Assume that $a = 4.075$. We prove that

$$X_3(u)\left[F(4.8) - F(u)\right] \leq 4.8f(4.8) - uf(u) \quad \text{for } 0 \leq u \leq 4.8, \tag{1}$$

where

$$X_3(u) \equiv -\frac{29}{5000}\left(u - \frac{383}{100}\right)^2 + \frac{10083}{5000}.$$

It is equivalently to prove that

$$C(u) \equiv 4.8f(4.8) - uf(u) - X_3(u)\left[F(4.8) - F(u)\right] > 0 \quad \text{for } 0 \leq u < 4.8. \tag{2}$$

We divide this proof into two steps.

Step 1. We prove that there exist four positive numbers $\tau_1 < \tau_2 < 3.83 < \tau_3 < \tau_4 < 4.8$ such that

$$C'(u)\begin{cases} < 0 & \text{for } u \in [0, \tau_1), \\ = 0 & \text{for } u = \tau_1, \\ > 0 & \text{for } u \in (\tau_1, \tau_2), \\ = 0 & \text{for } u = \tau_2, \\ < 0 & \text{for } u \in (\tau_2, \tau_3) \setminus \{3.83\}, \\ = 0 & \text{for } u = \tau_3, \\ > 0 & \text{for } u \in (\tau_3, \tau_4), \\ = 0 & \text{for } u = \tau_4, \\ < 0 & \text{for } u \in (\tau_4, 4.8]. \end{cases} \tag{3}$$

We observe that

$$X_3'(u)\begin{cases} > 0 & \text{for } 0 \leq u < 3.83, \\ = 0 & \text{for } u = 3.83, \\ < 0 & \text{for } 3.83 < u \leq 4.8. \end{cases} \tag{4}$$

We have that

$$\frac{C'(u)}{X_3'(u)} = \left[\frac{X_3(u)}{X_3'(u)} - \frac{(a+u)^2 + a^2u}{(a+u)^2 X_3'(u)}\right] f(u) - F(4.8) + F(u), \tag{5}$$

$$\left(\frac{C'(u)}{X_3'(u)}\right)' = \frac{1}{5800(40u+163)^4(100u-383)^2} C_1(u), \tag{6}$$

where

$$\begin{aligned} C_1(u) \quad \equiv \quad & 22272 \times 10^{10} \times u^6 + 31571024 \times 10^8 \times u^5 + 6544035104 \times 10^6 \times u^4 \\ & -526243475207480000u^3 + 3037169245712858800u^2 \\ & -6089140624842968180u + 3563308147692735097. \end{aligned}$$

By Figure 1, we observe that $C_1(u)$ has two prositive zeros r_1 and r_2. Furthermore, by the Intermediate Value Theorem, we can estimate that $0.995 < r_1 < 0.997$ and $4.350 < r_2 < 4.352$. So by (6), we have that

$$\left(\frac{C'(u)}{X_3'(u)}\right)'\begin{cases} > 0 & \text{for } u \in [0, r_1) \cup (r_2, 4.8] \\ = 0 & \text{for } u = r_1 \text{ or } u = r_2, \\ < 0 & \text{for } u \in (r_1, r_2) \setminus \{3.83\}. \end{cases} \tag{7}$$

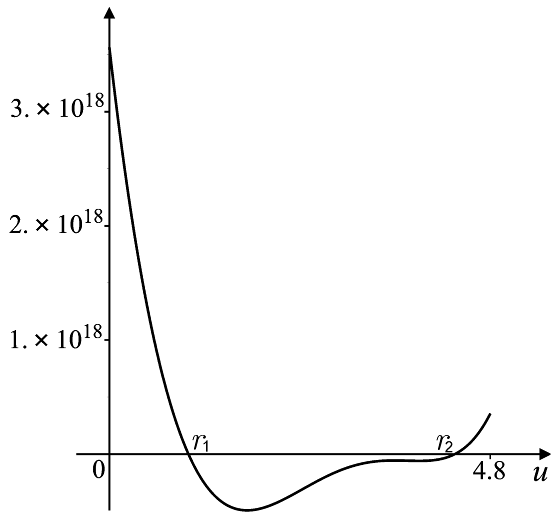

Figure 1: The graph of $C_1(u)$ on $[0, 4.8]$.

