The Proof of Assertion (3.54)

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1. Introduction

Assume that $\varepsilon_2 = \sqrt{\frac{25\sigma^3}{864\rho}} < \varepsilon < \varepsilon_4 = \sqrt{\frac{13\sigma^3}{400\rho}}$. We recall the assertion (3.54):

$$\begin{aligned}
& \frac{81}{200}\int_{\bar{p}_2}^{p_2} \frac{\theta(p_2)-\theta(u)}{[F(p_2)-F(u)]^{3/2}}du + \int_0^{p_2} \frac{-u\theta'(u)}{[F(p_2)-F(u)]^{3/2}}du \\
> & -\frac{81}{200}H_1(\tilde{p}_{2*},p_{2*})\sqrt{p_2^*-\tilde{p}_{2*}}I_1(\tilde{p}_2^*,p_{2*}) - H_3(p_{1_*},p_2^*)\frac{I_2(0,p_{2*})}{\sqrt{p_2^*}} \\
> & \ 0 \qquad (3.54)
\end{aligned}$$

for some positive numbers p_{1_*}, $\tilde{p}_{2*}$, $\tilde{p}_2^*$, p_{2*} and p_2^* satisfying $p_{1_*} \le p_1$ and

$$\frac{3\sigma}{25\varepsilon} \le \tilde{p}_{2*} < \bar{p}_2 < \tilde{p}_2^* \le \frac{23\sigma}{125\varepsilon} < \frac{39\sigma}{100\varepsilon} \le p_{2*} < p_2 < p_2^* \le \frac{417\sigma}{1000\varepsilon}.$$

To prove (3.54), we need the following lemma.

Lemma 1.1. *Assume that* $\varepsilon_2 \le \varepsilon \le \varepsilon_4$,

$$\begin{aligned}
& \frac{81}{200}\int_{\bar{p}_2}^{p_2} \frac{\theta(p_2)-\theta(u)}{[F(p_2)-F(u)]^{3/2}}du + \int_0^{p_2} \frac{-u\theta'(u)}{[F(p_2)-F(u)]^{3/2}}du \\
> & -\frac{81}{200}H_1(\tilde{p}_{2*},p_{2*})\sqrt{p_2^*-\tilde{p}_{2*}}I_1(\tilde{p}_2^*,p_{2*}) - H_3(p_{1_*},p_2^*)\frac{I_2(0,p_{2*})}{\sqrt{p_2^*}}
\end{aligned}$$

for any positive numbers p_{1_*}, $\tilde{p}_{2*}$, $\tilde{p}_2^*$, p_{2*} *and* p_2^* *satisfying* $p_{1_*} \le p_1$ *and*

$$\frac{3\sigma}{25\varepsilon} \le \tilde{p}_{2*} < \bar{p}_2 < \tilde{p}_2^* \le \frac{23\sigma}{125\varepsilon} < \frac{39\sigma}{100\varepsilon} \le p_{2*} < p_2 < p_2^* \le \frac{417\sigma}{1000\varepsilon}.$$

The proof of this lemma can be seen in Section 2. The first inequality of assertion (3.54) follows immediately by Lemma 1.1. For any $\varepsilon_2 < \varepsilon < \varepsilon_4$, we choice suitable numbers p_{1_*}, $\tilde{p}_{2*}$, $\tilde{p}_2^*$, p_{2*} and p_2^*, and then apply Lemma 1.1 to prove second inequality of assertion (3.54). We put the detail of proof of assertion (3.54) in Section 3.

We remark that most of the computation in this paper has been checked using the symbolic manipulator *Maple 16*.

2. The proof of Lemma 1.1

We divide the proof of Lemma 1.1 into three steps.

Step 1. We prove that, for $\varepsilon_2 \le \varepsilon \le \varepsilon_4$,

$$\int_{\bar{p}_2}^{p_2} \frac{\theta(p_2)-\theta(u)}{[F(p_2)-F(u)]^{3/2}}du > -H_1(\tilde{p}_{2*},p_{2*})\sqrt{p_2^*-\tilde{p}_{2*}}I_1(\tilde{p}_2^*,p_{2*}). \tag{2.1}$$

By Lemma 3.8(iii) in [1], we compute that

$$\hat{u}(p_2) < \hat{u}(\frac{39\sigma}{100\varepsilon}) = \frac{\sigma}{45\varepsilon}\left(-\frac{121}{100}+\sqrt{\frac{715463}{5000}}\right) \approx \frac{0.238\sigma}{\varepsilon},$$

$$\hat{u}(p_2) > \hat{u}(\frac{417\sigma}{1000\varepsilon}) = \frac{\sigma}{45\varepsilon}\left(-\frac{2263}{1000}+\sqrt{\frac{56237867}{500000}}\right) \approx \frac{0.185\sigma}{\varepsilon}.$$

