{i(p_2 - \bar{p}_2)}{20}, \quad \text{and} \quad \beta_i^* = \frac{ip_2^*}{20},
$$

where

$$
\bar{p}_2 \equiv \frac{1557897\sigma}{10^7\varepsilon_3}, \quad \bar{p}_2^* \equiv \frac{1557898\sigma}{10^7\varepsilon_3}, \quad p_2 \equiv \frac{4059387\sigma}{10^7\varepsilon_3}, \quad \text{and} \quad p_2^* \equiv \frac{4059388\sigma}{10^7\varepsilon_3}.
$$

By Lemmas 3.1 and 3.3 in [1], we compute and find that

$$
\bar{p}_2^* < \bar{p}_2 \left( \approx \frac{15578978\sigma}{10^8\varepsilon_3} \right) < \bar{p}_2^*
$$

(2.2)

and

$$
\frac{39\sigma}{100\varepsilon_3} < \frac{81\sigma}{200\varepsilon_3} < p_2 < p_2 \left( \approx \frac{40593877\sigma}{10^8\varepsilon_3} \right) < p_2^* < \frac{203\sigma}{500\varepsilon_3} < \frac{21\sigma}{50\varepsilon_3},
$$

(2.3)
So by (2.2)–(2.3), we observe that
\[ \alpha_{i*} < \alpha_i < \alpha_i^* \quad \text{and} \quad \beta_{i*} < \beta_i < \beta_i^* \quad \text{for} \quad i = 0, 1, \ldots, 20. \] (2.4)

By Lemma 3.1 in [1], \( H_2(u, p_2) \geq 0 \) for \( 0 < u < \bar{p}_2 \) and \( H_2(u, p_2) < 0 \) for \( \bar{p}_2 < u < p_2 \). Since
\[ \int H_2(u, \alpha) du = \sqrt{\alpha - u} I_1(u, \alpha), \]
we observe that
\[ \sqrt{\beta_{20}} - \alpha_{i+1} I_1(\alpha_{i+1}, \beta_{20}) - \sqrt{\beta_{20}} - \alpha_i I_1(\alpha_i, \beta_{20}) = \int_{\alpha_i}^{\alpha_{i+1}} H_2(u, \alpha) du > 0 \]
and
\[ \sqrt{\beta_{20}} - \beta_{i+1} I_1(\beta_{i+1}, \beta_{20}) - \sqrt{\beta_{20}} - \beta_i I_1(\beta_i, \beta_{20}) = \int_{\beta_i}^{\beta_{i+1}} H_2(u, \alpha) du < 0. \]

So by (2.1) and Lemma 3.8(i)(ii) in [1],
\[ T_{\varepsilon_3}(p_2) \leq \frac{1}{2\sqrt{2}p_2} \sum_{i=0}^{19} \left[ H_1(\alpha_{i*}, p_{2*}) \left( \sqrt{\beta_{20}} - \alpha_{i+1} I_1(\alpha_{i+1}, \beta_{20}) - \sqrt{\beta_{20}} - \alpha_i I_1(\alpha_i, \beta_{20}) \right) \right. \\
+ \left. H_1(\beta_{i+1}^*, p_{2*}^*) \left( \sqrt{\beta_{20}} - \beta_{i+1} I_1(\beta_{i+1}, \beta_{20}) - \sqrt{\beta_{20}} - \beta_i I_1(\beta_i, \beta_{20}) \right) \right]. \] (2.5)

In addition, we compute that \( I_1(0, p_{2*}^*) \approx 0.37\rho > 0 \) and \( I_1(\beta_{19}^*, p_{2*}^*) \approx 0.13\rho > 0 \). So by Lemma 3.8(iii)(iv) in [1] and (2.3), we observe that
\[ I_1(u, p_{2*}) \geq I_1(u, p_2) \geq I_1(u, p_{2*}^*) > 0 \quad \text{for} \quad 0 \leq u \leq \beta_{19}^*. \] (2.6)

By Lemma 3.8(iii) in [1], (2.3) and (2.4), we compute and observe that
\[ \beta_4 < \beta_4^* = \frac{10290981\sigma}{5 \times 10^7 \varepsilon_3} < \frac{7 (-131 + 7 \sqrt{13054}) \sigma}{22500 \varepsilon_3} = \frac{203 \sigma}{500 \varepsilon_3} \]
\[ < \frac{\hat{u}(p_{2*})}{\hat{u}(p_2)} < \frac{\hat{u}(p_{2*})}{\hat{u}(p_{2*})} < \frac{81 \sigma}{200 \varepsilon_3} = \frac{(-359b + \sqrt{5060566}) \sigma}{9000 \varepsilon_3} \]
\[ < \frac{8733077\sigma}{4 \times 10^7 \varepsilon_3} = \beta_5^* < \beta_5. \]
It follows that
\[ \beta_4 < \beta_4^* < \hat{u}(p_2) < \beta_5^* < \beta_5. \] (2.7)

By Lemma 3.8(iii)(iv) in [1], (2.2)–(2.7), we compute and observe that
\[ 2\sqrt{2}p_2T_{\varepsilon_1}^*(p_2) \leq \sum_{i=0}^{19} H_1(\alpha_{i*}, \beta_{20*}) \left[ \sqrt{\beta_{20}^* - \alpha_{i+1}I_1(\alpha_{i+1}, \beta_{20*})} - \sqrt{\beta_{20} - \alpha_i^*I_1(\alpha_{i*}, \beta_{20}^*)} \right] \]
\[ + \sum_{i=0}^{3} H_1(\beta_{i+1}^*, \beta_{20}) \left[ \sqrt{\beta_{20}^* - \beta_{i+1}I_1(\beta_{i+1}, \beta_{20*})} - \sqrt{\beta_{20} - \beta_i^*I_1(\beta_{i*}, \beta_{20})} \right] \]
\[ + \sum_{i=4}^{19} H_1(\beta_{i+1}^*, \beta_{20}) \left[ \sqrt{\beta_{20}^* - \beta_{i+1}I_1(\beta_{i+1}, \beta_{20*})} - \sqrt{\beta_{20} - \beta_i^*I_1(\beta_{i*}, \beta_{20})} \right] \]
\[ -H_1(\beta_{18}^*, \beta_{20})\sqrt{\beta_{20*} - \beta_{17}I_1(\beta_{17}, \beta_{20})}. \] (2.8)

Let \( k \equiv \tau/\sqrt{\sigma \rho} \). We compute and find that, for \( i = 0, 1, \ldots, 19 \),
\[ H_1(\alpha_{i*}, \beta_{20*}) \left[ \sqrt{\beta_{20}^* - \alpha_{i+1}I_1(\alpha_{i+1}, \beta_{20*})} - \sqrt{\beta_{20} - \alpha_i^*I_1(\alpha_{i*}, \beta_{20}^*)} \right] = \frac{c_i(k)}{(\sigma \rho)^{\frac{i}{2}}} \]
and
\[ \begin{cases} H_1(\beta_{i+1}^*, \beta_{20}) \left[ \sqrt{\beta_{20}^* - \beta_{i+1}I_1(\beta_{i+1}, \beta_{20*})} - \sqrt{\beta_{20} - \beta_i^*I_1(\beta_{i*}, \beta_{20})} \right] & \text{if } i = 0, 1, 2, 3, \\ H_1(\beta_{i+1}^*, \beta_{20}) \left[ \sqrt{\beta_{20}^* - \beta_{i+1}I_1(\beta_{i+1}, \beta_{20*})} - \sqrt{\beta_{20} - \beta_i^*I_1(\beta_{i*}, \beta_{20})} \right] & \text{if } i = 4, 5, \ldots, 18, \\ -H_1(\beta_{18}^*, \beta_{20})\sqrt{\beta_{20*} - \beta_{17}I_1(\beta_{17}, \beta_{20})} & \text{if } i = 19, \end{cases} \]
\[ = \frac{d_i(k)}{(\sigma \rho)^{\frac{i}{2}}}. \]

We note that functions \( c_i(k), d_i(k) \) for \( i = 0, 1, \ldots, 19 \) are all independent on \( \varepsilon, \sigma, \tau \) and \( \rho \). In fact, we see that the form of functions \( c_i(k), d_i(k) \) for \( i = 0, 1, \ldots, 5 \) are similar to the form of \( a_i(k) \) defined in proof of Lemma 3.9(i) in [1] for \( i = 1, 2, 3 \). So we can prove the following assertion
\[ \sum_{i=0}^{19} [c_i(k) + d_i(k)] < 0 \quad \text{for } k \geq 0. \] (2.9)
by applying similar technique used to prove
\[ a_1(k) + a_2(k) + a_3(k) > 0 \quad \text{for } k \geq 0 \]
in the proof of Lemma 3.9(i) in [1]. See Figure 2.1. Since the proof of assertion (2.9) is easy and lengthy, we put the proof in Appendix B and omit it here. So by (2.8) and (2.9), then $T'_s(p_2) < 0$.

