# The Proof of Lemma 3.9(ii)

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**Lemma 3.9(ii)**. Consider (1.1). If $\varepsilon = \varepsilon_3 = \sqrt{\frac{31\sigma^3}{1000\rho}}$, then $T'_{\varepsilon_3}(\gamma) > 0$ and $T'_{\varepsilon_3}(p_2) < 0$.

The proof of Lemma 3.9(ii) is similar the proof of Lemma 3.9(i). We recall the functions

$$H_1(u,\alpha) \equiv \frac{(\alpha-u)^{3/2}}{6\left[F(\alpha)-F(u)\right]^{3/2}}, \quad H_2(u,\alpha) \equiv \frac{6\left[\theta(\alpha)-\theta(u)\right]}{(\alpha-u)^{3/2}},$$

$$I_1(u,\alpha) \equiv \frac{2}{35}\left[-15\varepsilon u^3 - (39\varepsilon\alpha - 14\sigma)u^2 - \left(87\varepsilon\alpha^2 - 42\sigma\alpha\right)u - 279\varepsilon\alpha^3 + 154\alpha^2\sigma - 210\rho\right].$$

Next, we prove that $T'_{\varepsilon_3}(\gamma) > 0$ and $T'_{\varepsilon_3}(p_2) < 0$ in Sections 1 and 2, respectively. We remark that most of the computation in this paper has been checked using the symbolic manipulator *Maple 16*.

## 1. Proof of $T'_{\varepsilon_3}(\gamma) > 0$

Assume that $\varepsilon = \varepsilon_3$. By (3.34) in [1], we have that

$$\begin{aligned} T'_{\varepsilon_3}(\gamma) \;\geq\; & \frac{1}{2\sqrt{2}\gamma}\sum_{i=0}^{5}\Big[H_1(\alpha_{i+1},\beta_6)\left(\sqrt{\beta_6-\alpha_{i+1}}I_1(\alpha_{i+1},\beta_6) - \sqrt{\beta_6-\alpha_i}I_1(\alpha_i,\beta_6)\right) \\ & + H_1(\beta_i,\beta_6)\left(\sqrt{\beta_6-\beta_{i+1}}I_1(\beta_{i+1},\beta_6) - \sqrt{\beta_6-\beta_i}I_1(\beta_i,\beta_6)\right)\Big], \end{aligned} \tag{1.1}$$

where $\alpha_i = i\bar{\gamma}/6$ and $\beta_i = \bar{\gamma} + i(\gamma-\bar{\gamma})/6$ for $i = 0, 1, ..., 6$. Let

$$\alpha_{i*} \equiv \frac{i\tilde{\gamma}_*}{6}, \quad \alpha_i^* \equiv \frac{i\tilde{\gamma}^*}{6}, \quad \beta_{i*} \equiv \frac{(6-i)\tilde{\gamma}_* + i\gamma}{6}, \quad \text{and} \quad \beta_i^* \equiv \frac{(6-i)\tilde{\gamma}^* + i\gamma}{6}$$

for $i = 0, 1, ..., 6$, where

$$\tilde{\gamma}_* \equiv \frac{1799\sigma}{10000\varepsilon_3} \quad \text{and} \quad \tilde{\gamma}^* \equiv \frac{9\sigma}{50\varepsilon_3}.$$

By Lemma 3.3 in [1], we compute and observe that

$$\tilde{\gamma}_* < \bar{\gamma} = \frac{1 - 4\cos(\frac{y}{3} + \frac{\pi}{3})}{9}\bigg|_{\varepsilon=\varepsilon_3} \frac{\sigma}{\varepsilon_3} \left(\approx \frac{17996\sigma}{10^5\varepsilon_3}\right) < \tilde{\gamma}^* < \frac{21\sigma}{50\varepsilon_3}, \tag{1.2}$$

from which it follows that

$$\alpha_{i*} < \alpha_i < \alpha_i^* \ \text{ and } \ \beta_{i*} < \beta_i < \beta_i^* \ \text{ for } i = 0, 1, ..., 6. \tag{1.3}$$

By Lemma 3.8(i) in [1], $H_1(u, \gamma)$ is a strictly decreasing function of $u$ on $(0, \gamma)$. Also, by Lemma 3.1 in [1], $H_2(u, \gamma) > 0$ for $0 < u < \bar{\gamma}$ and $H_2(u, \gamma) < 0$ for $\bar{\gamma} < u < \gamma$. Since

$$\int H_2(u, \alpha) du = \sqrt{\alpha - u} I_1(u, \alpha),$$

we observe that

$$\sqrt{\beta_6 - \alpha_{i+1}} I_1(\alpha_{i+1}, \beta_6) - \sqrt{\beta_6 - \alpha_i} I_1(\alpha_i, \beta_6) = \int_{\alpha_i}^{\alpha_{i+1}} H_2(u, \alpha) du > 0$$

and

$$\sqrt{\beta_6 - \beta_{i+1}} I_1(\beta_{i+1}, \beta_6) - \sqrt{\beta_6 - \beta_i} I_1(\beta_i, \beta_6) = \int_{\beta_i}^{\beta_{i+1}} H_2(u, \alpha) du < 0.$$

So by (1.1),

$$\begin{aligned} T'_{\varepsilon_3}(\gamma) \ \geq \ & \frac{1}{2\sqrt{2}\gamma} \sum_{i=0}^{5} \Big[ H_1(\alpha_{i+1}^*, \gamma) \left(\sqrt{\gamma - \alpha_{i+1}} I_1(\alpha_{i+1}, \gamma) - \sqrt{\gamma - \alpha_i} I_1(\alpha_i, \gamma)\right) \\ & + H_1(\beta_{i*}, \gamma) \left(\sqrt{\gamma - \beta_{i+1}} I_1(\beta_{i+1}, \gamma) - \sqrt{\gamma - \beta_i} I_1(\beta_i, \gamma)\right) \Big]. \end{aligned} \tag{1.4}$$

By Lemma 3.8(iii) in [1],

$$I_1(u_2, \gamma) > I_1(u_1, \gamma) > I_1(0, \gamma) = \frac{964\rho}{1953} > 0 \ \text{ for } 0 < u_1 < u_2 < \gamma. \tag{1.5}$$

By Lemma 3.8(iii)(iv) in [1], (1.2)–(1.5), we compute and observe that

$$2\sqrt{2}\gamma T'_{\varepsilon_3}(\gamma) \ \geq \ \sum_{i=0}^{5} H_1(\alpha_{i+1}^*, \gamma) \left[\sqrt{\gamma - \alpha_{i+1}^*} I_1(\alpha_{i+1*}, \gamma) - \sqrt{\gamma - \alpha_{i*}} I_1(\alpha_i^*, \gamma)\right]$$

$$+\sum_{i=0}^{4} H_1(\beta_{i*},\gamma)\left[\sqrt{\gamma-\beta_i^*}I_1(\beta_{i*},\gamma)-\sqrt{\gamma-\beta_{i*}}I_1(\beta_i^*,\gamma)\right]$$
$$-H_1(\beta_{5*},\gamma)\sqrt{\gamma-\beta_{5*}}I_1(\beta_5^*,\gamma). \tag{1.6}$$

Let $k \equiv \tau/\sqrt{\sigma\rho}$. We compute and find that, for $i = 0, 1, ..., 5$ and $j = 0, 1, ..., 4$,

$$H_1(\alpha_{i+1}^*,\gamma)\left[\sqrt{\gamma-\alpha_{i+1}^*}I_1(\alpha_{i+1*},\gamma)-\sqrt{\gamma-\alpha_{i*}}I_1(\alpha_i^*,\gamma)\right]=\frac{c_i(k)}{(\sigma\rho)^{1/4}},$$

$$\sum_{i=0}^{4} H_1(\beta_{j*},\gamma)\left[\sqrt{\gamma-\beta_j^*}I_1(\beta_{j*},\gamma)-\sqrt{\gamma-\beta_*}I_1(\beta_j^*,\gamma)\right]=\frac{d_j(k)}{(\sigma\rho)^{1/4}},$$

and

$$-H_1(\beta_{5*},\gamma)\sqrt{\gamma-\beta_{5*}}I_1(\beta_5^*,\gamma)=\frac{d_5(k)}{(\sigma\rho)^{1/4}}.$$

So by (1.6),

$$T'_{\varepsilon_3}(\gamma) \geq \frac{1}{2\sqrt{2}\gamma\,(\sigma\rho)^{1/4}}\sum_{i=0}^{5}\left[c_i(k)+d_i(k)\right]. \tag{1.7}$$

We note that functions $c_i(k), d_i(k)$ for $i = 0, 1, ..., 5$ are all independent on $\varepsilon$, $\sigma$, $\tau$ and $\rho$. In fact, we see that the form of functions $c_i(k), d_i(k)$ for $i = 0, 1, ..., 5$ are similar to the form of $a_i(k)$ defined in the proof of Lemma 3.9(i) in [1] for $i = 1, 2, 3$. So we can prove the following assertion

$$\sum_{i=0}^{5}\left[c_i(k)+d_i(k)\right] > 0 \quad \text{for } k \geq 0 \tag{1.8}$$

by applying similar technique used to prove

$$a_1(k)+a_2(k)+a_3(k) > 0 \quad \text{for } k \geq 0$$

in the proof of Lemma 3.9(i) in [1]. See Figure 1.1. Since the proof of assertion (1.8) is easy and lengthy, we put the proof in Appendix A and omit it here. So by (1.7) and (1.8), $T'_{\varepsilon_2}(\gamma) > 0$.

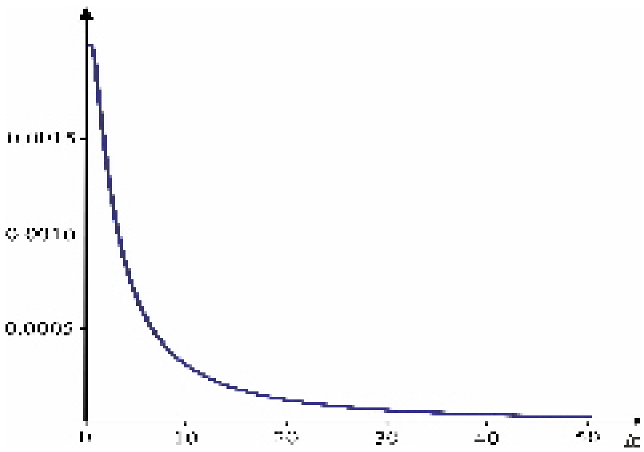

Figure 1.1: The graph of $\sum_{i=0}^{5}\left[a_i(k)+b_i(k)\right]$ for $k\geq 0$.