It follows that

$$\bar{p}_2 < \tilde{p}_2^* \le \frac{23\sigma}{125\varepsilon} < \hat{u}(\frac{417\sigma}{1000\varepsilon}) < \hat{u}(p_2) < \hat{u}(\frac{39\sigma}{100\varepsilon}) < \frac{39\sigma}{100\varepsilon} < p_{2*} < p_2 < p_2^*. \tag{2.2}$$

For the sake of convenience, we recall the functions

$$H_1(u,\alpha) \equiv \frac{(\alpha-u)^{3/2}}{6\,[F(\alpha)-F(u)]^{3/2}}, \quad H_2(u,\alpha) \equiv \frac{6\,[\theta(\alpha)-\theta(u)]}{(\alpha-u)^{3/2}},$$

$$\begin{aligned} I_1(u,\alpha) \quad \equiv \quad & \frac{2}{35}[-15\varepsilon u^3-(39\varepsilon\alpha-14\sigma)\,u^2-\left(87\varepsilon\alpha^2-42\sigma\alpha\right)u-279\varepsilon\alpha^3 \\ & +154\alpha^2\sigma-210\rho]. \end{aligned}$$

We note that $\int H_2(u,\alpha)du = \sqrt{\alpha-u}I_1(u,\alpha)$. By (2.2) and Lemma 3.8(i)–(iv) in [1], we observe that, for $\bar{p}_2 \le u \le p_2$,

$$H_1(u,p_2) \le H_1(\bar{p}_2,p_2) < H_1(\tilde{p}_{2*},p_{2*}) \ \text{ and } \ I_1(\bar{p}_2,p_2) < I_1(\tilde{p}_2^*,p_{2*}). \tag{2.3}$$

By Lemma 3.1 in [1], we see that $H_2(u,p_2) < 0$ for $\bar{p}_2 < u < p_2$. So by (2.3), we obtain that

$$\sqrt{p_2^*-\tilde{p}_{2*}}I_1(\tilde{p}_2^*,p_{2*}) \ge \sqrt{p_2-\bar{p}_2}I_1(\bar{p}_2,p_2) = -\int_{\bar{p}_2}^{p_2} H_2(u,p_2)du > 0. \tag{2.4}$$

By (2.3) and (2.4), we observe that

$$\begin{aligned}\int_{\bar{p}_2}^{p_2} \frac{\theta(p_2)-\theta(u)}{[F(p_2)-F(u)]^{3/2}}du &= \int_{\bar{p}_2}^{p_2} H_1(u,p_2)H_2(u,p_2)du \\ &\geq H_1(\tilde{p}_{2*},p_{2*})\int_{\bar{p}_2}^{p_2} H_2(u,p_2)du \\ &= -H_1(\tilde{p}_{2*},p_{2*})\sqrt{p_2^*-\tilde{p}_{2*}}I_1(\tilde{p}_2^*,p_{2*}).\end{aligned}$$

So inequality (2.1) holds.

Step2. We prove that, for $\varepsilon_2 \leq \varepsilon \leq \varepsilon_4$,

$$\int_0^{p_2} \frac{-u\theta'(u)}{[F(p_2)-F(u)]^{3/2}}du \geq -H_3(p_{1_*},p_2^*)\frac{I_2(0,p_{2*})}{\sqrt{p_2^*}}. \tag{2.5}$$

For the sake of convenience, we recall the functions

$$H_3(u,\alpha) \equiv \frac{u\,(p_2-u)^{3/2}}{[F(p_2)-F(u)]^{3/2}},\ \ H_4(u,\alpha) \equiv \frac{-\theta'(u)}{(p_2-u)^{3/2}}.$$

$$I_2(u,\alpha) \equiv \frac{2}{15}\left[6au^3+(12a\alpha-5b)u^2+(48a\alpha^2-20b\alpha)u-96a\alpha^3+40b\alpha^2-15d\right].$$

We note that $\int H_4(u,\alpha)du = I_2(u,\alpha)/\sqrt{\alpha-u}$. Clearly, $H_3(u,p_2)>0$ for $0<u<p_2$. By Lemma 3.1 in [1], we see that

$$H_4(u,p_2) = \frac{-\theta'(u)}{(p_2-u)^{3/2}}\begin{cases} <0 & \text{for } 0<u<p_1, \\ =0 & \text{for } u=p_2, \\ >0 & \text{for } p_1<u<p_2. \end{cases} \tag{2.6}$$

