1. The proof of assertion (4.27)

First, we recall the functions

$$A = A(u, \alpha) \equiv \alpha f(\alpha) - u f(u), \quad B = B(u, \alpha) \equiv F(\alpha) - F(u), \tag{1.1}$$

and

$$C = C(u, \alpha) \equiv \alpha^2 f'(\alpha) - u^2 f'(u), \tag{1.2}$$

where $f(u) = u^p + u^q$, $F(u) = \int_0^u f(t)dt$, and $0 < p \leq q$. In this note, we prove the assertion

$$H_{\lambda,\rho(\alpha,\lambda)}(u, \alpha) > 0 \text{ for } 0 < u < \alpha \text{ and } 0 < \lambda \leq 2, \tag{4.27}$$

where

$$\rho(\alpha, \lambda) \equiv \begin{cases} 0 & \text{if } (\alpha, \lambda) \in \Theta_1 \cup \Theta_2 \cup \Theta_3, \\ -1 & \text{if } (\alpha, \lambda) \in \Theta_4 \cup \Theta_5 \cup \Theta_6, \end{cases}$$

$$\Theta_1 \equiv (1, \infty) \times (0, 2.01], \quad \Theta_2 \equiv (0.75, 1] \times (1, 1.6], \quad \Theta_3 \equiv (0.71, 1] \times (1.6, 2.01],$$
$$\Theta_4 \equiv (0, 1] \times (0, 1], \quad \Theta_5 \equiv (0, 0.75] \times (1, 1.6], \quad \Theta_6 \equiv (0, 0.71] \times (1.6, 2.01].$$

Some parts of this proof can be obtained by applying the next Sturm's theorem based on Sturm sequences. Generally, Sturm's theorem is used to determine the exact number of distinct real zeros of univariate polynomials on some interval. See [5, pp. 18–19] for a complete proof of Sturm's theorem. See also [2, 3, 6] for some applications of Sturm's theorem.

Theorem 1.1 ([1, Theorem 8.8.15], [5, Theorem 1.3.9]). (*Sturm's Theorem*) *Assume that $\tau_1, \tau_2 \in \mathbb{R}$ and $Q(x)$ is a polynomial such that $Q(\tau_1) = Q(\tau_2) \neq 0$. Let the Sturm sequence $\{Q_0, Q_1, ..., Q_k\}$ of Q be defined by*

$$Q_0 = Q,\ Q_1 = Q',\ Q_2 = -rem(Q_0, Q_1),\ Q_3 = -rem(Q_1, Q_2),\ \cdots, 0 = -rem(Q_{k-1}, Q_k),$$

where $rem(Q_i, Q_j)$ is the remainder of the polynomial long division of Q_i by Q_j, and where k is the minimal number of polynomial divisions needed to obtain a zero remainder. Let $\sigma_Q(\zeta)$ denote the number of sign changes (ignoring zeroes) in the sequence

$$\{Q_0(\zeta), Q_1(\zeta), ..., Q_k(\zeta)\}.$$

Then Q has $\sigma_Q(\tau_1) - \sigma_Q(\tau_2)$ distinct real zeros on the interval (τ_1, τ_2).

Notice that a lot of Sturm sequences are needed to be computed in this section. All Sturm sequences can be computed precisely. To ensure the correctness of the computations, we apply the symbolic manipulator *Maple 16* to check them. These computations are based on symbolic and exact integer computations. The worth mentioning is that Reference [4] displays the detail of symbolic computations that appear in this section. Readers may refer it.

For $u \in \mathbf{R}$, let the function sgn be defined by

$$sgn(u) = \begin{cases} 1 & \text{if } u > 0, \\ 0 & \text{if } u = 0, \\ -1 & \text{if } u < 0. \end{cases}$$

Assuming that $s = \{s_i\}_{i=1}^m$ is a finite (real) sequence, we denote $sgn(s)$ by

$$sgn(s) = \{sgn(s_1), sgn(s_2), ..., sgn(s_m)\}.$$

Before we prove assertion (4.27), we have the following Lemma 1.2.