## 2. Proof of $T'_{\varepsilon_3}(p_2)<0$

Assume that $\varepsilon=\varepsilon_3$. By (3.35) in [1], we have that

$$\begin{aligned}T'_{\varepsilon_3}(p_2) \;&\leq\; \frac{1}{2\sqrt{2}p_2}\sum_{i=0}^{19}\Bigg[H_1(\alpha_i,\beta_{20})\left[\sqrt{\beta_{20}-\alpha_{i+1}}I_1(\alpha_{i+1},\beta_{20})-\sqrt{\beta_{20}-\alpha_i}I_1(\alpha_i,\beta_{20})\right]\\ &\quad +H_1(\beta_{i+1},\beta_{20})\left[\sqrt{\beta_{20}-\beta_{i+1}}I_1(\beta_{i+1},\beta_{20})-\sqrt{\beta_{20}-\beta_i}I_1(\beta_i,\beta_{20})\right]\Bigg], \quad (2.1)\end{aligned}$$

where $\alpha_i=i\bar{p}_2/20$ and $\beta_i=\bar{p}_2+i\left(p_2-\bar{p}_2\right)/20$ for $i=0,1,...,20$. Let

$$\alpha_{i*}\equiv\frac{i\tilde{p}_{2*}}{20},\quad \alpha_i^*\equiv\frac{i\tilde{p}_2^*}{20},\quad \beta_{i*}\equiv\tilde{p}_{2*}+\frac{i\left(p_{2*}-\tilde{p}_2^*\right)}{20},\quad\text{and } \beta_i^*\equiv\;\tilde{p}_2^*+\frac{i\left(p_2^*-\tilde{p}_{2*}\right)}{20},$$

where

$$\tilde{p}_{2*}\equiv\frac{1557897\sigma}{10^7\varepsilon_3},\quad \tilde{p}_2^*\equiv\frac{1557898\sigma}{10^7\varepsilon_3},\quad p_{2*}\equiv\frac{4059387\sigma}{10^7\varepsilon_3},\quad\text{and}\quad p_2^*\equiv\frac{4059388\sigma}{10^7\varepsilon_3}.$$

By Lemmas 3.1 and 3.3 in [1], we compute and find that

$$\tilde{p}_{2*}<\bar{p}_2\;\left(\approx\frac{15578978\sigma}{10^8\varepsilon_3}\right)<\tilde{p}_2^* \tag{2.2}$$

and

$$\frac{39\sigma}{100\varepsilon_3}<\frac{81\sigma}{200\varepsilon_3}<p_{2*}<p_2\;\left(\approx\frac{40593877\sigma}{10^8\varepsilon_3}\right)<p_2^*<\frac{203\sigma}{500\varepsilon_3}<\frac{21\sigma}{50\varepsilon_3}. \tag{2.3}$$

So by (2.2)–(2.3), we observe that

$$\alpha_{i*} < \alpha_i < \alpha_i^* \ \text{ and } \ \beta_{i*} < \beta_i < \beta_i^* \ \text{ for } i = 0, 1, ..., 20. \tag{2.4}$$

By Lemma 3.1 in [1], $H_2(u, p_2) \geq 0$ for $0 < u < \bar{p}_2$ and $H_2(u, p_2) < 0$ for $\bar{p}_2 < u < p_2$. Since

$$\int H_2(u, \alpha) du = \sqrt{\alpha - u} I_1(u, \alpha),$$

we observe that

$$\sqrt{\beta_{20} - \alpha_{i+1}} I_1(\alpha_{i+1}, \beta_{20}) - \sqrt{\beta_{20} - \alpha_i} I_1(\alpha_i, \beta_{20}) = \int_{\alpha_i}^{\alpha_{i+1}} H_2(u, \alpha) du > 0$$

and

$$\sqrt{\beta_{20} - \beta_{i+1}} I_1(\beta_{i+1}, \beta_{20}) - \sqrt{\beta_{20} - \beta_i} I_1(\beta_i, \beta_{20}) = \int_{\beta_i}^{\beta_{i+1}} H_2(u, \alpha) du < 0.$$

So by (2.1) and Lemma 3.8(i)(ii) in [1],

$$\begin{aligned} T'_{\varepsilon_3}(p_2) \ \leq \ & \frac{1}{2\sqrt{2}p_2} \sum_{i=0}^{19} \Big[ H_1(\alpha_{i*}, p_{2*}) \left[ \sqrt{\beta_{20} - \alpha_{i+1}} I_1(\alpha_{i+1}, \beta_{20}) - \sqrt{\beta_{20} - \alpha_i} I_1(\alpha_i, \beta_{20}) \right] \\ & + H_1(\beta_{i+1}^*, p_2^*) \left[ \sqrt{\beta_{20} - \beta_{i+1}} I_1(\beta_{i+1}, \beta_{20}) - \sqrt{\beta_{20} - \beta_i} I_1(\beta_i, \beta_{20}) \right] \Big]. \end{aligned} \tag{2.5}$$

In addition, we compute that $I_1(0, p_2^*)$ ($\approx 0.37\rho$) $> 0$ and $I_1(\beta_{19}^*, p_2^*)$ ($\approx 0.13\rho$) $> 0$. So by Lemma 3.8(iii)(iv) in [1] and (2.3), we observe that

$$I_1(u, p_{2*}) \geq I_1(u, p_2) \geq I_1(u, p_2^*) > 0 \ \text{ for } 0 \leq u \leq \beta_{19}^*. \tag{2.6}$$

By Lemma 3.8(iii) in [1], (2.3) and (2.4), we compute and observe that

$$\begin{aligned} \beta_4 \ & < \ \beta_4^* = \frac{10290981\sigma}{5 \times 10^7 \varepsilon_3} < \frac{7\left(-131 + 7\sqrt{13054}\right)\sigma}{22500\varepsilon_3} = \hat{u}(\frac{203\sigma}{500\varepsilon_3}) \\ & < \ \hat{u}(p_2^*) < \hat{u}(p_2) < \hat{u}(p_{2*}) < \hat{u}(\frac{81\sigma}{200\varepsilon_3}) = \frac{\left(-359b + \sqrt{5060566}\right)\sigma}{9000\varepsilon_3} \\ & < \ \frac{8733077\sigma}{4 \times 10^7 \varepsilon_3} = \beta_{5*} < \beta_5. \end{aligned}$$

It follows that

$$\beta_4 < \beta_4^* < \hat{u}(p_2) < \beta_{5*} < \beta_5. \tag{2.7}$$

By Lemma 3.8(iii)(iv) in [1], (2.2)–(2.7), we compute and observe that

$$\begin{aligned} 2\sqrt{2}p_2T'_{\varepsilon_3}(p_2) \;\leq\; & \sum_{i=0}^{19} H_1(\alpha_{i*},\beta_{20*})\left[\sqrt{\beta_{20}^*-\alpha_{i+1*}}I_1(\alpha_{i+1}^*,\beta_{20*})-\sqrt{\beta_{20*}-\alpha_i^*}I_1(\alpha_{i*},\beta_{20}^*)\right] \\ & +\sum_{i=0}^{3} H_1(\beta_{i+1}^*,\beta_{20}^*)\left[\sqrt{\beta_{20}^*-\beta_{i+1*}}I_1(\beta_{i+1}^*,\beta_{20*})-\sqrt{\beta_{20*}-\beta_i^*}I_1(\beta_{i*},\beta_{20}^*)\right] \\ & +\sum_{i=4}^{19} H_1(\beta_{i+1}^*,\beta_{20}^*)\left[\sqrt{\beta_{20}^*-\beta_{i+1*}}I_1(\beta_{i+1*},\beta_{20*})-\sqrt{\beta_{20*}-\beta_i^*}I_1(\beta_i^*,\beta_{20}^*)\right] \\ & -H_1(\beta_{18}^*,\beta_{20}^*)\sqrt{\beta_{20*}-\beta_{17}^*}I_1(\beta_{17}^*,\beta_{20}^*). \end{aligned} \tag{2.8}$$

Let $k \equiv \tau/\sqrt{\sigma\rho}$. We compute and find that, for $i = 0, 1, ..., 19$,

$$H_1(\alpha_{i*},\beta_{20*})\left[\sqrt{\beta_{20}^*-\alpha_{i+1*}}I_1(\alpha_{i+1}^*,\beta_{20*})-\sqrt{\beta_{20*}-\alpha_i^*}I_1(\alpha_{i*},\beta_{20}^*)\right] = \frac{c_i(k)}{(\sigma\rho)^{\frac{1}{4}}}$$

and

$$\begin{aligned} & \begin{cases} H_1(\beta_{i+1}^*,\beta_{20}^*)\left[\sqrt{\beta_{20}^*-\beta_{i+1*}}I_1(\beta_{i+1}^*,\beta_{20*})-\sqrt{\beta_{20*}-\beta_i^*}I_1(\beta_{i*},\beta_{20}^*)\right] & \text{if } i=0,1,2,3, \\ H_1(\beta_{i+1}^*,\beta_{20}^*)\left[\sqrt{\beta_{20}^*-\beta_{i+1*}}I_1(\beta_{i+1*},\beta_{20*})-\sqrt{\beta_{20*}-\beta_i^*}I_1(\beta_i^*,\beta_{20}^*)\right] & \text{if } i=4,5,...,18, \\ -H_1(\beta_{18}^*,\beta_{20}^*)\sqrt{\beta_{20*}-\beta_{17}^*}I_1(\beta_{17}^*,\beta_{20}^*) & \text{if } i=19, \end{cases} \\ = \;& \frac{d_i(k)}{(\sigma\rho)^{\frac{1}{4}}}. \end{aligned}$$

We note that functions $c_i(k), d_i(k)$ for $i = 0, 1, ..., 19$ are all independent on $\varepsilon$, $\sigma$, $\tau$ and $\rho$. In fact, we see that the form of functions $c_i(k), d_i(k)$ for $i = 0, 1, ..., 5$ are similar to the form of $a_i(k)$ defined in proof of Lemma 3.9(i) in [1] for $i = 1, 2, 3$. So we can prove the following assertion

$$\sum_{i=0}^{19}[c_i(k)+d_i(k)] < 0 \quad \text{for } k \geq 0. \tag{2.9}$$

by applying similar technique used to prove

$$a_1(k)+a_2(k)+a_3(k) > 0 \quad \text{for } k \geq 0$$

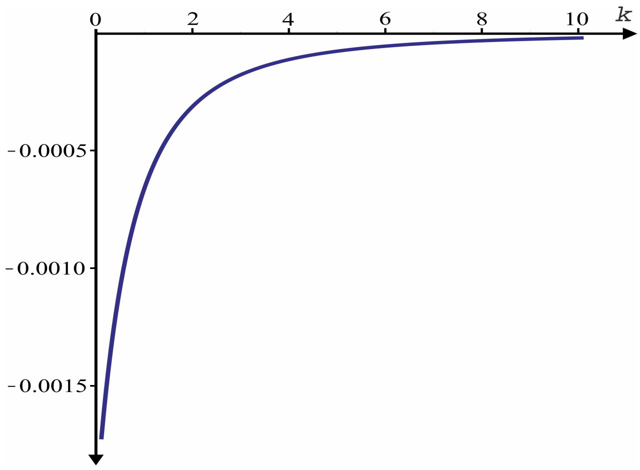


Figure 2.1: The graph of $\sum_{i=0}^{19}[c_i(k)+d_i(k)]$ for $k\geq 0$.

in the proof of Lemma 3.9(i) in [1]. See Figure 2.1. Since the proof of assertion (2.9) is easy and lengthy, we put the proof in Appendix B and omit it here. So by (2.8) and (2.9), then $T'_{\varepsilon_3}(p_2)<0$.