Since $I_2(u,p_2)$ is a polynomial of u and

$$I_2(p_2,p_2) = -15\left(2\varepsilon p_2^3-\sigma p_2^2+1\right) = -15\theta'(p_2)=0,$$

we see that $I_2(u,p_2)=(p_2-u)\,\bar{I}_2(u,p_2)$ where $\bar{I}_2(u,p_2)$ is a polynomial of u. Then

$$\lim_{u\to p_2}\frac{I_2(u,p_2)}{\sqrt{p_2-u}} = \lim_{u\to p_2}\frac{(p_2-u)\,\bar{I}_2(u,p_2)}{\sqrt{p_2-u}} = \lim_{u\to p_2}\left[\sqrt{p_2-u}\bar{I}_2(u,p_2)\right]=0. \tag{2.7}$$

By Lemma 3.4(iv) in [1], we see that $p_1 < \frac{27\sigma}{100\varepsilon}$. So by (2.6), (2.7) and Lemmas 3.8(v)–(vii) in [1], we observe that

$$\int_0^{p_2}\frac{-u\theta'(u)}{[F(p_2)-F(u)]^{3/2}}du$$

$$
\begin{aligned}
&= \int_0^{p_1} H_3(u,p_2)H_4(u,p_2)du + \int_{p_1}^{p_2} H_3(u,p_2)H_4(u,p_2)du \\
&> H_3(p_1,p_2)\int_0^{p_1} H_4(u,p_2)du + H_3(p_1,p_2)\int_{p_1}^{p_2} H_4(u,p_2)du \\
&= H_3(p_1,p_2)\int_0^{p_2} H_4(u,p_2)du \\
&= H_3(p_1,p_2)\left[\lim_{u\to p_2}\frac{I_2(p_2,p_2)}{\sqrt{p_2-u}} - \frac{I_2(0,p_2)}{\sqrt{p_2}}\right] \\
&\geq -H_3(p_{1_*},p_2^*)\frac{I_2(0,p_{2*})}{\sqrt{p_2^*}}.
\end{aligned}
$$

Step 3. Lemma 1.1 follows immediately by (2.1) and (2.5).
This proof of Lemma 1.1 is complete. ■

3. The proof of Assertion (3.54)

We divide this proof into four parts:

Part I: If $\varepsilon_2 \leq \varepsilon < \sqrt{\frac{859\sigma^3}{28800\rho}}$, then (3.54) holds.

Part II: If $\sqrt{\frac{859\sigma^3}{28800\rho}} \leq \varepsilon < \sqrt{\frac{1327\sigma^3}{43200\rho}}$, then (3.54) holds.

Part III: If $\sqrt{\frac{1327\sigma^3}{43200\rho}} \leq \varepsilon < \sqrt{\frac{2731\sigma^3}{86400\rho}}$, then (3.54) holds.

Part IV: If $\sqrt{\frac{2731\sigma^3}{86400\rho}} \leq \varepsilon \leq \varepsilon_4$, then (3.54) holds.

In folloiwng subsections, we prove Part I, Part II, Part III and Part IV, respectively. In fact, these proofs are similar.

3.1. The Proof of Part I

For the sake of convenience, we assume that

$$\left(\sqrt{\frac{25\sigma^3}{864\rho}} =\right) \varepsilon_2 \equiv \varepsilon_m \leq \varepsilon < \varepsilon_M \equiv \sqrt{\frac{859\sigma^3}{28800\rho}}.$$

We let

$$\tilde{p}_{2*} \equiv \frac{120\sigma}{1000\varepsilon}, \quad \tilde{p}_2^* \equiv \frac{136\sigma}{1000\varepsilon}, \quad p_{1_*} \equiv \frac{29\sigma}{125\varepsilon}, \quad p_{2*} \equiv \frac{412\sigma}{1000\varepsilon}, \quad \text{and} \quad p_2^* \equiv \frac{417\sigma}{1000\varepsilon}.$$

By Lemma 3.4 in [1], we compute and find that, for $\varepsilon_m \leq \varepsilon \leq \varepsilon_M$,

$$\frac{0.2326\sigma}{\varepsilon} \approx \frac{K(\varepsilon_m)\sigma}{\varepsilon} \leq p_1$$

$$\frac{0.1207\sigma}{\varepsilon} \approx \frac{R(\varepsilon_m)\sigma}{\varepsilon} \leq \bar{p}_2 \leq \frac{R(\varepsilon_M)\sigma}{\varepsilon} \approx \frac{0.1356\sigma}{\varepsilon},$$

$$\frac{0.4122\sigma}{\varepsilon} \approx \frac{L(\varepsilon_M)\sigma}{\varepsilon} \leq p_2 \leq \frac{L(\varepsilon_m)\sigma}{\varepsilon} \approx \frac{0.4166\sigma}{\varepsilon}.$$

So we have that

$$0 < p_{1_*} < p_1, \tag{3.1}$$

$$\tilde{p}_{2*} < \bar{p}_2 < \tilde{p}_2^* < \frac{23\sigma}{125\varepsilon} < \frac{39\sigma}{100\varepsilon} < p_{2*} < p_2 < p_2^* = \frac{417\sigma}{1000\varepsilon}. \tag{3.2}$$

