The proof of Lemma 3.2(ii)

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In this note, we prove Lemma 3.2(ii):

Lemma 3.2 (ii) Consider (1.1). Assume that $\eta \in (0,\infty)$ where $\eta \equiv \lim_{u\to 0^+} \frac{F(u)}{u^2}$. Assume that $\lim_{u\to 0^+} f''(u) = 0$ and $f^{(3)}(u)$ exists for $u > 0$. Then, for $\lambda > 0$,

$$\lim_{\alpha\to 0^+} T''_\lambda(\alpha) = \begin{cases} \frac{3\pi\sqrt{\eta}}{8\sqrt{2\lambda}}\left[\lambda - \frac{1}{12\eta^2}\lim_{u\to 0^+} f^{(3)}(u)\right] & \text{if } \lim_{u\to 0^+} f^{(3)}(u) \text{ exists,} \\ -\infty & \text{if } \lim_{u\to 0^+} f^{(3)}(u) = \infty. \end{cases}$$

Proof. First, we compute that

$$\begin{aligned} T''_\lambda(\alpha) &= \frac{1}{\alpha^2}\int_0^\alpha \frac{\left(3A^2B - CB^2 - 2AB^2\right)\lambda^3 + \left(3A^2 - 4AB - 2CB\right)\lambda^2}{\left[\lambda^2B^2 + 2\lambda B\right]^{5/2}} du \\ &= \frac{1}{\alpha}\int_0^1 \frac{1}{\left[\lambda^2 B^2(\alpha t,\alpha) + 2\lambda B(\alpha t,\alpha)\right]^{5/2}}\Big[(3\lambda^3 A^2(\alpha t,\alpha)B(\alpha t,\alpha) \\ &\quad -\lambda^3 C(\alpha t,\alpha)B^2(\alpha t,\alpha) - 2\lambda^3 A(\alpha t,\alpha)B^2(\alpha t,\alpha) + 3\lambda^2 A^2(\alpha t,\alpha) \\ &\quad -4\lambda^2 A(\alpha t,\alpha)B(\alpha t,\alpha) - 2\lambda^2 C(\alpha t,\alpha)B(\alpha t,\alpha)\Big] dt, \end{aligned} \tag{1}$$

where

$$A = A(u,\alpha) \equiv \alpha f(\alpha) - uf(u), \quad B = B(u,\alpha) \equiv F(\alpha) - F(u), \tag{2}$$

$$C = C(u,\alpha) \equiv \alpha^2 f'(\alpha) - u^2 f'(u). \tag{3}$$

Next, we consider two cases. Case 1: $\lim_{u\to 0^+} f^{(3)}(u)$ exists, and Case 2: $\lim_{u\to 0^+} f^{(3)}(u) = \infty$.

Case 1. Assume that $\lim_{u\to 0^+} f^{(3)}(u)$ exists. Since $\eta \in (0,\infty)$, we see that $f(0) = 0$ and $\lim_{u\to 0^+} f'(u) = 2\eta$. Let $\kappa \equiv \lim_{u\to 0^+} f^{(3)}(u)$. We have that

$$f(u) = 2\eta u + \frac{\kappa}{6}u^3 + o(u^4). \tag{4}$$

By L'Hopital's rule, we compute and find that, for $0 < t < 1$,

$$\lim_{\alpha\to 0}\frac{A(\alpha t,\alpha)}{\alpha^2} = \lim_{\alpha\to 0}\frac{C(\alpha t,\alpha)}{\alpha^2} = 2\left(1-t^2\right)\eta \text{ and } \lim_{\alpha\to 0}\frac{B(\alpha t,\alpha)}{\alpha^2} = \left(1-t^2\right)\eta. \tag{5}$$

By L'Hopital's rule, (4) and (5), we observe that

$$\lim_{\alpha\to 0}\frac{3A^2B-CB^2-2AB^2}{\alpha^6}=6\eta^3\left(1-t^2\right)^3 \tag{6}$$

and

$$\lim_{\alpha\to 0}\frac{3A^2-4AB-2CB}{\alpha^6}=\frac{-\kappa\eta\left(1-t^2\right)\left(1-t^4\right)}{6}. \tag{7}$$