## 3. Appendix A

In this section, we prove assertion (1.8). We apply symbolic manipulator *Maple 16* to compute and obtain $c_i(k)$ and $d_i(k)$ for $i=0,1,...,5$. To prove assertion (1.8), we observe that assertion (1.8) holds for $k\geq 0$ if, and only if, all of inequalities (R1), (R2),...,(P10) hold for $k\geq 0$ where

(R1) $\frac{33}{500}c_0(k)+d_0(k)>0$

(R2) $\frac{93}{500}c_0(k)+d_1(k)>0$

(R3) $\frac{152}{500}c_0(k)+d_2(k)>0$

(R4) $\frac{220}{500}c_0(k)+d_3(k)>0$

(R5) $\frac{380}{500}c_1(k)+d_4(k)>0$

(R6) $\frac{120}{500}c_1(k)+\frac{65}{500}d_5(k)>0$

(R7) $c_2(k)+\frac{209}{500}d_5(k)>0$

(R8) $c_3(k)+\frac{130}{500}d_5(k)>0$

(R9) $c_4(k)+\frac{83}{500}d_5(k)>0$

(R10) $c_5(k)+\frac{13}{500}d_5(k)>0$

**Remark 1.** *By symbolic manipulator Maple 16, we see that for* $i=0,1,...,5$,

$$\textit{denominators of } c_i(k)=[\tilde{c}_i(k)]^{3/2}>0$$

*and*

$$\text{denominators of } d_i(k) = \left[\tilde{d}_i(k)\right]^{3/2} > 0,$$

*where $\tilde{c}_i(k)$ and $\tilde{d}_i(k)$ are polynomials of $k$, cf. [1, p17, the definitions of $a_1(k)$, $a_2(k)$, and $a_1(k)$.]. So we observe that inequalities (R1)–(R10) holds for $k \geq 0$ if, and only if,*

$$Num(\text{left side of inequality (R}l\text{)}) > 0 \quad \text{for } k \geq 0 \text{ and } l = 1, 2, ..., 10,$$

*where*

$$Num\left(\frac{p}{q}\right) = p \quad \text{for } p \in \mathbf{R} \quad \text{and} \quad q \in \mathbf{R}\backslash\{0\}.$$

Next, we begin to prove (R1)–(R10).

**The proof of inequality (R1).** Since we compute that

$$(6.7)^{2/3}(93.3 + 52.4k) - (10)^{2/3}(71.5 + 37.2k)\,(\approx 0.28 + 13.56k) > 0,$$

we compute and observe that, for $k \geq 0$,

$$\begin{aligned}
& Num\left(\frac{33}{500}c_0(k) + d_0(k)\right) \\
= \; & 6\left(\frac{10}{31}\right)^{3/4}\left[\left(5109068236873491\sqrt{273} - 3435696 \times 10^{10}\sqrt{3}\right)\right. \\
& \times \left(2894575521409000 + 92382 \times 10^{9}k\sqrt{310}\right)^{3/2} \\
& + \left(2080574313641875 \times 10^{11}\sqrt{115} - 192980448 \times 10^{17}\sqrt{13809}\right) \\
& \left.\times \left(2213857000 + 65400000k\sqrt{310}\right)^{3/2}\right] \\
& \left(\approx 10^{26} \times \left[6.75\,(93.37 + 52.46k)^{3/2} - 9.91\,(71.41 + 37.14k)^{3/2}\right]\right) \\
> \; & 10^{26} \times \left[6.7\,(93.3 + 52.4k)^{3/2} - 10\,(71.5 + 37.2k)^{3/2}\right] \\
= \; & 10^{26} \times \left\{\left[(6.7)^{2/3}(93.3 + 52.4k)\right]^{3/2} - \left[(10)^{2/3}(71.5 + 37.2k)\right]^{3/2}\right\} > 0.
\end{aligned}$$

So inequality (R1) holds.

**The proof of inequality (R2)**. Since we compute that

$$(190.3)^{2/3}(5.06+2.8k)-(3.58)^{2/3}(71.5+37.2k)\left(\approx 8\times 10^{-2}+5.58k\right)>0,$$

we compute and observe that, for $k\geq 0$,

$$\begin{aligned}
& Num\left(\frac{93}{500}c_0(k)+d_1(k)\right) \\
& \left(\approx 10^{28}\times\left[190.34\,(5.07+2.85k)^{3/2}-3.57(71.41+37.14k)^{3/2}\right]\right) \\
> \;& 10^{28}\times\left[190.3\,(5.06+2.8k)^{3/2}-3.58(71.5+37.2k)^{3/2}\right] \\
= \;& 10^{28}\times\left\{\left[(190.3)^{2/3}(5.06+2.8k)\right]^{3/2}-\left[(3.58)^{2/3}(71.5+37.2k)\right]^{3/2}\right\} \\
> \;& 0.
\end{aligned}$$

So inequality (R2) holds.

**The proof of inequality (R3)**. Since we compute that

$$(31.1)^{2/3}(10.37+58.4k)-(1.72)^{2/3}(71.42+37.2k)\left(\approx 2.\times 10^{-2}+524.14k\right)>0,$$

we compute and observe that, for $k\geq 0$,

$$\begin{aligned}
& Num\left(\frac{152}{500}c_0(k)+d_2(k)\right) \\
& \left(\approx 10^{29}\times\left[31.11\,(10.38+58.41k)^{3/2}-1.71(71.41+37.14k)^{3/2}\right]\right) \\
> \;& 10^{29}\times\left[31.1\,(10.37+58.4k)^{3/2}-1.72(71.42+37.2k)^{3/2}\right] \\
= \;& 10^{29}\times\left\{\left[(31.1)^{2/3}(10.37+58.4k)\right]^{3/2}-\left[(1.72)^{2/3}(71.42+37.2k)\right]^{3/2}\right\} \\
> \;& 0.
\end{aligned}$$

So inequality (R3) holds.

**The proof of inequality (R4)**: Since we compute that

$$(45)^{2/3}(25.7+14.4k)-(9.6)^{2/3}(71.5+37.2k)\,(\approx 2.17+14.15k)>0,$$

we compute and observe that, for $k\geq 0$,

$$Num\left(\frac{220}{500}c_0(k)+d_3(k)\right)$$

$$
\begin{aligned}
&\left(\approx 10^{29} \times \left[45.02\,(25.75 + 14.47k)^{3/2} - 9.56(71.41 + 37.14k)^{3/2}\right]\right) \\
> &\ 10^{29} \times \left[45\,(25.7 + 14.4k)^{3/2} - 9.6(71.5 + 37.2k)^{3/2}\right] \\
= &\ 10^{29} \times \left\{\left[(45)^{2/3}\,(25.7 + 14.4k)\right]^{3/2} - \left[(9.6)^{2/3}\,(71.5 + 37.2k)\right]^{3/2}\right\} \\
> &\ 0.
\end{aligned}
$$

So inequality (R4) holds.

**The proof of inequality (R5).** Since we compute that

$$
(3)^{2/3}\,(90.8 + 50.9k) - (89.6)^{2/3}\,(9.4 + 5.1k)\,(\approx 0.65 + 3.75k) > 0,
$$

we compute and observe that, for $k \geq 0$,

$$
\begin{aligned}
&Num\left(\frac{380}{500}c_1(k) + d_4(k)\right) \\
&\left(\approx 10^{29} \times \left[3.01\,(90.85 + 50.97k)^{3/2} - 89.51(9.38 + 5.02k)^{3/2}\right]\right) \\
> &\ 10^{29} \times \left[3\,(90.8 + 50.9k)^{3/2} - 89.6(9.4 + 5.1k)^{3/2}\right] \\
= &\ 10^{29} \times \left\{\left[(3)^{2/3}\,(90.8 + 50.9k)\right]^{3/2} - \left[(89.6)^{2/3}\,(9.4 + 5.1k)\right]^{3/2}\right\} \\
> &\ 0.
\end{aligned}
$$

So inequality (R5) holds.

**The proof of inequality (R6).** Since we compute that

$$
(9.5)^{2/3}\,(7.56 + 4.2k) - (6.84)^{2/3}\,(9.4 + 5.1k)\left(\approx 3 \times 10^{-2} + 0.46k\right) > 0,
$$

we compute and observe that, for $k \geq 0$,

$$
\begin{aligned}
&Num\left(\frac{120}{500}c_1(k) + \frac{65}{500}d_5(k)\right) \\
&\left(\approx 10^{31} \times \left[9.50\,(7.58 + 4.24k)^{3/2} - 6.82(9.38 + 5.02k)^{3/2}\right]\right) \\
> &\ 10^{31} \times \left[9.5\,(7.56 + 4.2k)^{3/2} - 6.84(9.4 + 5.1k)^{3/2}\right] \\
= &\ 10^{31} \times \left\{\left[(9.5)^{2/3}\,(7.56 + 4.2k)\right]^{3/2} - \left[(6.84)^{2/3}\,(9.4 + 5.1k)\right]^{3/2}\right\}
\end{aligned}
$$

$$> \; 0.$$

So inequality (R6) holds.