By Lemma 1.1, (3.1) and (3.2), we see that

$$\begin{aligned}
& \frac{81}{200}\int_{\bar{p}_2}^{p_2} \frac{[\theta(p_2) - \theta(u)]}{[F(p_2) - F(u)]^{3/2}} du + \int_0^{p_2} \frac{-u\theta'(u)}{[F(p_2) - F(u)]^{3/2}} du \\
> \; & -\frac{81}{200} H_1(\tilde{p}_{2*}, p_{2*}) \sqrt{p_2^* - \tilde{p}_{2*}} I_1(\tilde{p}_2^*, p_{2*}) - H_3(p_{1_*}, p_2^*) \frac{I_2(0, p_{2*})}{\sqrt{p_2^*}} \\
= \; & \frac{\varepsilon^{\frac{7}{2}}\sqrt{\sigma}}{394210950} \frac{\Lambda_1}{(\Lambda_2)^{3/2}},
\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}
\Lambda_1 \equiv \; & 1078007562240000\sqrt{139}\left(29296875\varepsilon^2\rho - 148526\sigma^3\right) \\
& \times \left(\frac{10006909\sigma^3 + 49875000\varepsilon\sigma\tau}{\varepsilon^2} + 1875 \times 10^5\rho\right)^{3/2} \\
& + 123162763950\sqrt{11}\left(1093750000\varepsilon^2\rho - 35926011\sigma^3\right) \\
& \times \left(\frac{854470789\sigma^3 + 3894 \times 10^6\varepsilon\sigma\tau}{\varepsilon^2} + 12 \times 10^9\rho\right)^{3/2},
\end{aligned}$$

$$\Lambda_2 \equiv \left(10006909\sigma^3 + 49875000\varepsilon\sigma\tau + 1875 \times 10^5\varepsilon^2\rho\right)$$

$$\times \left(854470789\sigma^3 + 3894 \times 10^6 \varepsilon\sigma\tau a + 12 \times 10^9 \varepsilon^2 \rho\right).$$

Clearly, $\Lambda_2 > 0$. By (3.3), it is sufficient to prove that $\Lambda_1 > 0$. Since $\varepsilon_m \le \varepsilon \le \varepsilon_M$, we compute that

$$29296875\varepsilon^2\rho - 148526\sigma^3 \ge 29296875\varepsilon_m^2\rho - 148526\sigma^3 = \frac{201365137}{288}\sigma^3,$$

$$1093750000\varepsilon^2\rho - 35926011\sigma^3 \ge 1093750000\varepsilon_m^2\rho - 35926011\sigma^3 = -\frac{231020219}{54}\sigma^3.$$

So we observe that

$$\begin{aligned}
\Lambda_1 \ \ge\ & 1078007562240000\sqrt{139}\left(\frac{201365137}{288}\sigma^3\right) \\
& \times \left(\frac{10006909\sigma^3 + 49875000\varepsilon_m\sigma\tau}{\varepsilon_M^2} + 1875 \times 10^5\rho\right)^{3/2} \\
& +123162763950\sqrt{11}\left(\frac{-231020219}{54}\sigma^3\right) \\
& \times \left(\frac{854470789\sigma^3 + 3894 \times 10^6\varepsilon_M\sigma\tau}{\varepsilon_m^2} + 12 \times 10^9 d\right)^{3/2} \\
=\ & \frac{2368\sqrt{6}\sigma^3\rho^{3/2}}{55341075}\left\{[\Gamma_1(k)]^{3/2} - [\Gamma_2(k)]^{3/2}\right\}, \qquad (3.4)
\end{aligned}$$

where $k \equiv \tau/\sqrt{\sigma\rho}$,

$$\Gamma_1(k) \equiv \left(1145867506637952 \times 10^9\sqrt{119401}\right)^{2/3}\left(187192283 + 41562500\sqrt{6}k\right),$$

$$\Gamma_2(k) \equiv \left(378288171951727779431710\sqrt{11}\right)^{2/3}\left(10815237101 + 146025000\sqrt{1718}k\right).$$

We compute and observe that

$$\Gamma_1(k) - \Gamma_2(k)\ \left(\approx 3.9 \times 10^{25}k + 7.3 \times 10^{25}\right) > 0 \quad \text{for } k \ge 0. \qquad (3.5)$$

So by (3.4) and (3.5), $\Lambda_1 > 0$ for $k \ge 0$. It implies that (3.54) holds by (3.3).