3. Appendix A

In this section, we prove assertion (1.8). We apply symbolic manipulator Maple 16 to compute and obtain $c_i(k)$ and $d_i(k)$ for $i = 0, 1, ..., 5$. To prove assertion (1.8), we observe that assertion (1.8) holds for $k \geq 0$ if, and only if, all of inequalities (R1), (R2),..., (P10) hold for $k \geq 0$ where

- \textbf{(R1)} $\frac{33}{500}c_0(k) + d_0(k) > 0$
- \textbf{(R2)} $\frac{93}{500}c_0(k) + d_1(k) > 0$
- \textbf{(R3)} $\frac{152}{500}c_0(k) + d_2(k) > 0$
- \textbf{(R4)} $\frac{220}{500}c_0(k) + d_3(k) > 0$
- \textbf{(R5)} $\frac{380}{500}c_1(k) + d_4(k) > 0$
- \textbf{(R6)} $\frac{120}{500}c_1(k) + \frac{65}{500}d_5(k) > 0$
- \textbf{(R7)} $c_2(k) + \frac{209}{500}d_5(k) > 0$
- \textbf{(R8)} $c_3(k) + \frac{130}{500}d_5(k) > 0$
- \textbf{(R9)} $c_4(k) + \frac{83}{500}d_5(k) > 0$
- \textbf{(R10)} $c_5(k) + \frac{13}{500}d_5(k) > 0$

\textbf{Remark 1.} By symbolic manipulator Maple 16, we see that for $i = 0, 1, ..., 5$, denominators of $c_i(k) = [\hat{c}_i(k)]^{3/2} > 0$.
and

\[ \text{denominators of } d_i(k) = \left[ \tilde{d}_i(k) \right]^{3/2} > 0, \]

where \( \tilde{c}_i(k) \) and \( \tilde{d}_i(k) \) are polynomials of \( k \), cf. [1, p17, the definitions of \( a_1(k), a_2(k), \) and \( a_1(k) \)]. So we observe that inequalities (R1)–(R10) holds for \( k \geq 0 \) if, and only if,

\[ \text{Num(left side of inequality (Rl))} > 0 \quad \text{for} \quad k \geq 0 \quad \text{and} \quad l = 1, 2, ..., 10, \]

where

\[ \text{Num} \left( \frac{p}{q} \right) = p \quad \text{for} \quad p \in \mathbb{R} \quad \text{and} \quad q \in \mathbb{R}\setminus\{0\}. \]

Next, we begin to prove (R1)–(R10).

**The proof of inequality (R1).** Since we compute that

\[ (6.7)^{2/3} (93.3 + 52.4k) - (10)^{2/3} (71.5 + 37.2k) \approx 0.28 + 13.56k > 0, \]

we compute and observe that, for \( k \geq 0, \)

\[
\begin{align*}
\text{Num} \left( \frac{33}{500} c_0(k) + d_0(k) \right) &= 6 \left( \frac{10}{31} \right)^{3/4} \left[ \left( 5109068236873491\sqrt{273} - 3435696 \times 10^{10} \sqrt{3} \right) \\
&\quad \times \left( 2894575521409000 + 92382 \times 10^9 \sqrt{310} \right)^{3/2} \\
&\quad + \left( 2080574313641875 \times 10^{11} \sqrt{115} - 192980448 \times 10^{17} \sqrt{13809} \right) \\
&\quad \times \left( 2213857000 + 654000000 \sqrt{310} \right)^{3/2} \right] \\
&\approx 10^{26} \times \left[ 6.75 (93.37 + 52.46k)^{3/2} - 9.91 (71.41 + 37.14k)^{3/2} \right] \\
&> 10^{26} \times \left[ 6.7 (93.3 + 52.4k)^{3/2} - 10 (71.5 + 37.2k)^{3/2} \right] \\
&= 10^{26} \times \left\{ (6.7)^{2/3} (93.3 + 52.4k)^{3/2} - (10)^{2/3} (71.5 + 37.2k)^{3/2} \right\} > 0.
\end{align*}
\]

So inequality (R1) holds.
The proof of inequality (R2). Since we compute that
\[(190.3)^{2/3} (5.06 + 2.8k) - (3.58)^{2/3} (71.5 + 37.2k) \approx 8 \times 10^{-2} + 5.58k > 0,\]
we compute and observe that, for \(k \geq 0,\)
\[
Num \left( \frac{93}{500} c_0(k) + d_1(k) \right)
\approx 10^{28} \times \left[ 190.34 (5.07 + 2.85k)^{3/2} - 3.57(71.41 + 37.14k)^{3/2} \right]
> 10^{28} \times \left[ 190.3 (5.06 + 2.8k)^{3/2} - 3.58(71.5 + 37.2k)^{3/2} \right]
= 10^{28} \times \left\{ \left[ (190.3)^{2/3} (5.06 + 2.8k) \right]^{3/2} - \left[ (3.58)^{2/3} (71.5 + 37.2k) \right]^{3/2} \right\}
> 0.
\]
So inequality (R2) holds.

The proof of inequality (R3). Since we compute that
\[(31.1)^{2/3} (10.37 + 58.4k) - (1.72)^{2/3} (71.42 + 37.2k) \approx 2. \times 10^{-2} + 524.14k > 0,\]
we compute and observe that, for \(k \geq 0,\)
\[
Num \left( \frac{152}{500} c_0(k) + d_2(k) \right)
\approx 10^{29} \times \left[ 31.11 (10.38 + 58.41k)^{3/2} - 1.71(71.41 + 37.14k)^{3/2} \right]
> 10^{29} \times \left[ 31.1 (10.37 + 58.4k)^{3/2} - 1.72(71.42 + 37.2k)^{3/2} \right]
= 10^{29} \times \left\{ \left[ (31.1)^{2/3} (10.37 + 58.4k) \right]^{3/2} - \left[ (1.72)^{2/3} (71.42 + 37.2k) \right]^{3/2} \right\}
> 0.
\]
So inequality (R3) holds.

The proof of inequality (R4): Since we compute that
\[(45)^{2/3} (25.7 + 14.4k) - (9.6)^{2/3} (71.5 + 37.2k) \approx 2.17 + 14.15k > 0,\]
we compute and observe that, for \(k \geq 0,\)
\[
Num \left( \frac{220}{500} c_0(k) + d_3(k) \right)
\]
\[
\left( \approx 10^{29} \times \left[ 45.02 (25.75 + 14.47k)^{3/2} - 9.56(71.41 + 37.14k)^{3/2} \right] \right)
> 10^{29} \times \left[ 45 (25.7 + 14.4k)^{3/2} - 9.6(71.5 + 37.2k)^{3/2} \right]
= 10^{29} \times \left\{ \left[ (45)^{2/3} (25.7 + 14.4k) \right]^{3/2} - \left[ (9.6)^{2/3} (71.5 + 37.2k) \right]^{3/2} \right\}
> 0.
\]

So inequality (R4) holds.

**The proof of inequality (R5).** Since we compute that

\[
(3)^{2/3} (90.8 + 50.9k) - (89.6)^{2/3} (9.4 + 5.1k) \approx 0.65 + 3.75k > 0,
\]

we compute and observe that, for \( k \geq 0 \),

\[
Num \left( \frac{380}{500} c_1(k) + d_4(k) \right)
= 10^{29} \times \left[ 3.01 (90.85 + 50.97k)^{3/2} - 89.51(9.38 + 5.02k)^{3/2} \right]
> 10^{29} \times \left[ 3 (90.8 + 50.9k)^{3/2} - 89.6(9.4 + 5.1k)^{3/2} \right]
= 10^{29} \times \left\{ \left[ (3)^{2/3} (90.8 + 50.9k) \right]^{3/2} - \left[ (89.6)^{2/3} (9.4 + 5.1k) \right]^{3/2} \right\}
> 0.
\]

So inequality (R5) holds.

**The proof of inequality (R6).** Since we compute that

\[
(9.5)^{2/3} (7.56 + 4.2k) - (6.84)^{2/3} (9.4 + 5.1k) \approx 3 \times 10^{-2} + 0.46k > 0,
\]

we compute and observe that, for \( k \geq 0 \),

\[
Num \left( \frac{120}{500} c_1(k) + \frac{65}{500} d_5(k) \right)
= 10^{31} \times \left[ 9.50 (7.58 + 4.24k)^{3/2} - 6.82(9.38 + 5.02k)^{3/2} \right]
> 10^{31} \times \left[ 9.5 (7.56 + 4.2k)^{3/2} - 6.84(9.4 + 5.1k)^{3/2} \right]
= 10^{31} \times \left\{ \left[ (9.5)^{2/3} (7.56 + 4.2k) \right]^{3/2} - \left[ (6.84)^{2/3} (9.4 + 5.1k) \right]^{3/2} \right\}
\]
So inequality (R6) holds.