Since $\lim_{\alpha\to 0}B(\alpha t,\alpha)=0$, and by (5), we see that $\lim_{\alpha\to 0}B^2(\alpha t,\alpha)/\alpha^2=0$. So by (1), (6) and (7), we compute and find that

$$\lim_{\alpha\to 0^+}T_\lambda''(\alpha)=\frac{\eta\lambda^2}{6\left(2\lambda\eta\right)^{5/2}}\int_0^1\frac{36\eta^2\left(1-t^2\right)^3\lambda-\kappa\left(1-t^2\right)\left(1-t^4\right)}{\left(1-t^2\right)^{5/2}}dt=\frac{3\pi\sqrt{\eta}}{8\sqrt{2\lambda}}\left[\lambda-\frac{\kappa}{12\eta^2}\right].$$

Case 2. Assume that $\lim_{u\to 0^+}f^{(3)}(u)=\infty$. By similar argument in Case 1, it is sufficient to prove that

$$\lim_{\alpha\to 0}\frac{3A^2(\alpha t,\alpha)-4A(\alpha t,\alpha)B(\alpha t,\alpha)-2C(\alpha t,\alpha)B(\alpha t,\alpha)}{\alpha^6}=-\infty. \tag{8}$$

Let $g(u)=f(u)-2\eta u$ for $u\geq 0$. Then $g(u)\in C[0,\infty)\cap C^2(0,\infty)$, $g^{(3)}(u)$ exists for $u>0$,

$$g(0)=g'(0)=g''(0)=0 \quad\text{and}\quad \lim_{u\to 0^+}g'''(u)=\infty. \tag{9}$$

Let

$$G(u)\equiv 4\alpha g(\alpha)-\alpha^2 g'(\alpha)-6\int_0^\alpha g(s)ds.$$

By (9), we see that $G(0)=G'(0)=0$ and

$$\lim_{u\to 0^+}G''(u)=\lim_{u\to 0^+}\left[-u^2g'''(u)\right]=-\infty. \tag{10}$$

It follows that $G'(u)$ is strictly decreasing for sufficiently small $u>0$. In addition, we observe that

$$A(\alpha t,\alpha)=2\eta\left(1-t^2\right)\alpha^2+\alpha g(\alpha)-\alpha t g(\alpha t), \tag{11}$$

$$B(\alpha t,\alpha)=\eta\left(1-t^2\right)\alpha^2+\int_{\alpha t}^\alpha g(s)ds, \tag{12}$$

$$C(\alpha t,\alpha)=2\eta\left(1-t^2\right)\alpha^2+\alpha^2g'(\alpha)-\alpha^2t^2g'(\alpha t). \tag{13}$$

By L'Hopital's rule and (9), we find that

$$\lim_{\alpha\to 0}\frac{\alpha g(\alpha)-\alpha t g(\alpha t)}{\alpha^3}=\lim_{\alpha\to 0}\frac{\int_{\alpha t}^\alpha g(s)ds}{\alpha^3}\lim_{\alpha\to 0}\frac{\alpha^2g'(\alpha)-\alpha^2t^2g'(\alpha t)}{\alpha^3}=0. \tag{14}$$

By (9)–(14) and L'Hopital's rule, we compute that

$$\lim_{\alpha\to 0}\frac{3A^2(\alpha t,\alpha)-4A(\alpha t,\alpha)B(\alpha t,\alpha)-2C(\alpha t,\alpha)B(\alpha t,\alpha)}{2\eta\left(1-t^2\right)\alpha^6}$$

$$
\begin{aligned}
&= \lim_{\alpha\to 0} \frac{4\left[\alpha g(\alpha) - \alpha t g(\alpha t)\right] - \left[\alpha^2 g'(\alpha) - \alpha^2 t^2 g'(\alpha t)\right] - 6\int_{\alpha t}^{\alpha} g(s)ds}{\alpha^4} \\
&= \lim_{\alpha\to 0} \frac{G(\alpha) - G(\alpha t)}{\alpha^4} = \lim_{\alpha\to 0} \frac{G'(\alpha) - t^2 G'(\alpha t)}{4\alpha^3} \leq \lim_{\alpha\to 0} \frac{\left(1-t^2\right) G'(\alpha t)}{4\alpha^3} \\
&= \lim_{\alpha\to 0} \frac{\left(1-t^2\right) t G''(\alpha t)}{12\alpha^2} = \lim_{\alpha\to 0} \frac{-\left(1-t^2\right) t^3 g'''(\alpha t)}{12} = -\infty.
\end{aligned}
$$

It follows that (8) holds. So $\lim_{\alpha\to 0^+} T''_\lambda(\alpha) = -\infty$ if $\lim_{u\to 0^+} f^{(3)}(u) = \infty$.

By Cases 1 and 2, the proof of Lemma 3.2(ii) is complete. ■