**The proof of inequality (R7).** Since we compute that

$$(740)^{2/3}\,(7.5+4.2k)-(3)^{2/3}\,(80+44k)\;(\approx 447.19+252.09k)>0,$$

we compute and observe that, for $k \geq 0$,

$$\begin{aligned}
& Num\left(c_2(k)+\frac{209}{500}d_5(k)\right) \\
& \left(\approx 10^{31}\times\left[740.75\,(7.58+4.24k)^{3/2}-2.19(79.10+43.27k)^{3/2}\right]\right) \\
> \;& 10^{31}\times\left[740\,(7.5+4.2k)^{3/2}-3(80+44k)^{3/2}\right] \\
= \;& 10^{31}\times\left\{\left[(740)^{2/3}\,(7.5+4.2k)\right]^{3/2}-\left[(3)^{2/3}\,(80+44k)\right]^{3/2}\right\} \\
> \;& 0.
\end{aligned}$$

So inequality (R7) holds.

**The proof of inequality (R8).** Since we compute that

$$(13.6)^{2/3}\,(7.5+4.2k)-(1.4)^{2/3}\,(32.7+18.2k)\;(\approx 1.80+1.15k)>0,$$

we compute and observe that, for $k \geq 0$,

$$\begin{aligned}
& Num\left(c_3(k)+\frac{130}{500}d_5(k)\right) \\
& \left(\approx 10^{29}\times\left[13.63\,(7.58+4.24k)^{3/2}-1.36(32.63+18.10k)^{3/2}\right]\right) \\
> \;& 10^{29}\times\left[13.6\,(7.5+4.2k)^{3/2}-1.4(32.7+18.2k)^{3/2}\right] \\
= \;& 10^{29}\times\left\{\left[(13.6)^{2/3}\,(7.5+4.2k)\right]^{3/2}-\left[(1.4)^{2/3}\,(32.7+18.2k)\right]^{3/2}\right\} \\
> \;& 0.
\end{aligned}$$

So inequality (R8) holds.

**The proof of inequality (R9).** Since we compute that

$$(251.2)^{2/3}\,(7.55+4.22k)-(8.73)^{2/3}\,(70.7+39.55k)\;(\approx 1.51+2.41k)>0,$$

we compute and observe that, for $k \geq 0$,

$$\begin{aligned}
& Num\left(c_4(k) + \frac{83}{500}d_5(k) > 0\right) \\
& \left(\approx 10^{28} \times \left[251.25\,(7.58 + 4.24k)^{3/2} - 8.71(70.66 + 39.53k)^{3/2}\right]\right) \\
> \;& 10^{28} \times \left[251.2\,(7.55 + 4.22k)^{3/2} - 8.73(70.7 + 39.55k)^{3/2}\right] \\
= \;& 10^{28} \times \left\{\left[(251.2)^{2/3}\,(7.55 + 4.22k)\right]^{3/2} - \left[(8.73)^{2/3}\,(70.7 + 39.55k)\right]^{3/2}\right\} \\
> \;& 0.
\end{aligned}$$

So inequality (R9) holds.

**The proof of inequality (R10).** Since we compute that

$$(5.2)^{2/3}\,(7.5 + 4.2k) - (1.4)^{2/3}\,(11.7 + 6.6k)\;(\approx 7.86 + 4.34k) > 0,$$

we compute and observe that, for $k \geq 0$,

$$\begin{aligned}
& Num\left(c_5(k) + \frac{13}{500}d_5(k) > 0\right) \\
& \left(\approx 10^{31} \times \left[5.29\,(7.58 + 4.24k)^{3/2} - 1.36(11.67 + 6.55k)^{3/2}\right]\right) \\
> \;& 10^{31} \times \left[5.2\,(7.5 + 4.2k)^{3/2} - 1.4(11.7 + 6.6k)^{3/2}\right] \\
= \;& 10^{31} \times \left\{\left[(5.2)^{2/3}\,(7.5 + 4.2k)\right]^{3/2} - \left[(1.4)^{2/3}\,(11.7 + 6.6k)\right]^{3/2}\right\} \\
> \;& 0.
\end{aligned}$$

So inequality (R10) holds.

Thus the proof of assertion (1.8) is complete. ■

## 4. Appendix B

In this section, we prove assertion (2.9). We apply symbolic manipulator *Maple 16* to compute and obtain $c_i(k)$ and $d_i(k)$ for $i = 0, 1, ..., 19$. To prove assertion (2.9), we observe that assertion (2.9) holds for $k \geq 0$ if, and only if, all of inequalities (P1), (P2),...,(P39) hold for $k \geq 0$ where

(P1) $\frac{231}{500}c_0(k) + d_0(k) < 0$

(P2) $\frac{67.4}{500}c_0(k) + d_1(k) < 0$

(P3) $\frac{108.7}{500}c_0(k) + d_2(k) < 0$

(P4) $\frac{146.9}{500}c_0(k) + d_3(k) < 0$

(P5) $\frac{153.9}{500}c_0(k) + \frac{422.5}{500}d_4(k) < 0$

(P6) $\frac{30}{500}c_1(k) + \frac{77.5}{500}d_4(k) < 0$

(P7) $\frac{228.6}{500}c_1(k) + d_5(k) < 0$

(P8) $\frac{241.4}{500}c_1(k) + \frac{465.6}{500}d_6(k) < 0$

(P9) $\frac{19}{500}c_2(k) + \frac{34.4}{500}d_6(k) < 0$

(P10) $\frac{306.7}{500}c_2(k) + d_7(k) < 0$

(P11) $\frac{174.3}{500}c_2(k) + \frac{263}{500}d_8(k) < 0$

(P12) $\frac{168.8}{500}c_3(k) + \frac{237}{500}d_8(k) < 0$

(P13) $\frac{331.2}{500}c_3(k) + \frac{437.5}{500}d_9(k) < 0$

(P14) $\frac{50.9}{500}c_4(k) + \frac{62.5}{500}d_9(k) < 0$

(P15) $\frac{427.1}{500}c_4(k) + d_{10}(k) < 0$

(P16) $\frac{22}{500}c_4(k) + \frac{25.2}{500}d_{11}(k) < 0$

(P17) $\frac{453}{500}c_5(k) + \frac{474.8}{500}d_{11}(k) < 0$

(P18) $\frac{47}{500}c_5(k) + \frac{48.7}{500}d_{12}(k) < 0$

(P19) $\frac{476.4}{500}c_6(k) + \frac{451.3}{500}d_{12}(k) < 0$

(P20) $\frac{23.6}{500}c_6(k) + \frac{22.5}{500}d_{13}(k) < 0$

(P21) $c_7(k) + \frac{432.2}{500}d_{13}(k) < 0$

(P22) $\frac{57.6}{500}c_8(k) + \frac{45.3}{500}d_{13}(k) < 0$

(P23) $\frac{442.4}{500}c_8(k) + \frac{352.8}{500}d_{14}(k) < 0$

(P24) $\frac{205.2}{500}c_9(k) + \frac{147.2}{500}d_{14}(k) < 0$

(P25) $\frac{294.8}{500}c_9(k) + \frac{220}{500}d_{15}(k) < 0$

(P26) $\frac{421.5}{500}c_{10}(k) + \frac{280}{500}d_{15}(k) < 0$

(P27) $\frac{78.5}{500}c_{10}(k) + \frac{56}{500}d_{16}(k) < 0$

(P28) $c_{11}(k) + \frac{313.4}{500}d_{16}(k) < 0$

(P29) $\frac{239.8}{500}c_{12}(k) + \frac{130.6}{500}d_{16}(k) < 0$

(P30) $\frac{260.2}{500}c_{12}(k) + \frac{159.1}{500}d_{17}(k) < 0$

(P31) $c_{13}(k) + \frac{260.4}{500} d_{17}(k) < 0$ (P32) $\frac{185.5}{500} c_{14}(k) + \frac{80.5}{500} d_{17}(k) < 0$

(P33) $\frac{314.5}{500} c_{14}(k) + \frac{168}{500} d_{18}(k) < 0$ (P34) $c_{15}(k) + \frac{214.7}{500} d_{18}(k) < 0$

(P35) $\frac{356.5}{500} c_{16}(k) + \frac{117.2}{500} d_{18}(k) < 0$ (P36) $\frac{143.5}{500} c_{16}(k) + \frac{82.8}{500} d_{19}(k) < 0$

(P37) $c_{17}(k) + \frac{202.8}{500} d_{19}(k) < 0$ (P38) $c_{18}(k) + \frac{119.7}{500} d_{19}(k) < 0$

(P39) $c_{19}(k) + \frac{94.7}{500} d_{19}(k) < 0$

**Remark 2.** *By symbolic manipulator Maple 16, we see that for* $i = 0, 1, ..., 19$,

$$\text{denominators of } c_i(k) = [\tilde{c}_i(k)]^{3/2} > 0$$

*and*

$$\text{denominators of } d_i(k) = \left[\tilde{d}_i(k)\right]^{3/2} > 0,$$

*where* $\tilde{c}_i(k)$ *and* $\tilde{d}_i(k)$ *are polynomials of* $k$, *cf. [1, p17, the definitions of* $a_1(k)$, $a_2(k)$, *and* $a_1(k)$*.]. So we observe that inequalities (P1)–(P39) holds for* $k \geq 0$ *if, and only if,*

$$Num(\text{left side of inequality } (Pl)) > 0 \quad \text{for } k \geq 0 \text{ and } l = 1, 2, ..., 39,$$

*where*

$$Num\left(\frac{p}{q}\right) = p \quad \text{for } p \in \mathbf{R} \quad \text{and} \quad q \in \mathbf{R}\backslash\{0\}.$$

Next, we begin to prove (P1)–(P39).

**The proof of inequality (P1).** Since we compute that

$$(1027.62)^{2/3}(2.81 + 1.57k) - (6.46)^{2/3}(89.28 + 46.1k) \left(\approx -23.5 - 2.5 \times 10^{-2} k\right) < 0,$$

we compute and observe that, for $k \geq 0$,

$$Num\left(\frac{231}{500} c_0(k) + d_0(k)\right)$$
$$\left(\approx 10^{65} \times \left[1027.61\,(2.80 + 1.56k)^{3/2} - 6.460(89.29 + 46.11k)^{3/2}\right]\right)$$

$$
\begin{aligned}
&< 10^{65} \times \left[1027.62\,(2.81+1.57k)^{3/2} - 6.460(89.28+46.1k)^{3/2}\right] \\
&= 10^{65} \times \left\{ \left[(1027.62)^{2/3}\,(2.81+1.57k)\right]^{3/2} - \left[(6.46)^{2/3}\,(89.28+46.1k)\right]^{3/2} \right\} \\
&< 0.
\end{aligned}
$$

So inequality (P1) holds.