**The proof of inequality (R7).** Since we compute that

\[
(740)^{2/3} (7.5 + 4.2k) - (3)^{2/3} (80 + 44k) \approx 447.19 + 252.09k > 0,
\]

we compute and observe that, for \( k \geq 0 \),

\[
Num \left( c_2(k) + \frac{209}{500} d_5(k) \right)
\approx 10^{31} \times \left[ 740.75 (7.58 + 4.24k)^{3/2} - 2.19(79.10 + 43.27k)^{3/2} \right]
\]

\[
> 10^{31} \times \left[ 740 (7.5 + 4.2k)^{3/2} - 3(80 + 44k)^{3/2} \right]
\]

\[
= 10^{31} \times \left\{ \left( (740)^{2/3} (7.5 + 4.2k) \right)^{3/2} - \left[ (3)^{2/3} (80 + 44k) \right]^{3/2} \right\}
\]

\[
> 0.
\]

So inequality (R7) holds.

**The proof of inequality (R8).** Since we compute that

\[
(13.6)^{2/3} (7.5 + 4.2k) - (1.4)^{2/3} (32.7 + 18.2k) \approx 1.80 + 1.15k > 0,
\]

we compute and observe that, for \( k \geq 0 \),

\[
Num \left( c_3(k) + \frac{130}{500} d_5(k) \right)
\approx 10^{29} \times \left[ 13.63 (7.58 + 4.24k)^{3/2} - 1.36(32.63 + 18.10k)^{3/2} \right]
\]

\[
> 10^{29} \times \left[ 13.6 (7.5 + 4.2k)^{3/2} - 1.4(32.7 + 18.2k)^{3/2} \right]
\]

\[
= 10^{29} \times \left\{ \left( (13.6)^{2/3} (7.5 + 4.2k) \right)^{3/2} - \left[ (1.4)^{2/3} (32.7 + 18.2k) \right]^{3/2} \right\}
\]

\[
> 0.
\]

So inequality (R8) holds.

**The proof of inequality (R9).** Since we compute that

\[
(251.2)^{2/3} (7.55 + 4.22k) - (8.73)^{2/3} (70.7 + 39.55k) \approx 1.51 + 2.41k > 0,
\]

we compute and observe that, for \( k \geq 0 \),

\[
Num \left( c_4(k) + \frac{155}{500} d_5(k) \right)
\approx 10^{27} \times \left[ 251.23 (7.58 + 4.24k)^{3/2} - 1.51(70.7 + 39.55k)^{3/2} \right]
\]

\[
> 10^{27} \times \left[ 251.2 (7.5 + 4.2k)^{3/2} - 1.4(70.7 + 39.55k)^{3/2} \right]
\]

\[
= 10^{27} \times \left\{ \left( (251.2)^{2/3} (7.5 + 4.2k) \right)^{3/2} - \left[ (8.73)^{2/3} (70.7 + 39.55k) \right]^{3/2} \right\}
\]

\[
> 0.
\]
we compute and observe that, for $k \geq 0$,

$$
\text{Num} \left( c_4(k) + \frac{83}{500} d_5(k) > 0 \right)
\approx 10^{28} \times \left( 251.25 \left( 7.58 + 4.24k \right)^{3/2} - 8.71(70.66 + 39.53k)^{3/2} \right)
> 10^{28} \times \left( 251.2(7.55 + 4.22k)^{3/2} - 8.73(70.7 + 39.55k)^{3/2} \right)
= 10^{28} \times \left\{ \left( 251.2 \right)^{2/3} (7.55 + 4.22k) \right\}^{3/2} - \left( 8.73 \right)^{2/3} (70.7 + 39.55k)^{3/2}
> 0.
$$

So inequality (R9) holds.

**The proof of inequality (R10).** Since we compute that

$$(5.2)^{2/3} (7.5 + 4.2k) - (1.4)^{2/3} (11.7 + 6.6k) \approx 7.86 + 4.34k > 0,$$

we compute and observe that, for $k \geq 0$,

$$
\text{Num} \left( c_5(k) + \frac{13}{500} d_5(k) > 0 \right)
\approx 10^{31} \times \left[ 5.29 (7.58 + 4.24k)^{3/2} - 1.36(11.67 + 6.55k)^{3/2} \right]
> 10^{31} \times \left[ 5.2 (7.5 + 4.2k)^{3/2} - 1.4(11.7 + 6.6k)^{3/2} \right]
= 10^{31} \times \left\{ \left( 5.2 \right)^{2/3} (7.5 + 4.2k) \right\}^{3/2} - \left( 1.4 \right)^{2/3} (11.7 + 6.6k)^{3/2}
> 0.
$$

So inequality (R10) holds.

Thus the proof of assertion (1.8) is complete. ■

4. Appendix B

In this section, we prove assertion (2.9). We apply symbolic manipulator Maple 16 to compute and obtain $c_i(k)$ and $d_i(k)$ for $i = 0, 1, \ldots, 19$. To prove assertion (2.9), we observe that assertion (2.9) holds for $k \geq 0$ if, and only if, all of inequalities (P1), (P2),..., (P39) hold for $k \geq 0$ where
(P1) $\frac{231}{500} c_0(k) + d_0(k) < 0$
(P2) $\frac{574}{500} c_0(k) + d_1(k) < 0$
(P3) $\frac{108.7}{500} c_0(k) + d_2(k) < 0$
(P4) $\frac{146.9}{500} c_0(k) + d_3(k) < 0$
(P5) $\frac{153.9}{500} c_0(k) + \frac{422.5}{500} d_4(k) < 0$
(P6) $\frac{30}{500} c_1(k) + \frac{77.5}{500} d_4(k) < 0$
(P7) $\frac{228.6}{500} c_1(k) + d_5(k) < 0$
(P8) $\frac{241.4}{500} c_1(k) + \frac{465.6}{500} d_6(k) < 0$
(P9) $\frac{19}{500} c_2(k) + \frac{344}{500} d_6(k) < 0$
(P10) $\frac{306.7}{500} c_2(k) + d_7(k) < 0$
(P11) $\frac{174.3}{500} c_2(k) + \frac{263}{500} d_8(k) < 0$
(P12) $\frac{168.8}{500} c_3(k) + \frac{237}{500} d_8(k) < 0$
(P13) $\frac{331.2}{500} c_3(k) + \frac{437.5}{500} d_9(k) < 0$
(P14) $\frac{50.9}{500} c_4(k) + \frac{62.5}{500} d_9(k) < 0$
(P15) $\frac{427.1}{500} c_4(k) + d_{10}(k) < 0$
(P16) $\frac{22}{500} c_4(k) + \frac{26.2}{500} d_{11}(k) < 0$
(P17) $\frac{453}{500} c_5(k) + \frac{474.8}{500} d_{11}(k) < 0$
(P18) $\frac{47}{500} c_5(k) + \frac{48.7}{500} d_{12}(k) < 0$
(P19) $\frac{476.4}{500} c_6(k) + \frac{451.3}{500} d_{12}(k) < 0$
(P20) $\frac{23.6}{500} c_6(k) + \frac{22.5}{500} d_{13}(k) < 0$
(P21) $c_7(k) + \frac{432.2}{500} d_{13}(k) < 0$
(P22) $\frac{57.6}{500} c_8(k) + \frac{45.3}{500} d_{13}(k) < 0$
(P23) $\frac{442.4}{500} c_8(k) + \frac{352.8}{500} d_{14}(k) < 0$
(P24) $\frac{205.2}{500} c_9(k) + \frac{147.2}{500} d_{14}(k) < 0$
(P25) $\frac{294.8}{500} c_9(k) + \frac{220}{500} d_{15}(k) < 0$
(P26) $\frac{421.5}{500} c_{10}(k) + \frac{280}{500} d_{15}(k) < 0$
(P27) $\frac{78.5}{500} c_{10}(k) + \frac{56}{500} d_{16}(k) < 0$
(P28) $c_{11}(k) + \frac{313.4}{500} d_{16}(k) < 0$
(P29) $\frac{239.8}{500} c_{12}(k) + \frac{130.6}{500} d_{16}(k) < 0$
(P30) $\frac{260.2}{500} c_{12}(k) + \frac{159.1}{500} d_{17}(k) < 0$
(P31) \( c_{13}(k) + \frac{260.4}{500} d_{17}(k) < 0 \) \hspace{1cm} (P32) \( \frac{185.5}{500} c_{14}(k) + \frac{80.5}{500} d_{17}(k) < 0 \)