3.2. The Proof of Part II

Assume that

$$\sqrt{\frac{859\sigma^3}{28800\rho}} \equiv \varepsilon_m \le \varepsilon < \varepsilon_M \equiv \sqrt{\frac{1327\sigma^3}{43200\rho}}.$$

We let

$$\tilde{p}_{2*} \equiv \frac{135\sigma}{1000\varepsilon}, \quad \tilde{p}_2^* \equiv \frac{151\sigma}{1000\varepsilon}, \quad p_{1_*} \equiv \frac{239\sigma}{1000\varepsilon}, \quad p_{2*} \equiv \frac{407\sigma}{1000\varepsilon}, \quad \text{and} \quad p_2^* \equiv \frac{413\sigma}{1000\varepsilon}.$$

By Lemma 3.4 in [1], we compute and find that, for $\varepsilon_m \le \varepsilon < \varepsilon_M$,

$$\frac{0.2390\sigma}{\varepsilon} \approx \frac{K(\varepsilon_m)\sigma}{\varepsilon} \le p_1 < p_1,$$

$$\frac{0.1356\sigma}{\varepsilon} \approx \frac{R(\varepsilon_m)\sigma}{\varepsilon} \le \bar{p}_2 \le \frac{R(\varepsilon_M)\sigma}{\varepsilon} \approx \frac{0.1508\sigma}{\varepsilon},$$

$$\frac{0.4075\sigma}{\varepsilon} \approx \frac{L(\varepsilon_M)\sigma}{\varepsilon} \le p_2 \le \frac{L(\varepsilon_m)\sigma}{\varepsilon} \approx \frac{0.4122\sigma}{\varepsilon}.$$

So we have that

$$0 < p_{1_*} < p_1 < \frac{27\sigma}{100\varepsilon}, \tag{3.6}$$

$$\tilde{p}_{2*} < \bar{p}_2 < \tilde{p}_2^* < \frac{23\sigma}{125\varepsilon} < \frac{39\sigma}{100\varepsilon} < p_{2*} < p_2 < p_2^* < \frac{417\sigma}{1000\varepsilon}. \tag{3.7}$$

By Lemma 1.1, (3.6) and (3.7), we see that

$$\begin{aligned}
& \frac{81}{200}\int_{\bar{p}_2}^{p_2} \frac{[\theta(p_2)-\theta(u)]}{[F(p_2)-F(u)]^{3/2}}du + \int_0^{p_2} \frac{-u\theta'(u)}{[F(p_2)-F(u)]^{3/2}}du \\
> \; & -\frac{81}{200}H_1(\tilde{p}_{2*}, p_{2*})\sqrt{p_2^* - \tilde{p}_{2*}}I_1(\tilde{p}_2^*, p_{2*}) - H_3(p_{1_*}, p_2^*)\frac{I_2(0, p_{2*})}{\sqrt{p_2^*}} \\
= \; & \frac{4\varepsilon^{\frac{7}{2}}\sqrt{\sigma}}{45812025}\frac{\Lambda_1}{(\Lambda_2)^{3/2}},
\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}
\Lambda_1 \equiv \; & 282784800\sqrt{4130}\left(468750000\varepsilon^2\rho - 4803821\sigma^3\right) \\
& \times \left(\frac{164074219\sigma^3 + 813\times 10^5\varepsilon\sigma\tau}{\varepsilon^2} + 3\times 10^8\rho\right)^{3/2}
\end{aligned}$$

$$+1590331725\sqrt{834}\left(175\times 10^7\varepsilon^2\rho-584226471\sigma^3\right)$$
$$\times\left(\frac{7168553\sigma^3+3260000\varepsilon\sigma\tau}{\varepsilon^2}+10^8\rho\right)^{3/2},$$

$$\Lambda_2\equiv\left(164074219\sigma^3+813\times 10^6\varepsilon\sigma\tau+3\times 10^9\varepsilon^2\rho\right)(7168553\sigma^3+326\times 10^5\varepsilon\sigma\tau+10^9\varepsilon^2\rho).$$

Clearly, $\Lambda_2>0$. By (3.8), it is sufficient to prove that $\Lambda_1>0$. Since $\varepsilon_m\le\varepsilon<\varepsilon_M$, we compute that

$$468750000\varepsilon^2\rho-4803821\sigma^3\ge 468750000\varepsilon_m^2\rho-4803821\sigma^3=\frac{220255171}{24}\sigma^3,$$

$$175\times 10^7\varepsilon^2\rho-584226471\sigma^3\ge 175\times 10^7\varepsilon_m^2\rho-584226471\sigma^3=\frac{-560381989}{9}\sigma^3.$$