**The proof of inequality (P2).** Since we compute that

$$
(29.99)^{2/3}\,(35.8+20k) - (8.57)^{2/3}\,(89.2+46.1k)\,\left(\approx -27.9 - 3.1\times 10^{-3}k\right) < 0,
$$

we compute and observe that, for $k \geq 0$,

$$
\begin{aligned}
& Num\left(\frac{67.4}{500}c_0(k) + d_1(k)\right) \\
& \left(\approx 10^{64} \times \left[29.98\,(35.73+19.99k)^{3/2} - 8.572(89.29+46.11k)^{3/2}\right]\right) \\
&< 10^{64} \times \left[29.99\,(35.8+20k)^{3/2} - 8.57(89.2+46.1k)^{3/2}\right] \\
&= 10^{64} \times \left\{ \left[(29.99)^{2/3}\,(35.8+20k)\right]^{3/2} - \left[(8.57)^{2/3}\,(89.2+46.1k)\right]^{3/2} \right\} \\
&< 0.
\end{aligned}
$$

So inequality (P2) holds.

**The proof of inequality (P3).** Since we compute that

$$
(4.836)^{2/3}\,(97.3+54.46k) - (6.21)^{2/3}\,(89.25+46.1k)\,(\approx -23.9 - 0.1k) < 0,
$$

we compute and observe that, for $k \geq 0$,

$$
\begin{aligned}
& Num\left(\frac{108.7}{500}c_0(k) + d_2(k)\right) \\
& \left(\approx 10^{65} \times \left[4.835\,(97.29+54.45k)^{3/2} - 6.211(89.29+46.11k)^{3/2}\right]\right) \\
&< 10^{65} \times \left[4.836\,(97.3+54.46k)^{3/2} - 6.21(89.25+46.1k)^{3/2}\right] \\
&= 10^{65} \times \left\{ \left[(4.836)^{2/3}\,(97.3+54.46k)\right]^{3/2} - \left[(6.21)^{2/3}\,(89.25+46.1k)\right]^{3/2} \right\} \\
&< 0.
\end{aligned}
$$

So inequality (P3) holds.

**The proof of inequality (P4).** Since we compute that

$$(653.5)^{2/3}\,(4.7+2.6k)-(8.77)^{2/3}\,(89.25+46.1k)\,(\approx -25.6-0.2k)<0,$$

we compute and observe that, for $k \geq 0$,

$$\begin{aligned}
& Num\left(\frac{146.9}{500}c_0(k)+d_3(k)\right) \\
& \left(\approx 10^{63}\times\left[653.49\,(4.65+2.60k)^{3/2}-8.78(89.29+46.11k)^{3/2}\right]\right) \\
< \; & 10^{63}\times\left[653.5\,(4.65+2.60k)^{3/2}-8.77(89.25+46.1k)^{3/2}\right] \\
= \; & 10^{63}\times\left\{\left[(653.5)^{2/3}\,(4.7+2.6k)\right]^{3/2}-\left[(8.77)^{2/3}\,(89.25+46.1k)\right]^{3/2}\right\} \\
< \; & 0.
\end{aligned}$$

So inequality (P4) holds.

**The proof of inequality (P5).** Since we compute that

$$(6.847)^{2/3}\,(48.66+27.231k)-(3.108)^{2/3}\,(89.28+46.11k)\,(\approx -14.6-1.1k)<0,$$

we compute and observe that, for $k \geq 0$,

$$\begin{aligned}
& Num\left(\frac{153.9}{500}c_0(k)+\frac{422.5}{500}d_4(k)\right) \\
& \left(\approx 10^{62}\times\left[6.8464\,(48.65+27.230k)^{3/2}-3.1083(89.29+46.111k)^{3/2}\right]\right) \\
< \; & 10^{62}\times\left[6.847\,(48.66+27.231k)^{3/2}-3.108(89.28+46.11k)^{3/2}\right] \\
= \; & 10^{62}\times\left\{\left[(6.847)^{2/3}\,(48.66+27.231k)\right]^{3/2}-\left[(3.108)^{2/3}\,(89.28+46.11k)\right]^{3/2}\right\} \\
< \; & 0.
\end{aligned}$$

So inequality (P5) holds.

**The proof of inequality (P6).** Since we compute that

$$(9.21)^{2/3}\,(48.66+27.24k)-(5.69)^{2/3}\,(72.19+37.58k)\,(\approx -16.2-8.7k)<0,$$

we compute and observe that, for $k \geq 0$,

$$\begin{aligned}
& Num\left(\frac{30}{500}c_1(k) + \frac{77.5}{500}d_4(k)\right) \\
& \left(\approx 10^{67} \times \left[9.20\,(48.65 + 27.23k)^{3/2} - 5.70(72.20 + 37.59k)^{3/2}\right]\right) \\
< \;& 10^{67} \times \left[9.21\,(48.66 + 27.24k)^{3/2} - 5.69(72.19 + 37.58k)^{3/2}\right] \\
= \;& 10^{67} \times \left\{\left[(9.21)^{2/3}\,(48.66 + 27.24k)\right]^{3/2} - \left[(5.69)^{2/3}\,(72.19 + 37.58k)\right]^{3/2}\right\} \\
< \;& 0.
\end{aligned}$$

So inequality (P6) holds.

**The proof of inequality (P7)**. Since we compute that

$$(70.11)^{2/3}\,(12.94 + 7.24k) - (59.09)^{2/3}\,(72.19 + 37.58k)\;(\approx -875.1 - 447.0k) < 0,$$

we compute and observe that, for $k \geq 0$,

$$\begin{aligned}
& Num\left(\frac{228.6}{500}c_1(k) + d_5(k)\right) \\
& \left(\approx 10^{70} \times \left[70.10\,(12.93 + 7.23k)^{3/2} - 59.10(72.20 + 37.59k)^{3/2}\right]\right) \\
< \;& 10^{70} \times \left[70.11\,(12.94 + 7.24k)^{3/2} - 59.09(72.19 + 37.58k)^{3/2}\right] \\
= \;& 10^{70} \times \left\{\left[(70.11)^{2/3}\,(12.94 + 7.24k)\right]^{3/2} - \left[(59.09)^{2/3}\,(72.19 + 37.58k)\right]^{3/2}\right\} \\
< \;& 0.
\end{aligned}$$

So inequality (P7) holds.

**The proof of inequality (P8)**. Since we compute that

$$\begin{aligned}
& (740.3506)^{2/3}\,(3.166 + 1.7701k) - (7.5664)^{2/3}\,(72.208 + 37.596k) \\
& \left(\approx -19.1 - 3.7 \times 10^{-2}k\right) < 0,
\end{aligned}$$

we compute and observe that, for $k \geq 0$,

$$Num\left(\frac{241.4}{500}c_1(k) + \frac{465.6}{500}d_6(k)\right)$$

$$\begin{aligned}
&\left(\approx 10^{72} \times \left[740.3505\,(3.165+1.7700k)^{3/2} - 7.5665(72.209+37.597k)^{3/2}\right]\right)\\
< &\ 10^{72} \times \left[740.3506\,(3.166+1.7701k)^{3/2} - 7.5664(72.208+37.596k)^{3/2}\right]\\
= &\ 10^{72} \times \left\{\left[(740.3506)^{2/3}\,(3.166+1.7701k)\right]^{3/2} - \left[(7.5664)^{2/3}\,(72.208+37.596k)\right]^{3/2}\right\}\\
< &\ 0.
\end{aligned}$$

So inequality (P8) holds.

**The proof of inequality (P9)**. Since we compute that

$$(24.72)^{2/3}\,(3.166+1.771k) - (5.58)^{2/3}\,(9.125+4.787k)\ \left(\approx -1.8 - 3.1\times 10^{-2}k\right) < 0,$$

we compute and observe that, for $k \geq 0$,

$$\begin{aligned}
&Num\left(\frac{19}{500}c_2(k) + \frac{34.4}{500}d_6(k)\right)\\
&\left(\approx 10^{71} \times \left[24.71\,(3.165+1.770k)^{3/2} - 5.59(9.126+4.788k)^{3/2}\right]\right)\\
< &\ 10^{71} \times \left[24.72\,(3.167+1.771k)^{3/2} - 5.59(9.125+4.787k)^{3/2}\right]\\
= &\ 10^{71} \times \left\{\left[(24.72)^{2/3}\,(3.166+1.771k)\right]^{3/2} - \left[(5.58)^{2/3}\,(9.125+4.787k)\right]^{3/2}\right\}\\
< &\ 0
\end{aligned}$$

So inequality (P9) holds.

**The proof of inequality (P10)**. Since we compute that

$$\begin{aligned}
&(3.9903)^{2/3}\,(7.89+4.40474k) - (3.5213)^{2/3}\,(9.11++4.78810k)\\
&\left(\approx -1.2 - 1.0\times 10^{-3}k\right) < 0,
\end{aligned}$$

we compute and observe that, for $k \geq 0$,

$$\begin{aligned}
&Num\left(\frac{306.7}{500}c_2(k) + d_7(k)\right)\\
&\left(\approx 10^{67} \times \left[3.9902\,(7.88+4.40473k)^{3/2} - 3.5214(9.12+4.78811k)^{3/2}\right]\right)\\
< &\ 10^{67} \times \left[3.9903\,(7.89+4.40474k)^{3/2} - 3.5213(9.11+4.7881k)^{3/2}\right]\\
= &\ 10^{67} \times \left\{\left[(3.9903)^{2/3}\,(7.89+4.40474k)\right]^{3/2} - \left[(3.5213)^{2/3}\,(9.11++4.78810k)\right]^{3/2}\right\}
\end{aligned}$$

$$< \ 0.$$

So inequality (P10) holds.

**The proof of inequality (P11).** Since we compute that

$$(22.6770)^{2/3}(1.1+0.6128k)-(1.0387)^{2/3}(9.11+4.7880k)$$
$$\left(\approx -0.5-1.1\times 10^{-3}k\right)<0,$$

we compute and observe that, for $k \geq 0$,

$$\begin{aligned}
& Num\left(\frac{174.3}{500}c_2(k)+\frac{263}{500}d_8(k)\right) \\
& \left(\approx 10^{72}\times\left[22.6769\,(1.09+0.6127k)^{3/2}-1.0388(9.12+4.7881k)^{3/2}\right]\right) \\
< \ & 10^{72}\times\left[22.6770\,(1.10+0.6128k)^{3/2}-1.0387(9.11+4.7880k)^{3/2}\right] \\
= \ & 10^{72}\times\left\{\left[(22.6770)^{2/3}(1.1+0.6128k)\right]^{3/2}-\left[(1.0387)^{2/3}(9.11+4.7880k)\right]^{3/2}\right\} \\
< \ & 0.
\end{aligned}$$

So inequality (P11) holds.