(P33) \( \frac{314.5}{500} c_{14}(k) + \frac{168}{500} d_{18}(k) < 0 \) \hspace{1cm} (P34) \( c_{15}(k) + \frac{214.7}{500} d_{18}(k) < 0 \)

(P35) \( \frac{356.5}{500} c_{16}(k) + \frac{177.2}{500} d_{18}(k) < 0 \) \hspace{1cm} (P36) \( \frac{143.5}{500} c_{16}(k) + \frac{82.8}{500} d_{19}(k) < 0 \)

(P37) \( c_{17}(k) + \frac{228.8}{500} d_{19}(k) < 0 \) \hspace{1cm} (P38) \( c_{18}(k) + \frac{119.7}{500} d_{19}(k) < 0 \)

(P39) \( c_{19}(k) + \frac{94.7}{500} d_{19}(k) < 0 \)

**Remark 2.** By symbolic manipulator Maple 16, we see that for \( i = 0, 1, \ldots, 19, \)

denominators of \( c_i(k) = [\tilde{c}_i(k)]^{3/2} > 0 \)

and

\[ \text{denominators of } d_i(k) = [\tilde{d}_i(k)]^{3/2} > 0, \]

where \( \tilde{c}_i(k) \) and \( \tilde{d}_i(k) \) are polynomials of \( k \), cf. [1, p17, the definitions of \( a_1(k), \]

\( a_2(k), \) and \( a_1(k) \)]. So we observe that inequalities (P1)–(P39) holds for \( k \geq 0 \) if, and only if,

\[ \text{Num(left side of inequality (Pl)) > 0 for } k \geq 0 \text{ and } l = 1, 2, \ldots, 39, \]

where

\[ \text{Num} \left( \frac{p}{q} \right) = p \text{ for } p \in \mathbb{R} \text{ and } q \in \mathbb{R} \setminus \{0\}. \]

Next, we begin to prove (P1)–(P39).

**The proof of inequality (P1).** Since we compute that

\[ (1027.62)^{2/3} (2.81 + 1.57k) - (6.46)^{2/3} (89.28 + 46.1k) \left( \approx -23.5 - 2.5 \times 10^{-2}k \right) < 0, \]

we compute and observe that, for \( k \geq 0, \)

\[ \text{Num} \left( \frac{231}{500} c_0(k) + d_0(k) \right) \]

\[ \left( \approx 10^{65} \times \left[ 1027.61 (2.80 + 1.56k)^{3/2} - 6.460(89.29 + 46.11k)^{3/2} \right] \right) \]
\[
10^{65} \times \left[ 1027.62 \left( 2.81 + 1.57k \right)^{3/2} - 6.460(89.28 + 46.1k)^{3/2} \right]
\]
\[
= 10^{65} \times \left\{ \left( 1027.62 \right)^{2/3} \left( 2.81 + 1.57k \right)^{3/2} - \left( 6.46 \right)^{2/3} (89.28 + 46.1k)^{3/2} \right\}
\]
\[
< 0.
\]
So inequality (P1) holds.

**The proof of inequality (P2).** Since we compute that

\[
(29.99)^{2/3} (35.8 + 20k) - (8.57)^{2/3} (89.2 + 46.1k) \approx -27.9 - 3.1 \times 10^{-3}k < 0,
\]
we compute and observe that, for \( k \geq 0, \)

\[
\text{Num} \left( \frac{67.4}{500} c_0(k) + d_1(k) \right)
\]
\[
\approx 10^{64} \times \left[ 29.98 \left( 35.73 + 19.99k \right)^{3/2} - 8.572(89.29 + 46.11k)^{3/2} \right]
\]
\[
< 10^{64} \times \left[ 29.99 \left( 35.8 + 20k \right)^{3/2} - 8.57(89.2 + 46.1k)^{3/2} \right]
\]
\[
= 10^{64} \times \left\{ \left( 29.99 \right)^{2/3} (35.8 + 20k)^{3/2} - \left( 8.57 \right)^{2/3} (89.2 + 46.1k)^{3/2} \right\}
\]
\[
< 0.
\]
So inequality (P2) holds.

**The proof of inequality (P3).** Since we compute that

\[
(4.836)^{2/3} (97.3 + 54.46k) - (6.21)^{2/3} (89.25 + 46.1k) \approx -23.9 - 0.1k < 0,
\]
we compute and observe that, for \( k \geq 0, \)

\[
\text{Num} \left( \frac{108.7}{500} c_0(k) + d_2(k) \right)
\]
\[
\approx 10^{65} \times \left[ 4.835 \left( 97.29 + 54.45k \right)^{3/2} - 6.211(89.29 + 46.11k)^{3/2} \right]
\]
\[
< 10^{65} \times \left[ 4.836 \left( 97.3 + 54.46k \right)^{3/2} - 6.21(89.25 + 46.1k)^{3/2} \right]
\]
\[
= 10^{65} \times \left\{ \left( 4.836 \right)^{2/3} (97.3 + 54.46k)^{3/2} - \left( 6.21 \right)^{2/3} (89.25 + 46.1k)^{3/2} \right\}
\]
\[
< 0.
\]
So inequality (P3) holds.

**The proof of inequality (P4).** Since we compute that

\[(653.5)^{2/3} (4.7 + 2.6k) - (8.77)^{2/3} (89.25 + 46.1k) \approx -25.6 - 0.2k < 0,
\]

we compute and observe that, for \(k \geq 0,
\[
Num \left( \frac{146.9}{500} c_0(k) + d_3(k) \right)
\approx 10^{63} \times \left[ 653.49 (4.65 + 2.60k)^{3/2} - 8.78(89.29 + 46.11k)^{3/2} \right]
\]

\[< 10^{63} \times \left[ 653.5 (4.65 + 2.60k)^{3/2} - 8.77(89.25 + 46.1k)^{3/2} \right]
\]

\[= 10^{63} \times \left\{ \left[ (653.5)^{2/3} (4.7 + 2.6k) \right]^{3/2} - \left[ (8.77)^{2/3} (89.25 + 46.1k) \right]^{3/2} \right\}
\]

\[< 0.
\]

So inequality (P4) holds.

**The proof of inequality (P5).** Since we compute that

\[(6.847)^{2/3} (48.66 + 27.231k) - (3.108)^{2/3} (89.28 + 46.11k) \approx -14.6 - 1.1k < 0,
\]

we compute and observe that, for \(k \geq 0,
\[
Num \left( \frac{153.9}{500} c_0(k) + \frac{422.5}{500} d_4(k) \right)
\approx 10^{62} \times \left[ 6.8464 (48.65 + 27.230k)^{3/2} - 3.1083(89.29 + 46.111k)^{3/2} \right]
\]

\[< 10^{62} \times \left[ 6.847 (48.66 + 27.231k)^{3/2} - 3.108(89.28 + 46.11k)^{3/2} \right]
\]

\[= 10^{62} \times \left\{ \left[ (6.847)^{2/3} (48.66 + 27.231k) \right]^{3/2} - \left[ (3.108)^{2/3} (89.28 + 46.11k) \right]^{3/2} \right\}
\]

\[< 0.
\]

So inequality (P5) holds.

**The proof of inequality (P6).** Since we compute that

\[(9.21)^{2/3} (48.66 + 27.24k) - (5.69)^{2/3} (72.19 + 37.58k) \approx -16.2 - 8.7k < 0,
\]

\[< 0.
\]
we compute and observe that, for $k \geq 0$,

$$\text{Num} \left( \frac{30}{500} c_1(k) + \frac{77.5}{500} d_4(k) \right)$$

$$\approx 10^{67} \times \left[ 9.20 \left( 48.65 + 27.23k \right)^{3/2} - 5.70(72.20 + 37.59k)^{3/2} \right]$$

$$< 10^{67} \times \left[ 9.21 \left( 48.66 + 27.24k \right)^{3/2} - 5.69(72.19 + 37.58k)^{3/2} \right]$$

$$= 10^{67} \times \left\{ \left[ (9.21)^{2/3} (48.66 + 27.24k) \right]^{3/2} - \left[ (5.69)^{2/3} (72.19 + 37.58k) \right]^{3/2} \right\}$$

$$< 0.$$

So inequality (P6) holds.