So we observe that

$$\begin{aligned}\Lambda_1 &\ge 282784800\sqrt{4130}\left(\frac{220255171}{24}\sigma^3\right)\\ &\quad\times\left(\frac{164074219\sigma^3+813\times 10^5\varepsilon_m\sigma\tau}{\varepsilon_M^2}+3\times 10^8\rho\right)^{3/2}\\ &\quad+590331725\sqrt{834}\left(\frac{-560381989}{9}\sigma^3\right)\\ &\quad\times\left(\frac{7168553\sigma^3+3260000\varepsilon_M\sigma\tau}{\varepsilon_m^2}+10^8\rho\right)^{3/2}\\ &= \frac{640000\sqrt{15}\sigma^3\rho^{3/2}}{1299356051449}\left\{[\Gamma_1(k)]^{3/2}-[\Gamma_2(k)]^{3/2}\right\},\end{aligned}\qquad(3.9)$$

where $k\equiv\tau/\sqrt{\sigma\rho}$,

$$\Gamma_1(k)\equiv\left(574484764918313081310\sqrt{1096102}\right)^{2/3}\left(2306042971+30487500\sqrt{1718}k\right),$$

$$\Gamma_2(k)\equiv\left(139495826464945254102 42\sqrt{597005}\right)^{2/3}\left(91360727+81500\sqrt{3981}k\right).$$

We compute and observe that We compute and observe that

$$\Gamma_1(k)-\Gamma_2(k)\ \left(\approx 1.1\times 10^{25}+8.7\times 10^{24}k\right)>0\ \text{ for } k\ge 0.\qquad(3.10)$$

So by (3.9) and (3.10), $\Lambda_1>0$ for $k\ge 0$. It implies that (3.54) holds by (3.8).

3.3. The Proof of Part III

Assume that

$$\sqrt{\frac{1327\sigma^3}{43200\rho}} \equiv \varepsilon_m \le \varepsilon < \varepsilon_M \equiv \sqrt{\frac{2731\sigma^3}{86400\rho}}.$$

We let

$$\tilde{p}_{2*} \equiv \frac{150\sigma}{1000\varepsilon}, \quad \tilde{p}_2^* \equiv \frac{167\sigma}{1000\varepsilon}, \quad p_{1_*} \equiv \frac{245\sigma}{1000\varepsilon}, \quad p_{2*} \equiv \frac{402\sigma}{1000\varepsilon}, \quad \text{and} \quad p_2^* \equiv \frac{408\sigma}{1000\varepsilon}.$$

By Lemma 3.4 in [1], we compute and find that, for $\varepsilon_m \le \varepsilon < \varepsilon_M$,

$$\frac{0.2458\sigma}{\varepsilon} \approx \frac{K(\varepsilon_m)\sigma}{\varepsilon} \le p_1,$$

$$\frac{0.1508\sigma}{\varepsilon} \approx \frac{R(\varepsilon_m)\sigma}{\varepsilon} \le \bar{p}_2 \le \frac{R(\varepsilon_M)\sigma}{\varepsilon} \approx \frac{0.1665\sigma}{\varepsilon},$$

$$\frac{0.4023\sigma}{\varepsilon} \approx \frac{L(\varepsilon_M)\sigma}{\varepsilon} \le p_2 \le \frac{L(\varepsilon_m)\sigma}{\varepsilon} \approx \frac{0.4075\sigma}{\varepsilon}.$$

So we have that

$$0 < p_{1_*} < p_1 < \frac{27\sigma}{100\varepsilon}, \tag{3.11}$$

$$\tilde{p}_{2*} < \bar{p}_2 < \tilde{p}_2^* < \frac{23\sigma}{125\varepsilon} < \frac{39\sigma}{100\varepsilon} < p_{2*} < p_2 < p_2^* < \frac{417\sigma}{1000\varepsilon}. \tag{3.12}$$

By Lemma 1.1, (3.11) and (3.12), we see that

$$\begin{aligned} & \frac{81}{200}\int_{\bar{p}_2}^{p_2} \frac{[\theta(p_2)-\theta(u)]}{[F(p_2)-F(u)]^{3/2}}du + \int_0^{p_2} \frac{-u\theta'(u)}{[F(p_2)-F(u)]^{3/2}}du \\ > \ & -\frac{81}{200}H_1(\tilde{p}_{2*}, p_{2*})\sqrt{p_2^* - \tilde{p}_{2*}}I_1(\tilde{p}_2^*, p_{2*}) - H_3(p_{1_*}, p_2^*)\frac{I_2(0, p_{2*})}{\sqrt{p_2^*}} \\ = \ & \frac{\sqrt{2}\varepsilon^{\frac{7}{2}}\sqrt{\sigma}}{43449280}\frac{\Lambda_1}{(\Lambda_2)^{3/2}}, \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} \Lambda_1 \ \equiv \ & 769450967040000\sqrt{17}\left(9765625\varepsilon^2\rho - 148137\sigma^3\right) \\ & \times \left(\frac{3503853\sigma^3 + 1725000\varepsilon\sigma\tau}{\varepsilon^2} + 6250000\rho\right)^{3/2} \end{aligned}$$

$$+13092975\sqrt{129}\left(42\times 10^8\varepsilon^2\rho - 1423371673\sigma^3\right)$$
$$\times\left(\frac{862104049\sigma^3 + 3918\times 10^6\varepsilon\sigma\tau}{\varepsilon^2} + 12\times 10^9\rho\right)^{3/2},$$

$$\begin{aligned}\Lambda_2 \;\equiv\; & \left(3503853\sigma^3 + 1725000\varepsilon\sigma\tau + 6250000\varepsilon^2\rho\right) \\ & \times(862104049\sigma^3 + 3918\times 10^6\varepsilon\sigma\tau + 12\times 10^8\varepsilon^2\rho).\end{aligned}$$

