The Proof of Assertion (3.30)

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Assume that $\varepsilon_2 = \sqrt{\frac{25\sigma^3}{864\rho}} < \varepsilon < \varepsilon_4 = \sqrt{\frac{13\sigma^3}{400\rho}}$. We recall the assertion (3.30): for $0 < u < p_2$,

$$\begin{aligned} U(u) &\equiv \left[120N + 42 - \frac{6\rho N}{\varepsilon p_2^3} - \frac{120\sigma N}{\varepsilon (p_2 + u)}\right] D_{p_2} + 3(1+4N) A_{p_2} \\ &\quad + \left[2 - 16N + \frac{12\sigma N}{\varepsilon (p_2+u)}\right] B_{p_2} > 0, \end{aligned} \tag{3.30}$$

where $A_{p_2} \equiv \varepsilon (p_2^4 - u^4)$, $B_{p_2} \equiv \sigma (p_2^3 - u^3)$, $D_{p_2} \equiv \rho (p_2 - u)$, and

$$N \equiv -\frac{\varepsilon p_2}{3\varepsilon p_2 - \sigma}.$$

Proof of assertion (3.30). By Lemma 3.2 in [1], we have that

$$p_2^3 = \frac{\sigma p_2^2 - \rho}{2\varepsilon} \quad \text{and} \quad N = -\frac{\varepsilon p_2}{3\varepsilon p_2 - \sigma} < -1. \tag{1}$$

Since $p_2 < \sigma/(2\varepsilon)$ and $N < 0$, we compute and observe that, for $u > 0$,

$$\begin{aligned} \frac{\partial^5 U(u)}{\partial u^5} &= \frac{2880 p_2 \sigma (-N)}{\varepsilon (p_2+u)^6} \left(\sigma p_2^2 - 10\rho\right) < \frac{2880 p_2 \sigma (-N)}{\varepsilon (p_2+u)^6} \left[\sigma \left(\frac{\sigma}{2\varepsilon}\right)^2 - 10\rho\right] \\ &= \frac{1152000 p_2 \sigma \rho (-N)}{\varepsilon^3 (p_2+u)^6} \left(\frac{\sigma^3}{40\rho} - \varepsilon^2\right) \\ &< \frac{1152000 p_2 \sigma \rho (-N)}{\varepsilon^3 (p_2+u)^6} \left(\frac{\sigma^3}{40\rho} - \varepsilon_2^2\right) = \frac{13600 \sigma^4 p_2}{3\varepsilon^3 (p_2+u)^6} N < 0. \end{aligned} \tag{2}$$

By (1) and (2), we compute and observe that, for $0 < u < p_2$,

$$\begin{aligned} \frac{\partial^4 U(u)}{\partial u^4} &> \left.\frac{\partial^4 U(u)}{\partial u^4}\right|_{u=p_2} \\ &= \frac{18}{\varepsilon p_2^4} \left[-(16N+4)\varepsilon^2 p_2^4 + N\sigma \left(\sigma p_2^2 - 10\rho\right)\right] \end{aligned}$$

$$
\begin{aligned}
&> \frac{18}{\varepsilon p_2^4}\left[-\left(16N+4\right)\varepsilon^2 p_2^4 + N\sigma\left(\sigma p_2^2-\rho\right)\right] \\
&= \frac{18}{\varepsilon p_2}\left[-\left(16N+4\right)\varepsilon p_2 + 2N\sigma\right] \quad (\text{since } \sigma p_2^2-\rho = 2\varepsilon p_2^3) \\
&= \frac{12\left(2\varepsilon p_2+\sigma\right)}{\varepsilon\left(p_2-\gamma\right)} > 0 \quad (\text{since } N = -\frac{\varepsilon p_2}{3\varepsilon p_2-\sigma}).
\end{aligned} \tag{3}
$$

By Lemma 3.4(iv) in [1], we have that

$$
\frac{396\sigma}{1000\varepsilon} < p_2 < \frac{417\sigma}{1000\varepsilon}. \tag{4}
$$