**The proof of inequality (P7).** Since we compute that

$$(70.11)^{2/3} (12.94 + 7.24k) - (59.09)^{2/3} (72.19 + 37.58k) (\approx -875.1 - 447.0k) < 0,$$

we compute and observe that, for $k \geq 0$,

$$\text{Num} \left( \frac{228.6}{500} c_1(k) + d_5(k) \right)$$

$$\approx 10^{70} \times \left[ 70.10 \left( 12.93 + 7.23k \right)^{3/2} - 59.10(72.20 + 37.59k)^{3/2} \right]$$

$$< 10^{70} \times \left[ 70.11 \left( 12.94 + 7.24k \right)^{3/2} - 59.09(72.19 + 37.58k)^{3/2} \right]$$

$$= 10^{70} \times \left\{ \left[ (70.11)^{2/3} (12.94 + 7.24k) \right]^{3/2} - \left[ (59.09)^{2/3} (72.19 + 37.58k) \right]^{3/2} \right\}$$

$$< 0.$$

So inequality (P7) holds.

**The proof of inequality (P8).** Since we compute that

$$(740.3506)^{2/3} (3.166 + 1.7701k) - (7.5664)^{2/3} (72.208 + 37.596k)$$

$$\approx -19.1 - 3.7 \times 10^{-2}k < 0,$$

we compute and observe that, for $k \geq 0$,

$$\text{Num} \left( \frac{241.4}{500} c_1(k) + \frac{465.6}{500} d_6(k) \right)$$

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So inequality (P8) holds.

**The proof of inequality (P9).** Since we compute that

\[
(24.72)^{2/3} (3.166 + 1.771k) - (5.58)^{2/3} (9.125 + 4.787k) \approx -1.8 - 3.1 \times 10^{-2}k < 0,
\]

we compute and observe that, for \( k \geq 0 \),

\[
\text{Num} \left( \frac{19}{500} c_2(k) + \frac{34.4}{500} d_6(k) \right)
\]

\[
\approx 10^{71} \times \left[ 24.71 (3.165 + 1.770k)^{3/2} - 5.59(9.126 + 4.788k)^{3/2} \right]
\]

\[
< 10^{71} \times \left[ 24.72 (3.167 + 1.771k)^{3/2} - 5.59(9.125 + 4.787k)^{3/2} \right]
\]

\[
= 10^{71} \times \left\{ \left( (24.72)^{2/3} (3.166 + 1.771k) \right)^{3/2} - \left[ (5.58)^{2/3} (9.125 + 4.787k) \right]^{3/2} \right\}
\]

\[
< 0
\]

So inequality (P9) holds.

**The proof of inequality (P10).** Since we compute that

\[
(3.9903)^{2/3} (7.89 + 4.40474k) - (3.5213)^{2/3} (9.11 + +4.78810k) \approx -1.2 - 1.0 \times 10^{-3}k < 0,
\]

we compute and observe that, for \( k \geq 0 \),

\[
\text{Num} \left( \frac{306.7}{500} c_2(k) + d_7(k) \right)
\]

\[
\approx 10^{67} \times \left[ 3.9902 (7.88 + 4.40473k)^{3/2} - 3.5214(9.12 + 4.78811k)^{3/2} \right]
\]

\[
< 10^{67} \times \left[ 3.9903 (7.89 + 4.40474k)^{3/2} - 3.5213(9.11 + 4.7881k)^{3/2} \right]
\]

\[
= 10^{67} \times \left\{ \left( (3.9903)^{2/3} (7.89 + 4.40474k) \right)^{3/2} - \left[ (3.5213)^{2/3} (9.11 + +4.78810k) \right]^{3/2} \right\}
\]
So inequality (P10) holds.

**The proof of inequality (P11).** Since we compute that

\[
(22.6770)^{2/3} (1.1 + 0.6128k) - (1.0387)^{2/3} (9.11 + 4.7880k) \\
\approx -0.5 - 1.1 \times 10^{-3}k < 0,
\]

we compute and observe that, for \(k \geq 0\),

\[
Num \left( \frac{174.3}{500} c_2(k) + \frac{263}{500} d_5(k) \right) \\
\approx 10^{72} \times \left[ 22.6769 (1.09 + 0.6127k)^{3/2} - 1.0388(9.12 + 4.7881k)^{3/2} \right] \\
< 10^{72} \times \left[ 22.6770 (1.10 + 0.6128k)^{3/2} - 1.0387(9.11 + 4.7880k)^{3/2} \right] \\
= 10^{72} \times \left\{ (22.6770)^{2/3} (1.1 + 0.6128k) \right\}^{3/2} - \left\{ (1.0387)^{2/3} (9.11 + 4.7880k) \right\}^{3/2} \\
< 0.
\]

So inequality (P11) holds.

**The proof of inequality (P12).** Since we compute that

\[
(4750.622)^{2/3} (1.10 + 0.613k) - (9.360)^{2/3} (73.82 + 39.011k) \\
\approx -17.0 - 3.0 \times 10^{-3}k < 0,
\]

we compute and observe that, for \(k \geq 0\),

\[
Num \left( \frac{168.8}{500} c_3(k) + \frac{237}{500} d_5(k) \right) \\
\approx 10^{71} \times \left[ 4750.621 (1.09 + 0.612k)^{3/2} - 9.361(73.83 + 39.012k)^{3/2} \right] \\
< 10^{71} \times \left[ 4750.622 (1.10 + 0.613k)^{3/2} - 9.360(73.82 + 39.011k)^{3/2} \right] \\
= 10^{71} \times \left\{ (4750.622)^{2/3} (1.10 + 0.613k) \right\}^{3/2} - \left\{ (9.360)^{2/3} (73.82 + 39.011k) \right\}^{3/2} \\
< 0.
\]

So inequality (P12) holds.

**The proof of inequality (P13).** Since we compute that

\[
(93.2113)^{2/3} (6.72 + 3.745k) - (2.7725)^{2/3} (73.82 + 39.011k) \\
\approx -7.5 - 6.9 \times 10^{-4}k < 0,
\]

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we compute and observe that, for $k \geq 0$,

$$\text{Num} \left( \frac{331.2}{500} c_3(k) + \frac{437.5}{500} d_9(k) \right)$$

$$\left( \approx 10^{67} \times \left[ 93.2112 (6.71 + 3.744k)^{3/2} - 2.7726(73.83 + 39.012k)^{3/2} \right] \right)$$

$$< 10^{67} \times \left[ 93.2113 (6.72 + 3.745k)^{3/2} - 2.7725(73.82 + 39.011k)^{3/2} \right]$$

$$= 10^{67} \times \left\{ \left[ (93.2113)^{2/3} (6.72 + 3.745k) \right]^{3/2} - \left[ (2.7725)^{2/3} (73.82 + 39.011k) \right]^{3/2} \right\}$$

$$< 0.$$  

So inequality (P13) holds.

**The proof of inequality (P14).** Since we compute that

$$(1.472)^{2/3} (6.72 + 3.745k) - (3.959)^{2/3} (3.65 + 1.938k) \approx -0.4 - 3.9 \times 10^{-3}k < 0,$$

we compute and observe that, for $k \geq 0$,

$$\text{Num} \left( \frac{50.9}{500} c_4(k) + \frac{62.5}{500} d_9(k) \right)$$

$$\left( \approx 10^{63} \times \left[ 1.471 (6.71 + 3.744k)^{3/2} - 3.960(3.64 + 1.939k)^{3/2} \right] \right)$$

$$< 10^{63} \times \left[ 1.472 (6.72 + 3.745k)^{3/2} - 3.959(3.63 + 1.938k)^{3/2} \right]$$

$$= 10^{63} \times \left\{ \left[ (1.472)^{2/3} (6.72 + 3.745k) \right]^{3/2} - \left[ (3.959)^{2/3} (3.65 + 1.938k) \right]^{3/2} \right\}$$

$$< 0.$$  

So inequality (P14) holds.

**The proof of inequality (P15).** Since we compute that

$$(1.2344)^{2/3} (3.43 + 1.90648k) - (1.2031)^{2/3} (3.63 + 1.93947k)$$

$$\approx -0.1 - 7.4 \times 10^{-5}k < 0,$$

we compute and observe that, for $k \geq 0$,

$$\text{Num} \left( \frac{427.1}{500} c_4(k) + d_{10}(k) \right)$$

20
\[
\left( \approx 10^{70} \times \left[ 1.2343 (3.42 + 1.90647k)^{3/2} - 1.20332 (3.64 + 1.93948k)^{3/2} \right] \right) \\
< 10^{70} \times \left[ 1.2344 (3.43 + 1.90648k)^{3/2} - 1.20331 (3.63 + 1.93947k)^{3/2} \right] \\
= 10^{70} \times \left\{ \left[ (1.2344)^{2/3} (3.43 + 1.90648k)^{3/2} \right] - \left[ (1.2031)^{2/3} (3.63 + 1.93947k)^{3/2} \right] \right\} \\
< 0.
\]

So inequality (P15) holds.