Lemma 1.2. *Assume that* $\lambda_2 > \lambda_1 \geq 0$ *and*

$$H_{\lambda_1,-1}(u,\alpha) \geq 0 \ \text{ and } \ H_{\lambda_2,-1}(u,\alpha) > 0 \ \text{ for } 0 < u < \alpha. \tag{1.3}$$

Then $H_{\lambda,-1}(u,\alpha) > 0$ *for* $0 < u < \alpha$ *and* $\lambda_1 < \lambda \leq \lambda_2$.

Proof of Lemma 1.2. Let $\alpha > u > 0$ be given. We compute that

$$\begin{aligned} H_{\lambda,-1}(u,\alpha) &= -B^5\lambda^3 - 5B^4\lambda^2 + B\left(-8B^2 - 2AB - BC + 3A^2 + AB\right)\lambda \\ &\quad -4B^2 - 4AB - 2BC + 3A^2 + 2AB \end{aligned}$$

and

$$\frac{\partial}{\partial\lambda} H_{\lambda,-1}(u,\alpha) = -3B^5\lambda^2 - 10B^4\lambda + B\left(-8B^2 - 2AB - BC + 3A^2 + AB\right).$$

Let $t = u/\alpha$. It implies that $0 < t < 1$. Since $p = 1$ and $q = 3$, and by (1.1), we see that

$$\begin{aligned} \left.\frac{\partial}{\partial\lambda} H_{\lambda,-1}(u,\alpha)\right|_{\lambda=0} &= -8B^2 - 2AB - BC + 3A^2 + AB \\ &= \frac{3\alpha^6}{2}\left(t^2+1\right)\left(t^2-1\right)^2\left(t^2\alpha^2 + \alpha^2 + 1\right) > 0. \end{aligned} \tag{1.4}$$

Clearly, $\partial H_{\lambda,-1}(u,\alpha)/\partial\lambda$ is a strictly decreasing function of $\lambda > 0$. So by (1.4), we observe that either $H_{\lambda,-1}(u,\alpha)$ is a strictly increasing function of $\lambda > 0$, or $H_{\lambda,-1}(u,\alpha)$ is a strictly increasing and then strictly decreasing function of $\lambda > 0$. So if (1.3) holds, then $H_{\lambda,-1}(u,\alpha) > 0$ for $\lambda_1 < \lambda \leq \lambda_2$. The proof of Lemma 1.2 is complete. ■

Proof of assertion (4.27). We consider six cases. Case i: $(\alpha,\lambda) \in \Theta_i$ for $i = 1, 2, ..., 6$.

Case 1. Assume that $(\alpha,\lambda) \in \Theta_1 = (1,\infty) \times (0, 2.01]$. In this case, $\rho(\alpha,\lambda) = 0$. Since

$$\frac{B\lambda + 1}{B\lambda + 2} > \frac{1}{2} \ \text{ for } 0 < u < \alpha,$$

we observe that, for $\alpha > 1$, $0 < t < 1$, and $0 < \lambda \leq 2.01$,

$$\begin{aligned} H_{\lambda,0}(\alpha t,\alpha) &> [\lambda B(\alpha t,\alpha) + 2]\left[\frac{3}{2}A^2(\alpha t,\alpha) - 2A(\alpha t,\alpha)B(\alpha t,\alpha) - C(\alpha t,\alpha)B(\alpha t,\alpha)\right] \\ &= \frac{\lambda B(\alpha t,\alpha) + 2}{4}\alpha^6\left(1-t^2\right)^2\left(1+t^2\right)\left[\left(1+t^2\right)\alpha^2 - 1\right] > 0, \end{aligned}$$

from which it follows that $H_{\lambda,0}(u,\alpha) > 0$ for $0 < u < \alpha$, $\alpha > 1$, and $0 < \lambda \leq 2.01$. It implies that (4.27) holds for $(\alpha,\lambda) \in \Theta_1$.