**The proof of inequality (P12).** Since we compute that

$$(4750.622)^{2/3}(1.10+0.613k)-(9.360)^{2/3}(73.82+39.011k)\left(\approx -17.0-3.0\times 10^{-3}k\right)<0,$$

we compute and observe that, for $k \geq 0$,

$$\begin{aligned}
& Num\left(\frac{168.8}{500}c_3(k)+\frac{237}{500}d_8(k)\right) \\
& \left(\approx 10^{71}\times\left[4750.621\,(1.09+0.612k)^{3/2}-9.361(73.83+39.012k)^{3/2}\right]\right) \\
< \ & 10^{71}\times\left[4750.622\,(1.1+0.613k)^{3/2}-9.360(73.82+39.011k)^{3/2}\right] \\
= \ & 10^{71}\times\left\{\left[(4750.622)^{2/3}(1.10+0.613k)\right]^{3/2}-\left[(9.360)^{2/3}(73.82+39.011k)\right]^{3/2}\right\} \\
< \ & 0.
\end{aligned}$$

So inequality (P12) holds.

**The proof of inequality (P13).** Since we compute that

$$(93.2113)^{2/3}(6.72+3.745k)-(2.7725)^{2/3}(73.82+39.011k)\left(\approx -7.5-6.9\times 10^{-4}k\right)<0,$$

we compute and observe that, for $k \geq 0$,

$$
\begin{aligned}
& Num\left(\frac{331.2}{500}c_3(k)+\frac{437.5}{500}d_9(k)\right) \\
& \left(\approx 10^{67}\times\left[93.2112\,(6.71+3.744k)^{3/2}-2.7726(73.83+39.012k)^{3/2}\right]\right) \\
< \;& 10^{67}\times\left[93.2113\,(6.72+3.745k)^{3/2}-2.7725(73.82+39.011k)^{3/2}\right] \\
= \;& 10^{67}\times\left\{\left[(93.2113)^{2/3}\,(6.72+3.745k)\right]^{3/2}-\left[(2.7725)^{2/3}\,(73.82+39.011k)\right]^{3/2}\right\} \\
< \;& 0.
\end{aligned}
$$

So inequality (P13) holds.

**The proof of inequality (P14).** Since we compute that

$$(1.472)^{2/3}\,(6.72+3.745k)-(3.959)^{2/3}\,(3.65+1.938k)\left(\approx -0.4-3.9\times 10^{-3}k\right)<0,$$

we compute and observe that, for $k \geq 0$,

$$
\begin{aligned}
& Num\left(\frac{50.9}{500}c_4(k)+\frac{62.5}{500}d_9(k)\right) \\
& \left(\approx 10^{63}\times\left[1.471\,(6.71+3.744k)^{3/2}-3.960(3.64+1.939k)^{3/2}\right]\right) \\
< \;& 10^{63}\times\left[1.472\,(6.72+3.745k)^{3/2}-3.959(3.63+1.938k)^{3/2}\right] \\
= \;& 10^{63}\times\left\{\left[(1.472)^{2/3}\,(6.72+3.745k)\right]^{3/2}-\left[(3.959)^{2/3}\,(3.65+1.938k)\right]^{3/2}\right\} \\
< \;& 0.
\end{aligned}
$$

So inequality (P14) holds.

**The proof of inequality (P15).** Since we compute that

$$
\begin{aligned}
& (1.2344)^{2/3}\,(3.43+1.90648k)-(1.2031)^{2/3}\,(3.63+1.93947k) \\
& \left(\approx -0.1-7.4\times 10^{-5}k\right)<0,
\end{aligned}
$$

we compute and observe that, for $k \geq 0$,

$$Num\left(\frac{427.1}{500}c_4(k)+d_{10}(k)\right)$$

$$\left(\approx 10^{70} \times \left[1.2343\,(3.42 + 1.90647k)^{3/2} - 1.20332(3.64 + 1.93948k)^{3/2}\right]\right)$$
$$< \; 10^{70} \times \left[1.2344\,(3.43 + 1.90648k)^{3/2} - 1.20331(3.63 + 1.93947k)^{3/2}\right]$$
$$= \; 10^{70} \times \left\{\left[(1.2344)^{2/3}\,(3.43 + 1.90648k)\right]^{3/2} - \left[(1.2031)^{2/3}\,(3.63 + 1.93947k)\right]^{3/2}\right\}$$
$$< \; 0.$$

So inequality (P15) holds.

**The proof of inequality (P16).** Since we compute that

$$(6.36)^{2/3}\,(1.82 + 1.011k) - (2.41)^{2/3}\,(3.63 + 1.938k)\,\left(\approx -0.2 - 1.3 \times 10^{-2}k\right) < 0,$$

we compute and observe that, for $k \geq 0$,

$$Num\left(\frac{22}{500}c_4(k) + \frac{25.2}{500}d_{11}(k)\right)$$
$$\left(\approx 10^{65} \times \left[6.359\,(1.81 + 1.010k)^{3/2} - 2.412(3.64 + 1.939k)^{3/2}\right]\right)$$
$$< \; 10^{65} \times \left[6.360\,(1.82 + 1.011k)^{3/2} - 2.410(3.63 + 1.938k)^{3/2}\right]$$
$$= \; 10^{65} \times \left\{\left[(6.36)^{2/3}\,(1.82 + 1.011k)\right]^{3/2} - \left[(2.41)^{2/3}\,(3.63 + 1.938k)\right]^{3/2}\right\}$$
$$< \; 0.$$

So inequality (P16) holds.

**The proof of inequality (P17).** Since we compute that

$$(73.547)^{2/3}\,(1.82 + 1.0108k) - (4.544)^{2/3}\,(12.08 + 6.4684k)$$
$$\left(\approx -1.1 - 1.9 \times 10^{-3}k\right) < 0,$$

we compute and observe that, for $k \geq 0$,

$$Num\left(\frac{453}{500}c_5(k) + \frac{474.8}{500}d_{11}(k)\right)$$
$$\left(\approx 10^{66} \times \left[73.5460\,(1.81 + 1.0107k)^{3/2} - 4.5440(12.09 + 6.4685k)^{3/2}\right]\right)$$
$$< \; 10^{66} \times \left[73.5470\,(1.82 + 1.0108k)^{3/2} - 4.5440(12.08 + 6.4684k)^{3/2}\right]$$
$$= \; 10^{66} \times \left\{\left[(73.547)^{2/3}\,(1.82 + 1.0108k)\right]^{3/2} - \left[(4.544)^{2/3}\,(12.08 + 6.4684k)\right]^{3/2}\right\}$$

$$< \; 0.$$

So inequality (P17) holds.

**The proof of inequality (P18).** Since we compute that

$$(7.6306)^{2/3}(3.54+1.9746k)-(1.2922)^{2/3}(12.09+6.4685k)$$
$$\left(\approx -0.5-1.7\times 10^{-2}k\right)<0,$$

we compute and observe that, for $k \geq 0$,

$$\begin{aligned}
& Num\left(\frac{47}{500}c_5(k)+\frac{48.7}{500}d_{12}(k)\right) \\
& \left(\approx 10^{69}\times\left[7.630\,(3.54+1.974k)^{3/2}-1.2922(12.09+6.4685k)^{3/2}\right]\right) \\
< \;& 10^{69}\times\left[7.631\,(3.55+1.975k)^{3/2}-1.2920(12.08+6.468k)^{3/2}\right] \\
= \;& 10^{69}\times\left\{\left[(7.631)^{2/3}(3.55+1.975k)\right]^{3/2}-\left[(1.292)^{2/3}(12.08+6.468k)\right]^{3/2}\right\} \\
< \;& 0.
\end{aligned}$$

So inequality (P18) holds.

**The proof of inequality (P19).** Since we compute that

$$(5.0305)^{2/3}(3.55+1.9747k)-(1.1974)^{2/3}(9.54+5.1419k)$$
$$\left(\approx -0.3-5.5\times 10^{-4}k\right)<0,$$

we compute and observe that, for $k \geq 0$,

$$\begin{aligned}
& Num\left(\frac{476.4}{500}c_6(k)+\frac{451.3}{500}d_{12}(k)\right) \\
& \left(\approx 10^{73}\times\left[5.0304\,(3.54+1.9746k)^{3/2}-1.1975(9.55+5.1420k)^{3/2}\right]\right) \\
< \;& 10^{73}\times\left[5.0305\,(3.55+1.9747k)^{3/2}-1.1974(9.54+5.1419k)^{3/2}\right] \\
= \;& 10^{73}\times\left\{\left[(5.0305)^{2/3}(3.55+1.9747k)\right]^{3/2}-\left[(1.1974)^{2/3}(9.54+5.1419k)\right]^{3/2}\right\} \\
< \;& 0.
\end{aligned}$$

So inequality (P19) holds.

**The proof of inequality (P20).** Since we compute that

$$(2.5)^{2/3}(45.17+25.11k)-(27.0)^{2/3}(9.54+5.14k)\left(\approx -2.6-6.9\times 10^{-3}k\right)<0,$$

we compute and observe that, for $k \geq 0$,

$$\begin{aligned}
& Num\left(\frac{23.6}{500}c_6(k)+\frac{22.5}{500}d_{13}(k)\right) \\
& \left(\approx 10^{69}\times\left[2.49\,(45.16+25.109k)^{3/2}-27.06(9.55+5.142k)^{3/2}\right]\right) \\
< \ & 10^{69}\times\left[2.5\,(45.17+25.11k)^{3/2}-27.00(9.54+5.14k)^{3/2}\right] \\
= \ & 10^{69}\times\left\{\left[(2.5)^{2/3}(45.17+25.11k)\right]^{3/2}-\left[(27.0)^{2/3}(9.54+5.14k)\right]^{3/2}\right\} \\
< \ & 0.
\end{aligned}$$

So inequality (P20) holds.

**The proof of inequality (P21).** Since we compute that

$$\begin{aligned}
& (11.183)^{2/3}(45.17+25.11k)-(5.199)^{2/3}(77.39+41.844k) \\
& \left(\approx -6.3-7.9\times 10^{-3}k\right)<0,
\end{aligned}$$

we compute and observe that, for $k \geq 0$,

$$\begin{aligned}
& Num\left(c_7(k)+\frac{432.2}{500}d_{13}(k)\right) \\
& \left(\approx 10^{71}\times\left[11.182\,(45.16+25.109k)^{3/2}-5.1993(77.40+41.8442k)^{3/2}\right]\right) \\
< \ & 10^{71}\times\left[11.183\,(45.17+25.11k)^{3/2}-5.1990(77.39+41.844)^{3/2}\right] \\
= \ & 10^{71}\times\left\{\left[(11.183)^{2/3}(45.17+25.11k)\right]^{3/2}-\left[(5.199)^{2/3}(77.39+41.844k)\right]^{3/2}\right\} \\
< \ & 0.
\end{aligned}$$

So inequality (P21) holds.

