The Proof of Assertion (3.52)

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Assume that $\varepsilon_2 = \sqrt{\frac{25\sigma^3}{864\rho}} < \varepsilon < \varepsilon_4 = \sqrt{\frac{13\sigma^3}{400\rho}}$. We recall the assertion (3.52): for $0 < u < p_2$,

$$\frac{183}{100}\left[F(p_2) - F(u)\right] < p_2 f(p_2) - u f(u) < \frac{21}{10}\left[F(p_2) - F(u)\right]. \tag{3.52}$$

Proof of assertion (3.52). We let

$$J_1(u) \equiv p_2 f(p_2) - u f(u) - \frac{183}{100}\left[F(p_2) - F(u)\right],$$
$$J_2(u) \equiv \frac{52}{25}\left[F(p_2) - F(u)\right] - p_2 f(p_2) + u f(u).$$

It is sufficient to prove that $J_1(u) > 0$ and $J_2(u) > 0$ for $0 < u < p_2$. We compute that

$$J_1'(u) = \frac{217\varepsilon}{100}u^3 - \frac{117\sigma}{100}u^2 - \frac{17\tau}{100}u + \frac{83\rho}{100}.$$

By Lemma 3.4 in [1], we see that

$$\frac{396\sigma}{1000\varepsilon} < p_2 < \frac{417\sigma}{1000\varepsilon} \tag{1}$$

Since $\tau \geq 0$, and by (1), we find that, for $\varepsilon_2 < \varepsilon < \varepsilon_4$,

$$\begin{aligned} J_1'(p_2) &\leq \left[\frac{217}{100}\varepsilon\left(\frac{417\sigma}{1000\varepsilon}\right) - \frac{117}{100}\sigma\right]\left(\frac{396\sigma}{1000\varepsilon}\right)^2 + \frac{83}{100}\rho \\ &= \frac{1}{625 \times 10^7 \varepsilon^2}\left(51875 \times 10^5 \varepsilon^2 \rho - 259834311\sigma^3\right) \\ &\leq \frac{1}{625 \times 10^7 \varepsilon^2}\left[51875 \times 10\left(\varepsilon_4^2\right)\rho - 259834311\sigma^3\right] \\ &= \frac{-91240561\sigma^3}{625 \times 10^7 \varepsilon^2} < 0. \end{aligned} \tag{2}$$

We further compute that

$$J_1''(u) = \frac{651\varepsilon}{100}u^2 - \frac{117\sigma}{50}u - \frac{17\tau}{100}.$$

Since $J_1''(0) = -\frac{17}{100}\tau \le 0$ and $J_1''(u)$ is a quadratic polynomial of u, we observe that either $J_1'(u)$ is a strictly decreasing function of u on $(0, p_2)$, or $J_1'(u)$ is a strictly decreasing and then strictly increasing function of u on $(0, p_2)$. Since $J'(0) = \frac{83}{100}\rho > 0$, and by (2), there exists $u_1 \in (0, p_2)$ such that $J_1'(u) > 0$ for $0 < u < u_1$, $J_1'(u) = 0$ for $u = u_1$, and $J_1'(u) < 0$ for $u_1 < u < p_2$. Clearly, $J_1(p_2) = 0$. By (1), we compute that

$$\begin{aligned} J_1(0) &= \left(-\frac{217}{400}\varepsilon p_2^3 + \frac{39}{100}\sigma p_2^2 + \frac{17}{200}\tau p_2 - \frac{83}{100}\rho\right) p_2 \\ &> \left\{\left[-\frac{217}{400}\varepsilon\left(\frac{39\sigma}{100\varepsilon}\right) + \frac{39}{100}\sigma\right]\left(\frac{39\sigma}{100\varepsilon}\right)^2 - \frac{83}{100}\rho\right\} p_2 \\ &= \left(\frac{10855377\sigma^3}{4\times 10^8\varepsilon^2} - \frac{83\rho}{100}\right) p_2 > \left(\frac{10855377\sigma^3}{4\times 10^8\varepsilon_4^2} - \frac{83\rho}{100}\right) p_2 = \frac{5029\rho}{10^6} > 0. \end{aligned}$$

So $J_1(u) > 0$ for $0 < u < p_2$. It implies that

$$\frac{183}{100}\left[F(p_2) - F(u)\right] < p_2 f(p_2) - u f(u) \quad \text{for } 0 < u < p_2.$$

We compute that

$$J_2'(u) = -\frac{19\varepsilon}{10}u^3 + \frac{9\sigma}{10}u^2 - \frac{\tau}{10}u - \frac{11\rho}{10} < -\frac{19\varepsilon}{10}u^3 + \frac{9\sigma}{10}u^2 - \frac{11\rho}{10},$$

It is easy to see that the cubic polynomial $-\frac{19}{10}\varepsilon u^3 + \frac{9}{10}\sigma u^2 - \frac{11}{10}\rho$ of u is strictly increasing on $\left(0, \frac{6\sigma}{19\varepsilon}\right)$ and strictly decreasing for $\left(\frac{6\sigma}{19\varepsilon}, \infty\right)$. So we compute and observe that, for $u > 0$,

$$\begin{aligned} J_2'(u) &\le -\frac{19\varepsilon}{10}\left(\frac{6\sigma}{19\varepsilon}\right)^3 + \frac{9\sigma}{10}\left(\frac{6\sigma}{19\varepsilon}\right)^2 - \frac{11\rho}{10} = \frac{54\sigma^3}{1805\varepsilon^2} - \frac{11}{10}\rho \\ &< \frac{54\sigma^3}{1805\varepsilon_2^2} - \frac{11\rho}{10} = -\frac{5963\rho}{90250} < 0. \end{aligned} \tag{3}$$

Since $J_2(p_2) = 0$, and by (3), we see that $J_2(u) > 0$ for $0 < u < p_2$. It implies that

$$p_2 f(p_2) - u f(u) < \frac{21}{10}\left[F(p_2) - F(u)\right] \quad \text{for } 0 < u < p_2.$$

The proof of assertion (3.52) is complete. ■

References

[1] Shao-Yuan Huang, Shin-Hwa Wang, A variational property on the evolutionary bifurcation curves for a positone problem with cubic nonlinearity.