Clearly, $\Lambda_2 > 0$. By (3.13), it is sufficient to prove that $\Lambda_1 > 0$. Since $\varepsilon_m \le \varepsilon < \varepsilon_M$, we compute that

$$9765625\varepsilon^2\rho - 148137\sigma^3 \ge 9765625\varepsilon_m^2\rho - 148137\sigma^3 = \frac{262378639}{1728}\sigma^3,$$

$$42\times 10^8\varepsilon^2\rho - 1423371673\sigma^3 \ge 42\times 10^8\varepsilon_m^2\rho - 1423371673\sigma^3 = -\frac{1199095057}{9}\sigma^3.$$

So we observe that

$$\begin{aligned}\Lambda_1 \;\ge\; & 769450967040000\sqrt{17}\left(\frac{262378639\sigma^3}{1728}\right) \\ & \times\left(\frac{3503853\sigma^3 + 1725000\varepsilon_m\sigma\tau}{\varepsilon_M^2} + 6250000\rho\right)^{3/2} \\ & +13092975\sqrt{129}\left(-\frac{1199095057}{9}\sigma^3\right) \\ & \times\left(\frac{862104049\sigma^3 + 3918\times 10^6\varepsilon_M\sigma\tau}{\varepsilon_m^2} + 12\times 10^9\rho\right)^{3/2} \\ = \; & \frac{2240000\sqrt{50}\sigma^3\rho^{3/2}}{354608392788963}\left\{[\Gamma_1(k)]^{3/2} - [\Gamma_2(k)]^{3/2}\right\}, \qquad (3.14)\end{aligned}$$

where $k \equiv \tau/\sqrt{\sigma\rho}$,

$$\Gamma_1(k) \equiv \left(7935481006726758475468800\sqrt{92854}\right)^{2/3}\left(591775499 + 5175000\sqrt{3981}k\right),$$

$$\Gamma_2(k) \equiv \left(1806596094675628241304\sqrt{57061}\right)^{2/3}\left(11076436441 + 48975000\sqrt{16386}k\right).$$

We compute and observe that We compute and observe that

$$\Gamma_1(k) - \Gamma_2(k)\;\left(\approx 7.7\times 10^{26} + 4.2\times 10^{26}k\right) > 0 \quad \text{for } k \ge 0. \qquad (3.15)$$

So by (3.14) and (3.15), $\Lambda_1 > 0$ for $k \ge 0$. It implies that (3.54) holds by (3.13).

3.4. The Proof of Part IV

Assume that
$$\sqrt{\frac{2731\sigma^3}{86400\rho}} \equiv \varepsilon_m \le \varepsilon < \varepsilon_4 \left(= \sqrt{\frac{13\sigma^3}{400\rho}} \right).$$
We let
$$\tilde{p}_{2*} \equiv \frac{166\sigma}{1000\varepsilon}, \quad \tilde{p}_2^* \equiv \frac{184\sigma}{1000\varepsilon}, \quad p_{1_*} \equiv \frac{252\sigma}{1000\varepsilon}, \quad p_{2*} \equiv \frac{396\sigma}{1000\varepsilon}, \quad \text{and} \quad p_2^* \equiv \frac{403\sigma}{1000\varepsilon}.$$
By Lemma 3.4 in [1], we compute and find that, for $\varepsilon_m \le \varepsilon \le \varepsilon_M$,
$$\frac{0.2529\sigma}{\varepsilon} \approx \frac{K(\varepsilon_m)\sigma}{\varepsilon} \le p_1,$$
$$\frac{0.1665\sigma}{\varepsilon} \approx \frac{R(\varepsilon_m)\sigma}{\varepsilon} \le \bar{p}_2 \le \frac{R(\varepsilon_M)\sigma}{\varepsilon} \approx \frac{0.1830\sigma}{\varepsilon},$$
$$\frac{0.3967\sigma}{\varepsilon} \approx \frac{L(\varepsilon_M)\sigma}{\varepsilon} \le p_2 \le \frac{L(\varepsilon_m)\sigma}{\varepsilon} \approx \frac{0.4023\sigma}{\varepsilon}.$$
So we have that
$$0 < p_{1_*} < p_1 < \frac{27\sigma}{100\varepsilon}, \tag{3.16}$$
$$\tilde{p}_{2*} < \bar{p}_2 < \tilde{p}_2^* < \frac{23\sigma}{125\varepsilon} < \frac{39\sigma}{100\varepsilon} < p_{2*} < p_2 < p_2^* < \frac{417\sigma}{1000\varepsilon}. \tag{3.17}$$
By Lemma 1.1, (3.16) and (3.17), we see that
$$\begin{aligned} & \frac{81}{200}\int_{\bar{p}_2}^{p_2} \frac{[\theta(p_2) - \theta(u)]}{[F(p_2) - F(u)]^{3/2}} du + \int_0^{p_2} \frac{-u\theta'(u)}{[F(p_2) - F(u)]^{3/2}} du \\ > & -\frac{81}{200} H_1(\tilde{p}_{2*}, p_{2*}) \sqrt{p_2^* - \tilde{p}_{2*}} I_1(\tilde{p}_2^*, p_{2*}) - H_3(p_{1_*}, p_2^*) \frac{I_2(0, p_{2*})}{\sqrt{p_2^*}} \\ = & \frac{216\varepsilon^{\frac{7}{2}}\sqrt{\sigma}}{244933325} \frac{\Lambda_1}{(\Lambda_2)^{3/2}}, \end{aligned} \tag{3.18}$$
where
$$\begin{aligned} \Lambda_1 \equiv & \ 139408998400\sqrt{6045}\left(9765625\varepsilon^2\rho - 202554\sigma^3\right) \\ & \times \left(\frac{86197601\sigma^3 + 4215 \times 10^5 \varepsilon\sigma\tau}{\varepsilon^2} + 15 \times 10^7 \rho \right)^{3/2} \end{aligned}$$