**The proof of inequality (P22).** Since we compute that

$$\begin{aligned}
& (2.339)^{2/3}(45.17+25.11k)-(544.958)^{2/3}(1.21+0.664k) \\
& \left(\approx -1.1-5.5\times 10^{-2}k\right)<0,
\end{aligned}$$

we compute and observe that, for $k \geq 0$,

$$\begin{aligned}
& Num\left(\frac{57.6}{500}c_8(k)+\frac{45.3}{500}d_{13}(k)\right) \\
& \left(\approx 10^{68}\times\left[2.338\,(45.16+25.109k)^{3/2}-544.959(1.22+0.6648k)^{3/2}\right]\right) \\
< \;& 10^{68}\times\left[2.339\,(45.17+25.110k)^{3/2}-544.958(1.21+0.664k)^{3/2}\right] \\
= \;& 10^{68}\times\left\{\left[(2.339)^{2/3}\,(45.17+25.11k)\right]^{3/2}-\left[(544.958)^{2/3}\,(1.21+0.664k)\right]^{3/2}\right\} \\
< \;& 0.
\end{aligned}$$

So inequality (P22) holds.

**The proof of inequality (P23).** Since we compute that

$$\begin{aligned}
& (1.7959)^{2/3}\,(98.06+54.477k)-(1.3323)^{2/3}\,(122.42+66.485k) \\
& \left(\approx -3.3-1.0\times 10^{-2}k\right)<0,
\end{aligned}$$

we compute and observe that, for $k \geq 0$,

$$\begin{aligned}
& Num\left(\frac{442.4}{500}c_8(k)+\frac{352.8}{500}d_{14}(k)\right) \\
& \left(\approx 10^{66}\times\left[1.7958\,(98.05+54.476k)^{3/2}-1.3324(122.43+66.486k)^{3/2}\right]\right) \\
< \;& 10^{66}\times\left[1.7959\,(98.06+54.477k)^{3/2}-1.3323(122.42+66.485k)^{3/2}\right] \\
= \;& 10^{66}\times\left\{\left[(1.7959)^{2/3}\,(98.06+54.477k)\right]^{3/2}-\left[(1.3323)^{2/3}\,(122.42+66.485k)\right]^{3/2}\right\} \\
< \;& 0.
\end{aligned}$$

So inequality (P23) holds.

**The proof of inequality (P24).** Since we compute that

$$(3.931)^{2/3}\,(98.1+54.477k)-(5.558)^{2/3}\,(79.3+43.258k)\,\left(\approx -4.4-4.0\times 10^{-2}k\right)<0,$$

we compute and observe that, for $k \geq 0$,

$$Num\left(\frac{205.2}{500}c_9(k)+\frac{147.2}{500}d_{14}(k)\right)$$

$$
\begin{aligned}
& \left(\approx 10^{68} \times \left[3.930\,(98.05+54.476k)^{3/2} - 5.559(79.33+43.259k)^{3/2}\right]\right) \\
< \;& 10^{68} \times \left[3.931\,(98.1+54.477k)^{3/2} - 5.558(79.3+43.258k)^{3/2}\right] \\
= \;& 10^{68} \times \left\{\left[(3.931)^{2/3}\,(98.1+54.477k)\right]^{3/2} - \left[(5.558)^{2/3}\,(79.3+43.258k)\right]^{3/2}\right\} \\
< \;& 0.
\end{aligned}
$$

So inequality (P24) holds.

**The proof of inequality (P25).** Since we compute that

$$
(5.648)^{2/3}\,(73.1+40.57k) - (5.132)^{2/3}\,(79.3+43.25k)\,\left(\approx -4.1 - 1.2\times 10^{-2}k\right) < 0,
$$

we compute and observe that, for $k \geq 0$,

$$
\begin{aligned}
& Num\left(\frac{294.8}{500}c_9(k) + \frac{220}{500}d_{15}(k)\right) \\
& \left(\approx 10^{68} \times \left[5.647\,(73.06+40.56k)^{3/2} - 5.133(79.33+43.259k)^{3/2}\right]\right) \\
< \;& 10^{68} \times \left[5.648\,(73.1+40.57k)^{3/2} - 5.132(79.3+43.25k)^{3/2}\right] \\
= \;& 10^{68} \times \left\{\left[(5.648)^{2/3}\,(73.1+40.57k)\right]^{3/2} - \left[(5.132)^{2/3}\,(79.3+43.25k)\right]^{3/2}\right\} \\
< \;& 0.
\end{aligned}
$$

So inequality (P25) holds.

**The proof of inequality (P26).** Since we compute that

$$
\begin{aligned}
& (20.8432)^{2/3}\,(73.1+40.566k) - (6.5331)^{2/3}\,(160.6+87.934k) \\
& \left(\approx -7.6 - 7.2\times 10^{-2}k\right) < 0,
\end{aligned}
$$

we compute and observe that, for $k \geq 0$,

$$
\begin{aligned}
& Num\left(\frac{421.5}{500}c_{10}(k) + \frac{280}{500}d_{15}(k)\right) \\
& \left(\approx 10^{62} \times \left[20.8431\,(73.06+40.565k)^{3/2} - 6.5332(160.67+87.935k)^{3/2}\right]\right) \\
< \;& 10^{62} \times \left[20.8432\,(73.1+40.566k)^{3/2} - 6.5331(160.6+87.934k)^{3/2}\right] \\
= \;& 10^{62} \times \left\{\left[(20.8432)^{2/3}\,(73.1+40.566k)\right]^{3/2} - \left[(6.5331)^{2/3}\,(160.6+87.934k)\right]^{3/2}\right\}
\end{aligned}
$$

$$< \ 0.$$

So inequality (P26) holds.

**The proof of inequality (P27).** Since we compute that

$$\begin{aligned}&(388.182)^{2/3}(3.81+2.112k)-(1.446)^{2/3}(160.66+87.934k)\\&\left(\approx -2.6-5.6\times 10^{-2}k\right)<0,\end{aligned}$$

we compute and observe that, for $k \geq 0$,

$$\begin{aligned}&Num\left(\frac{78.5}{500}c_{10}(k)+\frac{56}{500}d_{16}(k)\right)\\&\left(\approx 10^{66}\times\left[388.181\,(3.80+2.111k)^{3/2}-1.447(160.67+87.935k)^{3/2}\right]\right)\\<\ &10^{66}\times\left[388.182\,(3.81+2.112k)^{3/2}-1.446(160.66+87.934k)^{3/2}\right]\\=\ &10^{66}\times\left\{\left[(388.182)^{2/3}(3.81+2.112k)\right]^{3/2}-\left[(1.446)^{2/3}(160.66+87.934k)\right]^{3/2}\right\}\\<\ &0.\end{aligned}$$

So inequality (P27) holds.

**The proof of inequality (P28).** Since we compute that

$$\begin{aligned}&(788.425)^{2/3}(3.81+2.1111k)-(8.102)^{2/3}(81.3+44.6755k)\\&\left(\approx -2.7-4.7\times 10^{-2}k\right)<0,\end{aligned}$$

we compute and observe that, for $k \geq 0$,

$$\begin{aligned}&Num\left(c_{11}(k)+\frac{313.4}{500}d_{16}(k)\right)\\&\left(\approx 10^{72}\times\left[788.4241\,(3.80+2.1110k)^{3/2}-8.1021(81.35+44.6756k)^{3/2}\right]\right)\\<\ &10^{72}\times\left[788.425\,(3.81+2.1111k)^{3/2}-8.102(81.3+44.6755k)^{3/2}\right]\\=\ &10^{72}\times\left\{\left[(788.425)^{2/3}(3.81+2.1111k)\right]^{3/2}-\left[(8.102)^{2/3}(81.3+44.6755k)\right]^{3/2}\right\}\\<\ &0.\end{aligned}$$

So inequality (P28) holds.

**The proof of inequality (P29).** Since we compute that

$$(82.053)^{2/3}(3.81 + 2.112k) - (3.375)^{2/3}(32.17 + 17.726k)$$
$$\left(\approx -0.4 - 3.3 \times 10^{-3}k\right) < 0,$$

we compute and observe that, for $k \geq 0$,

$$\begin{aligned}
& Num\left(\frac{239.8}{500}c_{12}(k) + \frac{130.6}{500}d_{16}(k)\right) \\
& \left(\approx 10^{69} \times \left[82.052\,(3.80 + 2.111k)^{3/2} - 3.376(32.18 + 17.727k)^{3/2}\right]\right) \\
< \; & 10^{69} \times \left[82.053\,(3.81 + 2.112k)^{3/2} - 3.375(32.17 + 17.726k)^{3/2}\right] \\
= \; & 10^{69} \times \left\{\left[(82.053)^{2/3}(3.81 + 2.112k)\right]^{3/2} - \left[(3.375)^{2/3}(32.17 + 17.726k)\right]^{3/2}\right\} \\
< \; & 0.
\end{aligned}$$

So inequality (P29) holds.

**The proof of inequality (P30).** Since we compute that

$$(8.904)^{2/3}(16.12 + 8.939k) - (3.190)^{2/3}(32.17 + 17.726k)$$
$$\left(\approx -0.4 - 1.1 \times 10^{-3}k\right) < 0,$$

we compute and observe that, for $k \geq 0$,

$$\begin{aligned}
& Num\left(\frac{260.2}{500}c_{12}(k) + \frac{159.1}{500}d_{17}(k)\right) \\
& \left(\approx 10^{67} \times \left[8.903\,(16.11 + 8.938k)^{3/2} - 3.191(32.18 + 17.727k)^{3/2}\right]\right) \\
< \; & 10^{67} \times \left[8.904\,(16.12 + 8.939k)^{3/2} - 3.19(32.17 + 17.726k)^{3/2}\right] \\
= \; & 10^{67} \times \left\{\left[(8.904)^{2/3}(16.12 + 8.939k)\right]^{3/2} - \left[(3.190)^{2/3}(32.17 + 17.726k)\right]^{3/2}\right\} \\
< \; & 0.
\end{aligned}$$

So inequality (P30) holds.