We compute that

$$22.500 < F(4.08)\ \ (\approx 22.509) < 22.510 \quad \text{for } a = 4.075. \tag{8}$$

By (5) and (8), we further compute and obtain that

$$\frac{C'(0)}{X_3'(0)} = \frac{46576019}{2221400} - F(4.8) < \frac{46576019}{2221400} - 22.5 = -\frac{3405481}{2221400} < 0, \tag{9}$$

$$\frac{C'(4.8)}{X_3'(4.8)} = \frac{203862301}{2836066600} f(4.8) > 0, \tag{10}$$

$$\lim_{u \to 3.83^-} \frac{C'(u)}{X_3'(u)} = -\infty \quad \text{and} \quad \lim_{u \to 3.83^-} \frac{C'(u)}{X_3'(u)} = +\infty. \tag{11}$$

We observe that

$$\left(\frac{1}{X_3'(u)}\right)' = \frac{25 \times 10^6}{29(100u - 383)^2} > 0 \quad \text{for } u \geq 0, \tag{12}$$

$$\left(\frac{X_3(u)}{X_3'(u)}\right)' = \frac{290000u^2 - 2221400u + 105083981}{58(100u - 383)^2} > 0 \quad \text{for } 0 \leq u \leq 5, \tag{13}$$

$$\left[\left(1 + \frac{(4.075)^2 u}{(4.075 + u)^2}\right) f(u)\right]' = \frac{4330747\,(243u + 326)}{(40u + 163)^4} f(u) > 0 \quad \text{for } u \geq 0. \tag{14}$$

Since $f(u)$ and $F(u)$ are strictly increasing on $(0, \infty)$, and by (4), (8), (12)–(14), we have that

$$
\begin{aligned}
\frac{C'(r_1)}{X_3'(r_1)} &= -\left[1+\frac{(4.075)^2\, r_1}{(4.075+r_1)^2}\right]\frac{f(r_1)}{X_3'(r_1)} - F(4.8) + F(r_1) + \frac{X_3(r_1)}{X_3'(r_1)} f(r_1) \\
&> -\left[1+\frac{(4.075)^2\, 0.997}{(4.075+0.997)^2}\right]\frac{f(0.997)}{X_3'(0.997)} - F(4.8) + F(0.995) \\
&\quad + \frac{X_3(0.995)}{X_3'(0.995)} f(0.995) \ \ (\approx 0.966) \\
&> 0, \qquad (15)
\end{aligned}
$$

$$
\begin{aligned}
\frac{C'(r_2)}{X_3'(r_2)} &= -\left[1+\frac{(4.075)^2\, r_2}{(4.075+r_2)^2}\right]\frac{f(r_2)}{X_3'(r_2)} - F(4.8) + F(r_2) + \frac{X_3(r_2)}{X_3'(r_2)} f(r_2) \\
&< -\left[1+\frac{(4.075)^2\, 4.352}{(4.075+4.352)^2}\right]\frac{f(4.352)}{X_3'(4.35)} - F(4.8) + F(4.352) \\
&\quad + \frac{X_3(4.352)}{X_3'(4.352)} f(4.35) \ \ (\approx -3.866) \\
&< 0. \qquad (16)
\end{aligned}
$$

By (7), (9)–(11), (15) and (16), there exist positive numbers $\tau_1 \in (0, r_1)$, $\tau_2 \in (r_1, 3.83)$, $\tau_3 \in (3.83, r_2)$, $\tau_4 \in (r_2, 4.8)$ such that

$$
\frac{C'(u)}{X_3'(u)} \begin{cases} < 0 & \text{for } u \in [0,\tau_1) \cup (\tau_2, 3.83) \cup (\tau_3, \tau_4)\,, \\ = 0 & \text{for } u \in \{\tau_1, \tau_2, \tau_3, \tau_4\}\,, \\ > 0 & \text{for } u \in (\tau_1, \tau_2) \cup (3.83, \tau_3) \cup (\tau_4, 4.8)\,. \end{cases} \qquad (17)
$$

By (4) and (17), (3) holds.