By (1), (3), and (4), we observe that, for $0 < u < p_2$,

$$
\begin{aligned}
\frac{\partial^3 U(u)}{\partial u^3} &< \left.\frac{\partial^3 U(u)}{\partial u^3}\right|_{u=p_2} \\
&= \frac{3}{\varepsilon p_2^3}\left[\left(-96N-24\right)\varepsilon^2 p_2^4 + \left(32\sigma N-4\sigma\right)\varepsilon p_2^3 - 3\sigma N\left(\sigma p_2^2-10\rho\right)\right] \\
&= \frac{1}{\varepsilon p_2^2\left(p_2-\gamma\right)}\left[\left(24\varepsilon^2 p_2^2-20\varepsilon\sigma p_2+7\sigma^2\right)p_2^2-30\sigma\rho\right] \\
&\le \frac{1}{\varepsilon p_2^2\left(p_2-\gamma\right)}\left\{\left[24\varepsilon^2\left(\frac{417\sigma}{1000\varepsilon}\right)^2-20\varepsilon\sigma\gamma+7\sigma^2\right]p_2^2-30\sigma\rho\right\} \\
&= \frac{\sigma}{\varepsilon p_2^2\left(p_2-\gamma\right)}\left(\frac{1690001}{375000}\sigma p_2^2-30\rho\right) \\
&\le \frac{\sigma}{\varepsilon p_2^2\left(p_2-\gamma\right)}\left[\frac{1690001}{375000}\sigma\left(\frac{417\sigma}{1000\varepsilon_2}\right)^2-30\rho\right] \\
&= -\frac{5574540755001}{97656250000}\frac{\sigma\rho}{\varepsilon p_2^2\left(p_2-\gamma\right)} < 0.
\end{aligned} \tag{5}
$$

Similarly, by (1), (4), and (5), we compute and observe that, for $0 < u < p_2$,

$$
\begin{aligned}
\frac{\partial^2 U(u)}{\partial u^2} &> \left.\frac{\partial^2 U(u)}{\partial u^2}\right|_{u=p_2} \\
&= \frac{6}{\varepsilon p_2^2}\left[-6\varepsilon^2 p_2^4 - 2\sigma\varepsilon p_2^3 - N\left(24\varepsilon^2 p_2^4 - 16\varepsilon\sigma p_2^3 + 3\sigma^2 p_2^2 + 10\sigma\rho\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2}{\varepsilon p_2 (p_2-\gamma)}\left[\left(6\varepsilon^2 p_2^2 - 16\sigma\varepsilon p_2 + 5\sigma^2\right)p_2^2 + 10\sigma\rho\right] \quad (\text{since } N = \frac{-p_2}{3(p_2-\gamma)}) \\
&> \frac{2}{\varepsilon p_2 (p_2-\gamma)}\left\{\left[6\varepsilon^2\gamma^2 - 16\sigma\varepsilon\left(\frac{417\sigma}{1000\varepsilon}\right) + 5\sigma^2\right]p_2^2 + 10\sigma\rho\right\} \\
&= \frac{2}{\varepsilon p_2 (p_2-\gamma)}\left(-\frac{377}{375}\sigma^2 p_2^2 + 10\sigma\rho\right) \\
&> \frac{2}{\varepsilon p_2 (p_2-\gamma)}\left[-\frac{377}{375}\sigma^2\left(\frac{417\sigma}{1000\varepsilon_2}\right)^2 + 10\sigma\rho\right] \\
&= \frac{386557123\sigma\rho}{48828125\varepsilon p_2 (p_2-\gamma)} > 0.
\end{aligned}
$$

So by (1), we compute and observe that, for $0 < u < p_2$,

$$
\begin{aligned}
\frac{\partial U(u)}{\partial u} &< \left.\frac{\partial U(u)}{\partial u}\right|_{u=p_2} \\
&= \frac{6}{\sigma p_2^3}\Big[-2\sigma^2 p_2^6 - \sigma\sigma p_2^5 - 7\rho\sigma p_2^3 + N(-8p_2^6\sigma^2 + 8\sigma\sigma p_2^5 \\
&\quad -3\sigma^2 p_2^4 + 10\sigma\rho p_2^2 - 20\rho\sigma p_2^3 + \rho^2)\Big] \\
&= \frac{6}{\sigma p_2^3}\left[-2\sigma^2 p_2^6 - \sigma\sigma p_2^5 - 7\rho\sigma p_2^3 + \left(\frac{-\sigma p_2}{3\sigma p_2 - \sigma}\right)(-8p_2^6\sigma^2\right. \\
&\quad \left. +8\sigma\sigma p_2^5 - 3\sigma^2 p_2^4 + 10\sigma\rho p_2^2 - 20\rho\sigma p_2^3 + \rho^2\right] \\
&= \frac{6\left(2\sigma p_2^3 - \sigma p_2^2 + \rho\right)}{\left(3\sigma p_2 - \sigma\right)p_2^2}\theta(p_2) = 0.
\end{aligned}
$$

It follows that $U(u)$ is a strictly decreasing function of u on $(0, p_2)$. Clearly, $U(p_2) = 0$. So $U(u) > U(p_2) = 0$ for $0 < u < p_2$. Then (3.30) holds.

References

[1] Shao-Yuan Huang, Shin-Hwa Wang, A variational property on the evolutionary bifurcation curves for a positone problem with cubic nonlinearity.