Case 2. Assume that $(\alpha, \lambda) \in \Theta_2 = (0.75, 1] \times (1, 1.6]$. In this case, $\rho(\alpha, \lambda) = 0$. We have that

$$H_{\lambda,0}(u, \alpha) = B\left(-2AB - BC + 3A^2\right)\lambda - 4AB - 2BC + 3A^2. \tag{1.5}$$

By (1.5), we compute and find that, for $0.75 < \alpha \leq 1$ and $0 < t < 1$,

$$\begin{aligned} H_{1,0}(\alpha t, \alpha) &= \frac{\alpha^6\left(1-t^2\right)^2}{16}\Big[7\left(1-t^2\right)\left(1+t^2\right)^3\alpha^6 + 25\left(1-t^2\right)\left(1+t^2\right)^2\alpha^4 \\ &\quad +4\left(9-5t^2\right)\left(1+t^2\right)\alpha^2 + 4\left(1-5t^2\right)\Big] \\ &> \frac{\alpha^6\left(1-t^2\right)^2}{16}\left[7\left(1-t^2\right)\left(1+t^2\right)^3\left(\frac{75}{100}\right)^6 + 25\left(1-t^2\right)\left(1+t^2\right)^2\left(\frac{75}{100}\right)^4\right. \\ &\quad \left. +4\left(9-5t^2\right)\left(1+t^2\right)\left(\frac{75}{100}\right)^2 + 4\left(1-5t^2\right)\right] \\ &= \frac{\alpha^6\left(1-t^2\right)^2}{16}\left(-\frac{5103}{4096}t^8 - \frac{21303}{2048}t^6 - \frac{4905}{256}t^4 - \frac{1225}{2048}t^2 + \frac{136831}{4096}\right) > 0 \end{aligned}$$

and

$$\begin{aligned} H_{1.6,0}(\alpha t, \alpha) &= \frac{32\alpha^6\left(1-t^2\right)^2}{125}\Big[7\left(1-t^2\right)\left(1+t^2\right)^3\alpha^6 + 25\left(1-t^2\right)\left(1+t^2\right)^2\alpha^4 \\ &\quad +\left(33-23t^2\right)\left(t^2+1\right)\alpha^2 + \left(7-17t^2\right)\Big] \\ &\geq \frac{32\alpha^6\left(1-t^2\right)^2}{125}\left[7\left(1-t^2\right)\left(1+t^2\right)^3\left(\frac{75}{100}\right)^6 + 25\left(1-t^2\right)\left(1+t^2\right)^2\left(\frac{75}{100}\right)^4\right. \\ &\quad \left. +\left(33-23t^2\right)\left(1+t^2\right)\left(\frac{75}{100}\right)^2 + 7 - 17t^2\right] \\ &= \frac{32\alpha^6\left(1-t^2\right)^2}{125}\left(-\frac{5103}{4096}t^8 - \frac{21303}{2048}t^6 - \frac{5337}{256}t^4 - \frac{1993}{2048}t^2 + \frac{142207}{4096}\right) > 0. \end{aligned}$$

Thus we have that

$$H_{1,0}(u, \alpha) > 0 \;\text{ and }\; H_{1.6,0}(u, \alpha) > 0 \;\text{ for } 0 < u < \alpha \text{ and } 0.75 < \alpha \leq 1. \tag{1.6}$$

Let $\alpha > u > 0$ be given. Then we observe that $H_{\lambda,0}(u, \alpha)$ is either monotone increasing or monotone decreasing for $\lambda > 0$ by (1.5). So by (1.6), we obtain that

$$H_{\lambda,0}(u, \alpha) > 0 \;\text{ for } 0 < u < \alpha,\ 0.75 < \alpha \leq 1 \text{ and } 1 < \lambda \leq 1.6,$$

which implies that (4.27) holds for $(\alpha, \lambda) \in \Theta_2$.