$$+629828550\sqrt{158}\left(3281250000\varepsilon^2\rho-112994561\sigma^3\right)$$
$$\times\left(\frac{173191391\sigma^3+786\times10^6\varepsilon\sigma\tau}{\varepsilon^2}+24\times10^7\rho\right)^{3/2},$$

$$\Lambda_2 \equiv \left(86197601\sigma^3+4215\times10^5\varepsilon\sigma\tau+15\times10^7\varepsilon^2\rho\right)$$
$$\times(173191391\sigma^3+786\times10^6\varepsilon\sigma\tau+24\times10^7\varepsilon^2\rho).$$

Clearly, $\Lambda_2>0$. By (3.18), it is sufficient to prove that $\Lambda_1>0$. Since $\varepsilon_m\le\varepsilon\le\varepsilon_M$, we compute that

$$9765625\varepsilon^2\rho-202554\sigma^3\ge 9765625\varepsilon_m^2\rho-202554\sigma^3=\frac{366770251}{3456}\sigma^3,$$

$$3281250000\varepsilon^2\rho-112994561\sigma^3\ge 3281250000\varepsilon_m^2\rho-112994561\sigma^3=-\frac{668030267}{72}\sigma^3.$$

So we observe that

$$\begin{aligned}\Lambda_1 \ \ge\ & 139408998400\sqrt{6045}\left(\frac{366770251}{3456}\sigma^3\right)\\ &\times\left(\frac{86197601\sigma^3+4215\times10^5\varepsilon_m\sigma\tau}{\varepsilon_M^2}+15\times10^7\rho\right)^{3/2}\\ &+629828550\sqrt{158}\left(3281250000\varepsilon^2\rho-112994561\sigma^3\right)\\ &\times\left(\frac{173191391\sigma^3+786\times10^6\varepsilon_M\sigma\tau}{\varepsilon_m^2}+24\times10^7\rho\right)^{3/2}\\ =\ & \frac{160000\sigma^3\rho^{3/2}}{306292511187}\left\{[\Gamma_1(k)]^{3/2}-[\Gamma_2(k)]^{3/2}\right\} \qquad (3.19)\end{aligned}$$

where $k\equiv\tau/\sqrt{\sigma\rho}$,

$$\Gamma_1(k)\equiv\left(580968989324055276855456 0\sqrt{155}\right)^{2/3}\left(404842803+1756250\sqrt{16386}k\right),$$

$$\Gamma_2(k)\equiv\left(115191438633705501873 0\sqrt{647247}\right)^{2/3}\left(2241472519+353700000\sqrt{13}k\right).$$

We compute and observe that We compute and observe that

$$\Gamma_1(k)-\Gamma_2(k)\ \left(\approx 4.8\times10^{25}+2.6\times10^{25}k\right)>0\ \text{ for } k\ge0. \qquad (3.20)$$

So by (3.19) and (3.20), $\Lambda_1>0$ for $k\ge0$. It implies that (3.54) holds by (3.18).

The proof of assertion (3.54) is complete. ■

References

[1] Shao-Yuan Huang, Shin-Hwa Wang, A variational property on the evolutionary bifurcation curves for a positone problem with cubic nonlinearity.