**The proof of inequality (P16).** Since we compute that
\[
(6.36)^{2/3} (1.82 + 1.011k) - (2.41)^{2/3} (3.63 + 1.938k) (\approx -0.2 - 1.3 \times 10^{-2}k) < 0,
\]
we compute and observe that, for \( k \geq 0, \)
\[
Num \left( \frac{22}{500} c_4(k) + \frac{25.2}{500} d_{11}(k) \right) \\
\left( \approx 10^{65} \times \left[ 6.359 (1.81 + 1.010k)^{3/2} - 2.412 (3.64 + 1.939k)^{3/2} \right] \right) \\
< 10^{65} \times \left[ 6.360 (1.82 + 1.011k)^{3/2} - 2.410 (3.63 + 1.938k)^{3/2} \right] \\
= 10^{65} \times \left\{ \left[ (6.36)^{2/3} (1.82 + 1.011k)^{3/2} \right] - \left[ (2.41)^{2/3} (3.63 + 1.938k)^{3/2} \right] \right\} \\
< 0.
\]

So inequality (P16) holds.

**The proof of inequality (P17).** Since we compute that
\[
(73.547)^{2/3} (1.82 + 1.0108k) - (4.544)^{2/3} (12.08 + 6.4684k) \\
(\approx -1.1 - 1.9 \times 10^{-3}k) < 0,
\]
we compute and observe that, for \( k \geq 0, \)
\[
Num \left( \frac{453}{500} c_5(k) + \frac{474.8}{500} d_{11}(k) \right) \\
\left( \approx 10^{66} \times \left[ 73.5460 (1.81 + 1.0107k)^{3/2} - 4.5440 (12.09 + 6.4685k)^{3/2} \right] \right) \\
< 10^{66} \times \left[ 73.5470 (1.82 + 1.0108k)^{3/2} - 4.5440 (12.08 + 6.4684k)^{3/2} \right] \\
= 10^{66} \times \left\{ \left[ (73.547)^{2/3} (1.82 + 1.0108k)^{3/2} \right] - \left[ (4.544)^{2/3} (12.08 + 6.4684k)^{3/2} \right] \right\}
\]
< 0.

So inequality (P17) holds.

**The proof of inequality (P18).** Since we compute that

\[(7.6306)^{2/3} (3.54 + 1.9746k) - (1.2922)^{2/3} (12.09 + 6.4685k)\]
\[\approx -0.5 - 1.7 \times 10^{-2}k < 0,
\]

we compute and observe that, for \(k \geq 0,
\[
Num \left( \frac{47}{500} c_5(k) + \frac{48.7}{500} d_{12}(k) \right)
\approx 10^{69} \times \left[ 7.630 (3.54 + 1.974k)^{3/2} - 1.2922(12.09 + 6.4685k)^{3/2} \right]
< 10^{69} \times \left[ 7.631 (3.55 + 1.975k)^{3/2} - 1.2920(12.08 + 6.468k)^{3/2} \right]
= 10^{69} \times \left\{ \left[(7.631)^{2/3} (3.55 + 1.975k)^{3/2} - (1.292)^{2/3} (12.08 + 6.468k)^{3/2} \right] \right\}
< 0.

So inequality (P18) holds.

**The proof of inequality (P19).** Since we compute that

\[(5.0305)^{2/3} (3.55 + 1.9747k) - (1.1974)^{2/3} (9.54 + 5.1419k)\]
\[\approx -0.3 - 5.5 \times 10^{-4}k < 0,
\]

we compute and observe that, for \(k \geq 0,
\[
Num \left( \frac{476.4}{500} c_6(k) + \frac{451.3}{500} d_{12}(k) \right)
\approx 10^{73} \times \left[ 5.0304 (3.54 + 1.9746k)^{3/2} - 1.1975(9.55 + 5.1420k)^{3/2} \right]
< 10^{73} \times \left[ 5.0305 (3.55 + 1.9747k)^{3/2} - 1.1974(9.54 + 5.1419k)^{3/2} \right]
= 10^{73} \times \left\{ \left[(5.0305)^{2/3} (3.55 + 1.9747k)^{3/2} - (1.1974)^{2/3} (9.54 + 5.1419k)^{3/2} \right] \right\}
< 0.

So inequality (P19) holds.
The proof of inequality (P20). Since we compute that
\[(2.5)^{2/3} (45.17 + 25.11k) - (27.0)^{2/3} (9.54 + 5.14k) (\approx -2.6 - 6.9 \times 10^{-3}k) < 0,\]
we compute and observe that, for \(k \geq 0,\)
\[
\begin{align*}
Num \left( \frac{23.6}{500} c_6(k) + \frac{22.5}{500} d_{13}(k) \right) \\
\approx 10^{69} \times \left[ 2.49 (45.16 + 25.109k)^{3/2} - 27.06(9.55 + 5.142k)^{3/2} \right] \\
< 10^{69} \times \left[ 2.5 (45.17 + 25.11k)^{3/2} - 27.00(9.54 + 5.14k)^{3/2} \right] \\
= 10^{69} \times \left\{ \left[ (2.5)^{2/3} (45.17 + 25.11k) \right]^{3/2} - \left[ (27.0)^{2/3} (9.54 + 5.14k) \right]^{3/2} \right\} \\
< 0.
\end{align*}
\]
So inequality (P20) holds.

The proof of inequality (P21). Since we compute that
\[(11.183)^{2/3} (45.17 + 25.11k) - (5.199)^{2/3} (77.39 + 41.844k) \]
\[(\approx -6.3 - 7.9 \times 10^{-3}k) < 0,\]
we compute and observe that, for \(k \geq 0,\)
\[
\begin{align*}
Num \left( c_7(k) + \frac{432.2}{500} d_{13}(k) \right) \\
\approx 10^{71} \times \left[ 11.182 (45.16 + 25.109k)^{3/2} - 5.1993(77.40 + 41.8442k)^{3/2} \right] \\
< 10^{71} \times \left[ 11.183 (45.17 + 25.11k)^{3/2} - 5.1990(77.39 + 41.844)\right]^{3/2} \\
= 10^{71} \times \left\{ \left[ (11.183)^{2/3} (45.17 + 25.11k) \right]^{3/2} - \left[ (5.199)^{2/3} (77.39 + 41.844) \right]^{3/2} \right\} \\
< 0.
\end{align*}
\]
So inequality (P21) holds.

The proof of inequality (P22). Since we compute that
\[(2.339)^{2/3} (45.17 + 25.11k) - (544.958)^{2/3} (1.21 + 0.664k) \]
\[(\approx -1.1 - 5.5 \times 10^{-2}k) < 0,\]
we compute and observe that, for \( k \geq 0 \),

\[
\text{Num} \left( \frac{57.6}{500} c_8(k) + \frac{45.3}{500} d_{13}(k) \right) \\
\left( \approx 10^{68} \times \left[ 2.338 \left( 45.16 + 25.109k \right)^{3/2} - 544.959(1.22 + 0.664k)^{3/2} \right] \right)
\]

\[
< 10^{68} \times \left[ 2.339 \left( 45.17 + 25.110k \right)^{3/2} - 544.958(1.21 + 0.664k)^{3/2} \right]
\]

\[
= 10^{68} \times \left\{ \left( (2.339)^{2/3} (45.17 + 25.11k) \right)^{3/2} - \left( (544.958)^{2/3} (1.21 + 0.664k) \right)^{3/2} \right\}
\]

\[
< 0.
\]

So inequality (P22) holds.

**The proof of inequality (P23).** Since we compute that

\[
(1.7959)^{2/3} (98.06 + 54.477k) - (1.3323)^{2/3} (122.42 + 66.485k)
\]

\[
(\approx -3.3 - 1.0 \times 10^{-2}k) < 0,
\]

we compute and observe that, for \( k \geq 0 \),

\[
\text{Num} \left( \frac{442.4}{500} c_8(k) + \frac{352.8}{500} d_{14}(k) \right) \\
\left( \approx 10^{66} \times \left[ 1.7958 \left( 98.05 + 54.476k \right)^{3/2} - 1.3324(122.43 + 66.486k)^{3/2} \right] \right)
\]

\[
< 10^{66} \times \left[ 1.7959 \left( 98.06 + 54.477k \right)^{3/2} - 1.3323(122.42 + 66.485k)^{3/2} \right]
\]

\[
= 10^{66} \times \left\{ \left( (1.7959)^{2/3} (98.06 + 54.477k) \right)^{3/2} - \left( (1.3323)^{2/3} (122.42 + 66.485k) \right)^{3/2} \right\}
\]

\[
< 0.
\]

So inequality (P23) holds.