Step 2. We prove that $C(u) > 0$ for $0 \le u < 4.8$. Since $C'(\tau_1) = C'(\tau_3) = 0$, we see that

$$
F(4.8) - F(\tau_1) = \left[X_3(\tau_1) - 1 - \frac{a^2 \tau_1}{(a+\tau_1)^2}\right]\frac{f(\tau_1)}{X_3'(\tau_1)}, \qquad (18)
$$

$$
F(4.8) - F(\tau_3) = \left[X_3(\tau_3) - 1 - \frac{a^2 \tau_3}{(a+\tau_3)^2}\right]\frac{f(\tau_3)}{X_3'(\tau_3)}. \qquad (19)
$$

It is easy to see that, for $0 \le u \le 4.2$,

$$
\begin{aligned}
&-14848 \times 10^8 u^5 - 30843472 \times 10^6 u^4 - 61282340640000 u^3 + 4835098792031600 u^2 \\
&-1646093669927116 0u - 848916071455141 > 0\,.
\end{aligned}
$$

See Figure 2. So we compute and obtain that, for $0 \le u \le 4.2$,

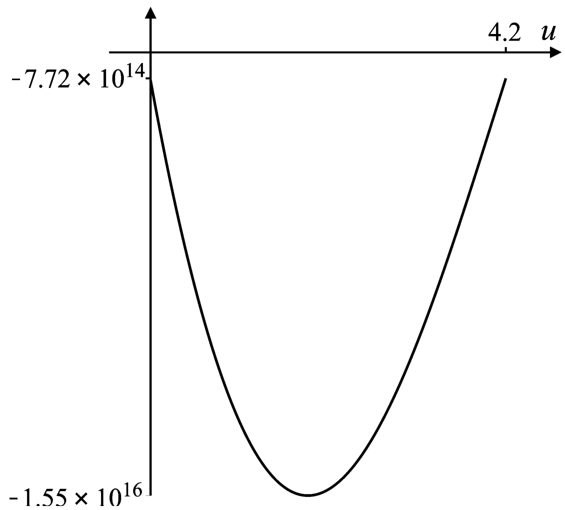

Figure 2: The graph of $-14848\times10^8u^5-30843472\times10^6u^4-61282340640000u^3+4835098792031600u^2-16460936699271160u-848916071455141$ on $[0,4.2]$.

$$
\begin{aligned}
& \frac{d}{du}\left[X_3(u)-1-\frac{a^2u}{(a+u)^2}\right]f(u) \\
= \; & \frac{f(u)}{5\times10^7\times(40u+163)^4}(-14848\times10^8u^5-30843472\times10^6u^4 \\
& -61282340640000u^3+4835098792031600u^2-16460936699271160u \\
& -848916071455141) \\
< \; & 0. \qquad (20)
\end{aligned}
$$

By the Intermediate Value Theorem, we can estimate that $0.3184<\tau_1<0.3186$ and $4.17443<\tau_3<4.17445$. So by (13), (18)–(20), we have that

$$
\begin{aligned}
C(\tau_1) &= 4.8f(4.8)-\tau_1f(\tau_1)-X_3(\tau_1)\left[F(4.8)-F(\tau_1)\right] \\
&= 4.8f(4.8)-\tau_1f(\tau_1)-\frac{X_3(\tau_1)}{X_3'(\tau_1)}\left[X_3(\tau_1)-1-\frac{a^2\tau_1}{(a+\tau_1)^2}\right]f(\tau_1) \\
&> 4.8f(4.8)-0.3186f(0.3186)-\frac{X_3(0.3186)}{X_3'(0.3186)}\Bigg[X_3(0.3184)-1 \\
&\quad -\frac{(4.075)^2\times0.3184}{(4.075+0.3184)^2}\Bigg]f(0.3184)\ \ (\approx 0.002) \\
&> 0, \qquad (21)
\end{aligned}
$$

$$
\begin{aligned}
C(\tau_9) &= 4.8f(4.8) - \tau_3 f(\tau_3) - X_3(\tau_3)\left[F(4.8) - F(\tau_3)\right] \\
&= 4.8f(4.8) - \tau_3 f(\tau_3) - \frac{X_3(\tau_3)}{X_3'(\tau_3)}\left[X_3(\tau_3) - 1 - \frac{a^2\tau_3}{(a+\tau_3)^2}\right] f(\tau_3) \\
&> 4.8f(4.8) - 4.175f(4.175) - \frac{X_3(4.175)}{X_3'(4.175)}\Bigg[X_3(4.173) \\
&\quad -1 - \frac{(4.075)^2 \times 4.173}{(4.075+4.173)^2}\Bigg] f(4.173) \ \ (\approx 0.018) \\
&> 0. \qquad (22)
\end{aligned}
$$

Since $C(4.8) = 0$, and by (3), (21) and (22), we see that $C(u) > 0$ for $0 \leq u < 4.8$.

The proof is complete.. ■