**The proof of inequality (P31).** Since we compute that

$$(61.136)^{2/3}(16.12 + 8.939k) - (5.222)^{2/3}(83.45 + 46.090k)$$

$$\left(\approx -1.0 - 3.5 \times 10^{-3}k\right) < 0,$$

we compute and observe that, for $k \geq 0$,

$$\begin{aligned}
& Num\left(c_{13}(k) + \frac{260.4}{500}d_{17}(k)\right) \\
& \left(\approx 10^{70} \times \left[61.135\,(16.11 + 8.938k)^{3/2} - 5.223(83.46 + 46.091k)^{3/2}\right]\right) \\
< \; & 10^{70} \times \left[61.136\,(16.12 + 8.939k)^{3/2} - 5.222(83.45 + 46.090k)^{3/2}\right] \\
= \; & 10^{70} \times \left\{\left[(61.136)^{2/3}\,(16.12 + 8.939k)\right]^{3/2} - \left[(5.222)^{2/3}\,(83.45 + 46.090k)\right]^{3/2}\right\} \\
< \; & 0.
\end{aligned}$$

So inequality (P31) holds.

**The proof of inequality (P32).** Since we compute that

$$\begin{aligned}
& (8.533)^{2/3}\,(16.12 + 8.939k) - (16.145)^{2/3}\,(10.55 + 5.848k) \\
& \left(\approx -7.9 \times 10^{-2} - 2.9 \times 10^{-2}k\right) < 0,
\end{aligned}$$

we compute and observe that, for $k \geq 0$,

$$\begin{aligned}
& Num\left(\frac{185.5}{500}c_{14}(k) + \frac{80.5}{500}d_{17}(k)\right) \\
& \left(\approx 10^{69} \times \left[8.532\,(16.11 + 8.938k)^{3/2} - 16.146(10.56 + 5.849k)^{3/2}\right]\right) \\
< \; & 10^{69} \times \left[8.533\,(16.12 + 8.939k)^{3/2} - 16.145(10.55 + 5.848k)^{3/2}\right] \\
= \; & 10^{69} \times \left\{\left[(8.533)^{2/3}\,(16.12 + 8.939k)\right]^{3/2} - \left[(16.145)^{2/3}\,(10.55 + 5.848k)\right]^{3/2}\right\} \\
< \; & 0.
\end{aligned}$$

So inequality (P32) holds.

**The proof of inequality (P33).** Since we compute that

$$\begin{aligned}
& (14.466)^{2/3}\,(3.93 + 2.1793k) - (3.290)^{2/3}\,(10.55 + 5.8498k) \\
& \left(\approx -4.9 \times 10^{-3} - 1.6 \times 10^{-3}k\right) < 0,
\end{aligned}$$

we compute and observe that, for $k \geq 0$,

$$
\begin{aligned}
& Num\left(\frac{314.5}{500}c_{14}(k)+\frac{168}{500}d_{18}(k)\right) \\
& \left(\approx 10^{72}\times\left[14.465\,(3.92+2.1792k)^{3/2}-3.291(10.56+5.8499k)^{3/2}\right]\right) \\
< \; & 10^{72}\times\left[14.466\,(3.93+2.1793k)^{3/2}-3.290(10.55+5.849k)^{3/2}\right] \\
= \; & 10^{72}\times\left\{\left[(14.466)^{2/3}\,(3.93+2.1793k)\right]^{3/2}-\left[(3.290)^{2/3}\,(10.55+5.8498k)\right]^{3/2}\right\} \\
< \; & 0.
\end{aligned}
$$

So inequality (P33) holds.

**The proof of inequality (P34).** Since we compute that

$$
\begin{aligned}
& (27.3913)^{2/3}\,(3.92971+2.1793k)-(4.2066)^{2/3}\,(13.70320+7.6010k) \\
& \left(\approx -5.9\times 10^{-4}-4.5\times 10^{-3}k\right)<0,
\end{aligned}
$$

we compute and observe that, for $k \geq 0$,

$$
\begin{aligned}
& Num\left(c_{15}(k)+\frac{214.7}{500}d_{18}(k)\right) \\
& \left(\approx 10^{69}\times\left[27.3912\,(3.92970+2.1792k)^{3/2}-4.2067(13.70321+7.6011k)^{3/2}\right]\right) \\
< \; & 10^{69}\times\left[27.3913\,(3.92971+2.1793k)^{3/2}-4.2066(13.7032+7.601k)^{3/2}\right] \\
= \; & 10^{69}\times\left\{\left[(27.3913)^{2/3}\,(3.92971+2.1793k)\right]^{3/2}-\left[(4.2066)^{2/3}\,(13.7032+7.601k)\right]^{3/2}\right\} \\
< \; & 0.
\end{aligned}
$$

So inequality (P34) holds.

**The proof of inequality (P35).** Since we compute that

$$
\begin{aligned}
& (4.6568)^{2/3}\,(3.92971+2.1793k)-(22.9832)^{2/3}\,(1.35567+0.7532k) \\
& \left(\approx -2.2\times 10^{-4}-1.1\times 10^{-2}k\right)<0,
\end{aligned}
$$

we compute and observe that, for $k \geq 0$,

$$
Num\left(\frac{356.5}{500}c_{16}(k)+\frac{117.2}{500}d_{18}(k)\right)
$$

$$
\begin{aligned}
& \left(\approx 10^{71} \times \left[4.6567\,(3.92970 + 2.1792k)^{3/2} - 22.9833(1.35568 + 0.7533k)^{3/2}\right]\right) \\
< \;& 10^{71} \times \left[4.6568\,(3.92971 + 2.1793k)^{3/2} - 22.9832(1.35567 + 0.7532k)^{3/2}\right] \\
= \;& 10^{71} \times \left\{\left[(4.6568)^{2/3}\,(3.92971 + 2.1793k)\right]^{3/2} - \left[(22.9832)^{2/3}\,(1.35567 + 0.7532k)\right]^{3/2}\right\} \\
< \;& 0.
\end{aligned}
$$

So inequality (P35) holds.

**The proof of inequality (P36).** Since we compute that

$$
\begin{aligned}
& (1.8745)^{2/3}\,(99.79 + 55.34k) - (1.1844)^{2/3}\,(135.55 + 75.32k) \\
& \left(\approx -0.1 - 3.0 \times 10^{-2}k\right) < 0,
\end{aligned}
$$

we compute and observe that, for $k \geq 0$,

$$
\begin{aligned}
& Num\left(\frac{143.5}{500}c_{16}(k) + \frac{82.8}{500}d_{19}(k)\right) \\
& \left(\approx 10^{62} \times \left[1.8744\,(99.78 + 55.33k)^{3/2} - 1.1845(135.56 + 75.33k)^{3/2}\right]\right) \\
< \;& 10^{62} \times \left[1.8745\,(99.79 + 55.34k)^{3/2} - 1.1844(135.55 + 75.32k)^{3/2}\right] \\
= \;& 10^{62} \times \left\{\left[(1.8745)^{2/3}\,(99.79 + 55.34k)\right]^{3/2} - \left[(1.1844)^{2/3}\,(135.55 + 75.32k)\right]^{3/2}\right\} \\
< \;& 0.
\end{aligned}
$$

So inequality (P36) holds.

**The proof of inequality (P37).** Since we compute that

$$
\begin{aligned}
& (2.398)^{2/3}\,(99.789 + 55.34k) - (2.901)^{2/3}\,(87.899 + 48.91k) \\
& \left(\approx -1.3 \times 10^{-2} - 0.34k\right) < 0,
\end{aligned}
$$

we compute and observe that, for $k \geq 0$,

$$
\begin{aligned}
& Num\left(c_{17}(k) + \frac{202.8}{500}d_{19}(k)\right) \\
& \left(\approx 10^{65} \times \left[2.3978\,(99.788 + 55.33k)^{3/2} - 2.9012(87.900 + 48.92k)^{3/2}\right]\right) \\
< \;& 10^{65} \times \left[2.398\,(99.789 + 55.34k)^{3/2} - 2.901(87.899 + 48.91k)^{3/2}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= 10^{65} \times \left\{ \left[ (2.398)^{2/3} (99.789 + 55.34k) \right]^{3/2} - \left[ (2.901)^{2/3} (87.899 + 48.91k) \right]^{3/2} \right\} \\
&< 0.
\end{aligned}
$$

So inequality (P37) holds.

**The proof of inequality (P38).** Since we compute that

$$
\begin{aligned}
&(6.3797)^{2/3} (99.7889 + 55.34k) - (171.2397)^{2/3} (11.1315 + 6.19k) \\
&\left(\approx -6.1 \times 10^{-3} - 0.98k\right) < 0,
\end{aligned}
$$

we compute and observe that, for $k \geq 0$,

$$
\begin{aligned}
& Num \left( c_{18}(k) + \frac{119.7}{500} d_{19}(k) \right) \\
& \left( \approx 10^{63} \times \left[ 6.3796 \, (99.7888 + 55.33k)^{3/2} - 171.2398 (11.1315 + 6.20k)^{3/2} \right] \right) \\
<\ & 10^{63} \times \left[ 6.3797 \, (99.7889 + 55.34k)^{3/2} - 171.2397 (11.1316 + 6.19k)^{3/2} \right] \\
=\ & 10^{63} \times \left\{ \left[ (6.3797)^{2/3} (99.7889 + 55.34k) \right]^{3/2} - \left[ (171.2397)^{2/3} (11.1315 + 6.19k) \right]^{3/2} \right\} \\
<\ & 0.
\end{aligned}
$$

So inequality (P38) holds.

**The proof of inequality (P39).** Since we compute that

$$
(4.9)^{2/3} (99.8 + 55.4k) - (13.4)^{2/3} (90.1 + 50.2k) \left(\approx 220.40 - 123.3k\right) < 0,
$$

we compute and observe that, for $k \geq 0$,

$$
\begin{aligned}
& Num \left( c_{19}(k) + \frac{94.7}{500} d_{19}(k) \right) \\
& \left( \approx 10^{64} \times \left[ 4.8 \, (99.7 + 55.3k)^{3/2} - 13.5 (90.2 + 50.3k)^{3/2} \right] \right) \\
<\ & 10^{64} \times \left[ 4.9 \, (99.8 + 55.4k)^{3/2} - 13.4 (90.1 + 50.2k)^{3/2} \right] \\
=\ & 10^{64} \times \left\{ \left[ (4.9)^{2/3} (99.8 + 55.4k) \right]^{3/2} - \left[ (13.4)^{2/3} (90.1 + 50.2k) \right]^{3/2} \right\} \\
<\ & 0.
\end{aligned}
$$

So inequality (P39) holds.

Thus the proof of assertion (2.9) is complete. ■

## References

[1] Shao-Yuan Huang, Shin-Hwa Wang, A variational property on the evolutionary bifurcation curves for a positone problem with cubic nonlinearity.