**The proof of inequality (P24).** Since we compute that

\[
(3.931)^{2/3} (98.1 + 54.477k) - (5.558)^{2/3} (79.3 + 43.258k) \approx -4.4 - 4.0 \times 10^{-2}k < 0,
\]

we compute and observe that, for \( k \geq 0 \),

\[
\text{Num} \left( \frac{205.2}{500} c_9(k) + \frac{147.2}{500} d_{14}(k) \right)
\]
\[
\left( \approx 10^{68} \times \left[ \frac{3.930 (98.05 + 54.476k)}{3/2} - \frac{5.559(79.33 + 43.259k)}{3/2} \right] \right)
< 10^{68} \times \left[ \frac{3.931 (98.1 + 54.477k)}{3/2} - \frac{5.558(79.3 + 43.258k)}{3/2} \right]
= 10^{68} \times \left\{ \left( \frac{3.931}{2} (98.1 + 54.477k) \right)^{3/2} - \left( \frac{5.558}{2} (79.3 + 43.258k) \right)^{3/2} \right\}
< 0.
\]
So inequality (P24) holds.

**The proof of inequality (P25).** Since we compute that

\[
(5.648)^{2/3} (73.1 + 40.57k) - (5.132)^{2/3} (79.3 + 43.25k) \approx -4.1 - 1.2 \times 10^{-2}k < 0,
\]
we compute and observe that, for \( k \geq 0 \),

\[
Num \left( \frac{294.8}{500} c_9(k) + \frac{220}{500} d_{15}(k) \right)
\left( \approx 10^{68} \times \left[ \frac{5.647 (73.06 + 40.56k)}{3/2} - \frac{5.133(79.33 + 43.259k)}{3/2} \right] \right)
< 10^{68} \times \left[ \frac{5.648 (73.1 + 40.57k)}{3/2} - \frac{5.132(79.3 + 43.25k)}{3/2} \right]
= 10^{68} \times \left\{ \left( \frac{5.648}{2} (73.1 + 40.57k) \right)^{3/2} - \left( \frac{5.132}{2} (79.3 + 43.25k) \right)^{3/2} \right\}
< 0.
\]
So inequality (P25) holds.

**The proof of inequality (P26).** Since we compute that

\[
(20.8432)^{2/3} (73.1 + 40.566k) - (6.5331)^{2/3} (160.6 + 87.934k)
\approx -7.6 - 7.2 \times 10^{-2}k < 0,
\]
we compute and observe that, for \( k \geq 0 \),

\[
Num \left( \frac{421.5}{500} c_{10}(k) + \frac{280}{500} d_{15}(k) \right)
\left( \approx 10^{62} \times \left[ \frac{20.8431 (73.06 + 40.565k)}{3/2} - \frac{6.5332(160.67 + 87.935k)}{3/2} \right] \right)
< 10^{62} \times \left[ \frac{20.8432 (73.1 + 40.566k)}{3/2} - \frac{6.5331(160.6 + 87.934k)}{3/2} \right]
= 10^{62} \times \left\{ \left( \frac{20.8432}{2} (73.1 + 40.566k) \right)^{3/2} - \left( \frac{6.5331}{2} (160.6 + 87.934k) \right)^{3/2} \right\}
\]
< 0.

So inequality (P26) holds.

The proof of inequality (P27). Since we compute that

\[(388.182)^{2/3} (3.81 + 2.112k) - (1.446)^{2/3} (160.66 + 87.934k)\]
\[(\approx -2.6 - 5.6 \times 10^{-2}k) < 0,\]

we compute and observe that, for \(k \geq 0\),

\[
\begin{aligned}
&Num\left(\frac{78.5}{500} c_{10}(k) + \frac{56}{500} d_{16}(k)\right) \\
&\quad \left(\approx 10^{66} \times \left[388.181 (3.80 + 2.111k)^{3/2} - 1.447 (160.67 + 87.935k)^{3/2}\right]\right) \\
&< 10^{66} \times \left[388.182 (3.81 + 2.112k)^{3/2} - 1.446 (160.66 + 87.934k)^{3/2}\right] \\
&= 10^{66} \times \left\{\left[(388.182)^{2/3} (3.81 + 2.112k)\right]^{3/2} - \left[(1.446)^{2/3} (160.66 + 87.934k)\right]^{3/2}\right\} \\
&< 0.
\end{aligned}
\]

So inequality (P27) holds.

The proof of inequality (P28). Since we compute that

\[(788.425)^{2/3} (3.81 + 2.1111k) - (8.102)^{2/3} (81.3 + 44.6755k)\]
\[(\approx -2.7 - 4.7 \times 10^{-2}k) < 0,\]

we compute and observe that, for \(k \geq 0\),

\[
\begin{aligned}
&Num\left(c_{11}(k) + \frac{313.4}{500} d_{16}(k)\right) \\
&\quad \left(\approx 10^{72} \times \left[788.4241 (3.80 + 2.1110k)^{3/2} - 8.1021 (81.35 + 44.6756k)^{3/2}\right]\right) \\
&< 10^{72} \times \left[788.425 (3.81 + 2.1111k)^{3/2} - 8.102 (81.3 + 44.6755k)^{3/2}\right] \\
&= 10^{72} \times \left\{\left[(788.425)^{2/3} (3.81 + 2.1111k)\right]^{3/2} - \left[(8.102)^{2/3} (81.3 + 44.6755k)\right]^{3/2}\right\} \\
&< 0.
\end{aligned}
\]

So inequality (P28) holds.
The proof of inequality (P29). Since we compute that
\[ (82.053)^{2/3} (3.81 + 2.112k) - (3.375)^{2/3} (32.17 + 17.726k) \]
\[ (\approx -0.4 - 3.3 \times 10^{-3}k) < 0, \]
we compute and observe that, for \( k \geq 0 \),
\[
\begin{align*}
\text{Num} & \left( \frac{239.8}{500} c_{12}(k) + \frac{130.6}{500} d_{16}(k) \right) \\
& \left( \approx 10^{69} \times \left[ 82.052 (3.80 + 2.111k)^{3/2} - 3.376(32.18 + 17.727k)^{3/2} \right] \right) \\
& < 10^{69} \times \left[ 82.053 (3.81 + 2.112k)^{3/2} - 3.375(32.17 + 17.726k)^{3/2} \right] \\
& = 10^{69} \times \left\{ \left[ (82.053)^{2/3} (3.81 + 2.112k) \right]^{3/2} - \left[ (3.375)^{2/3} (32.17 + 17.726k) \right]^{3/2} \right\} \\
& < 0.
\end{align*}
\]
So inequality (P29) holds.

The proof of inequality (P30). Since we compute that
\[ (8.904)^{2/3} (16.12 + 8.939k) - (3.190)^{2/3} (32.17 + 17.726k) \]
\[ (\approx -0.4 - 1.1 \times 10^{-3}k) < 0, \]
we compute and observe that, for \( k \geq 0 \),
\[
\begin{align*}
\text{Num} & \left( \frac{260.2}{500} c_{12}(k) + \frac{159.1}{500} d_{17}(k) \right) \\
& \left( \approx 10^{67} \times \left[ 8.903 (16.11 + 8.938k)^{3/2} - 3.191(32.18 + 17.727k)^{3/2} \right] \right) \\
& < 10^{67} \times \left[ 8.904 (16.12 + 8.939k)^{3/2} - 3.19(32.17 + 17.726k)^{3/2} \right] \\
& = 10^{67} \times \left\{ \left[ (8.904)^{2/3} (16.12 + 8.939k) \right]^{3/2} - \left[ (3.190)^{2/3} (32.17 + 17.726k) \right]^{3/2} \right\} \\
& < 0.
\end{align*}
\]
So inequality (P30) holds.

The proof of inequality (P31). Since we compute that
\[ (61.136)^{2/3} (16.12 + 8.939k) - (5.222)^{2/3} (83.45 + 46.090k) \]
\[ (\approx -1.0 - 3.5 \times 10^{-3}k) < 0, \]

we compute and observe that, for \( k \geq 0 \),
\[
Num \left( c_{13}(k) + \frac{260.4}{500} d_{17}(k) \right) \\
\approx 10^{70} \times \left[ 61.135 (16.11 + 8.938k)^{3/2} - 5.223(83.46 + 46.091k)^{3/2} \right] \\
< 10^{70} \times \left[ 61.136 (16.12 + 8.939k)^{3/2} - 5.222(83.45 + 46.090k)^{3/2} \right] \\
= 10^{70} \times \left\{ \left( 61.136 \right)^{2/3} (16.12 + 8.939k) \right\}^{3/2} - \left( 5.222 \right)^{2/3} (83.45 + 46.090k) \right\}^{3/2} \\
< 0.
\]

So inequality (P31) holds.

**The proof of inequality (P32).** Since we compute that
\[
(8.533)^{2/3} (16.12 + 8.939k) - (16.145)^{2/3} (10.55 + 5.848k) \\
(\approx -7.9 \times 10^{-2} - 2.9 \times 10^{-2}k) < 0,
\]

we compute and observe that, for \( k \geq 0 \),
\[
Num \left( \frac{185.5}{500} c_{14}(k) + \frac{80.5}{500} d_{17}(k) \right) \\
\approx 10^{69} \times \left[ 8.532 (16.11 + 8.938k)^{3/2} - 16.146(10.56 + 5.849k)^{3/2} \right] \\
< 10^{69} \times \left[ 8.533 (16.12 + 8.939k)^{3/2} - 16.145(10.55 + 5.848k)^{3/2} \right] \\
= 10^{69} \times \left\{ \left( 8.533 \right)^{2/3} (16.12 + 8.939k) \right\}^{3/2} - \left( 16.145 \right)^{2/3} (10.55 + 5.848k) \right\}^{3/2} \\
< 0.
\]

So inequality (P32) holds.

**The proof of inequality (P33).** Since we compute that
\[
(14.466)^{2/3} (3.93 + 2.1793k) - (3.290)^{2/3} (10.55 + 5.8498k) \\
(\approx -4.9 \times 10^{-3} - 1.6 \times 10^{-3}k) < 0,
\]
we compute and observe that, for \( k \geq 0 \),

\[
\text{Num} \left( \frac{314.5}{500} c_{14}(k) + \frac{168}{500} d_{18}(k) \right)
\]

\[
\approx 10^{72} \times \left[ 14.465 (3.92 + 2.1792k)^{3/2} - 3.291 (10.56 + 5.8499k)^{3/2} \right]
\]

\[
< 10^{72} \times \left[ 14.466 (3.93 + 2.1793k)^{3/2} - 3.290 (10.55 + 5.849k)^{3/2} \right]
\]

\[
= 10^{72} \times \left\{ \left[ (14.466)^{2/3} (3.93 + 2.1793k) \right]^{3/2} - \left[ (3.290)^{2/3} (10.55 + 5.8498k) \right]^{3/2} \right\}
\]

\[< 0.\]

So inequality (P33) holds.

The proof of inequality (P34). Since we compute that

\[
(27.3913)^{2/3} (3.92971 + 2.1793k) - (4.2066)^{2/3} (13.70320 + 7.6010k)
\]

\[
\approx -5.9 \times 10^{-4} - 4.5 \times 10^{-3}k < 0,
\]

we compute and observe that, for \( k \geq 0 \),

\[
\text{Num} \left( c_{15}(k) + \frac{214.7}{500} d_{18}(k) \right)
\]

\[
\approx 10^{69} \times \left[ 27.3912 (3.92970 + 2.1792k)^{3/2} - 4.2067 (13.70321 + 7.6011k)^{3/2} \right]
\]

\[
< 10^{69} \times \left[ 27.3913 (3.92971 + 2.1793k)^{3/2} - 4.2066 (13.7032 + 7.601k)^{3/2} \right]
\]

\[
= 10^{69} \times \left\{ \left[ (27.3913)^{2/3} (3.92971 + 2.1793k) \right]^{3/2} - \left[ (4.2066)^{2/3} (13.7032 + 7.601k) \right]^{3/2} \right\}
\]

\[< 0.\]

So inequality (P34) holds.

The proof of inequality (P35). Since we compute that

\[
(4.6568)^{2/3} (3.92971 + 2.1793k) - (22.9832)^{2/3} (1.35567 + 0.7532k)
\]

\[
\approx -2.2 \times 10^{-4} - 1.1 \times 10^{-2}k < 0,
\]

we compute and observe that, for \( k \geq 0 \),

\[
\text{Num} \left( \frac{356.5}{500} c_{16}(k) + \frac{117.2}{500} d_{18}(k) \right)
\]
\[
\left( \approx 10^{71} \times \left[ 4.6567 \left(3.92970 + 2.1792k\right)^{3/2} - 22.9833(1.35568 + 0.7533k)^{3/2}\right] \right)
< 10^{71} \times \left[ 4.6568 \left(3.92971 + 2.1793k\right)^{3/2} - 22.9832(1.35567 + 0.7532k)^{3/2}\right]
= 10^{71} \times \left\{ \left[ (4.6568)^{2/3} \left(3.92971 + 2.1793k\right) \right]^{3/2} - \left[ (22.9832)^{2/3} (1.35567 + 0.7532k) \right]^{3/2} \right\}
< 0.
\]

So inequality (P35) holds.

**The proof of inequality (P36).** Since we compute that

\[
\left(1.8745\right)^{2/3} \left(99.79 + 55.34k\right) - \left(1.1844\right)^{2/3} \left(135.55 + 75.32k\right)
\approx -0.1 - 3.0 \times 10^{-2}k < 0,
\]

we compute and observe that, for \( k \geq 0, \)

\[
Num \left( \frac{143.5}{500} c_{16}(k) + \frac{82.8}{500} d_{19}(k) \right)
\left( \approx 10^{62} \times \left[ 1.8744 \left(99.78 + 55.33k\right)^{3/2} - 1.1845(135.56 + 75.33k)^{3/2}\right] \right)
< 10^{62} \times \left[ 1.8745 \left(99.79 + 55.34k\right)^{3/2} - 1.1844(135.55 + 75.32k)^{3/2}\right]
= 10^{62} \times \left\{ \left[ (1.8745)^{2/3} \left(99.79 + 55.34k\right) \right]^{3/2} - \left[ (1.1844)^{2/3} (135.55 + 75.32k) \right]^{3/2} \right\}
< 0.
\]

So inequality (P36) holds.

**The proof of inequality (P37).** Since we compute that

\[
\left(2.398\right)^{2/3} \left(99.789 + 55.34k\right) - \left(2.901\right)^{2/3} \left(87.899 + 48.91k\right)
\approx -1.3 \times 10^{-2} - 0.34k < 0,
\]

we compute and observe that, for \( k \geq 0, \)

\[
Num \left( c_{17}(k) + \frac{202.8}{500} d_{19}(k) \right)
\left( \approx 10^{65} \times \left[ 2.3978 \left(99.788 + 55.33k\right)^{3/2} - 2.9012(87.900 + 48.92k)^{3/2}\right] \right)
< 10^{65} \times \left[ 2.398 \left(99.789 + 55.34k\right)^{3/2} - 2.901(87.899 + 48.91k)^{3/2}\right]
\]

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\[ 10^{65} \times \left\{ \left[ (2.398)^{2/3} (99.789 + 55.34k) \right]^{3/2} - \left[ (2.901)^{2/3} (87.899 + 48.91k) \right]^{3/2} \right\} < 0. \]

So inequality (P37) holds.

**The proof of inequality (P38).** Since we compute that

\[
(6.3797)^{2/3} (99.7889 + 55.34k) - (171.2397)^{2/3} (11.1315 + 6.19k)
\]

\[
(\approx -6.1 \times 10^{-3} - 0.98k) < 0,
\]

we compute and observe that, for \( k \geq 0 \),

\[
Num \left( c_{18}(k) + \frac{119.7}{500} d_{19}(k) \right)
\]

\[
(\approx 10^{63} \times \left[ 6.3796 (99.7888 + 55.33k)^{3/2} - 171.2398 (11.1315 + 6.20k)^{3/2} \right])
\]

\[
< 10^{63} \times \left[ 6.3797 (99.7889 + 55.34k)^{3/2} - 171.2397 (11.1316 + 6.19k)^{3/2} \right]
\]

\[
= 10^{63} \times \left\{ \left[ (6.3797)^{2/3} (99.7889 + 55.34k) \right]^{3/2} - \left[ (171.2397)^{2/3} (11.1315 + 6.19k) \right]^{3/2} \right\}
\]

\[
< 0.
\]

So inequality (P38) holds.

**The proof of inequality (P39).** Since we compute that

\[
(4.9)^{2/3} (99.8 + 55.4k) - (13.4)^{2/3} (90.1 + 50.2k) (\approx 220.40 - 123.3k) < 0,
\]

we compute and observe that, for \( k \geq 0 \),

\[
Num \left( c_{19}(k) + \frac{94.7}{500} d_{19}(k) \right)
\]

\[
(\approx 10^{64} \times \left[ 4.8 (99.7 + 55.3k)^{3/2} - 13.5 (90.2 + 50.3k)^{3/2} \right])
\]

\[
< 10^{64} \times \left[ 4.9 (99.8 + 55.4k)^{3/2} - 13.4 (90.1 + 50.2k)^{3/2} \right]
\]

\[
= 10^{64} \times \left\{ \left[ (4.9)^{2/3} (99.8 + 55.4k) \right]^{3/2} - \left[ (13.4)^{2/3} (90.1 + 50.2k) \right]^{3/2} \right\}
\]

\[
< 0.
\]

So inequality (P39) holds.

Thus the proof of assertion (2.9) is complete. \( \blacksquare \)
References