Case 3. Assume that $(\alpha,\lambda)\in\Theta_3=(0.71,1]\times(1.6,2.01]$. In this case, $\rho(\alpha,\lambda)=0$. By (1.5), we compute and find that, for $0.71<\alpha\le 1$ and $0<t<1$,

$$\begin{aligned}
H_{1.6,0}(\alpha t,\alpha) &= \frac{32\alpha^6\left(1-t^2\right)^2}{125}\Big[7\left(1-t^2\right)\left(1+t^2\right)^3\alpha^6+25\left(1-t^2\right)\left(1+t^2\right)^2\alpha^4 \\
&\quad +\left(33-23t^2\right)\left(t^2+1\right)\alpha^2+\left(7-17t^2\right)\Big] \\
&\ge \frac{32\alpha^6\left(1-t^2\right)^2}{125}\left[7\left(1-t^2\right)\left(1+t^2\right)^3\left(\frac{71}{100}\right)^6+25\left(1-t^2\right)\left(1+t^2\right)^2\left(\frac{71}{100}\right)^4\right. \\
&\quad \left.+\left(33-23t^2\right)\left(1+t^2\right)\left(\frac{71}{100}\right)^2+7-17t^2\right] \\
&= \frac{32\alpha^6\left(1-t^2\right)^2}{125}\left[-\frac{896701987447}{10^{12}}t^8-\frac{4073162112447}{5\times10^{11}}t^6-\frac{71788881}{4\times10^6}t^4\right. \\
&\quad \left.-\frac{1906337887553}{5\times10^{11}}t^2+\frac{30884922237447}{10^{12}}\right]>0
\end{aligned}$$

and

$$\begin{aligned}
H_{2.01,0}(\alpha t,\alpha) &= \frac{(1-t^2)^2\alpha^6}{1600}\Big[1407\left(1-t^2\right)\left(1+t^2\right)^3\alpha^6+5025\left(1-t^2\right)\left(t^2+1\right)^2\alpha^4 \\
&\quad +4\left(1607-1207t^2\right)\left(1+t^2\right)\alpha^2+1612-3212t^2\Big] \\
&> \frac{(1-t^2)^2\alpha^6}{1600}\left[1407\left(1-t^2\right)\left(1+t^2\right)^3\left(\frac{71}{100}\right)^6+5025\left(1-t^2\right)\left(t^2+1\right)^2\left(\frac{71}{100}\right)^4\right. \\
&\quad \left.+4\left(1607-1207t^2\right)\left(1+t^2\right)\left(\frac{71}{100}\right)^2+1612-3212t^2\right] \\
&= \frac{(1-t^2)^2\alpha^6}{1600}\left[-\frac{180237099476847}{10^{12}}t^8-\frac{818705584601847}{5\times10^{11}}t^6-\frac{14842927081}{4\times10^6}t^4\right. \\
&\quad \left.-\frac{384014415398153}{5\times10^{11}}t^2+\frac{6309528869726847}{10^{12}}\right]>0.
\end{aligned}$$

So we have that

$$H_{1.6,0}(u,\alpha)>0 \text{ and } H_{2.01,0}(u,\alpha)>0 \text{ for } 0<u<\alpha \text{ and } 0.71\le\alpha\le 1. \tag{1.7}$$

Let $\alpha>u>0$ be given. Then we observe that $H_{\lambda,0}(u,\alpha)$ is either monotone increasing or monotone decreasing for $\lambda>0$ by (1.5). So by (1.7), we obtain that

$$H_{\lambda,0}(u,\alpha)>0 \text{ for } 0<u<\alpha,\ 0.71<\alpha\le 1 \text{ and } 1.6<\lambda\le 2.01,$$

which implies that (4.27) holds for $(\alpha,\lambda)\in\Theta_3$.

Case 4. Assume that $(\alpha,\lambda)\in\Theta_4=(0,1]\times(0,1]$. In this case, $\rho(\alpha,\lambda)=-1$. For any $0<u<\alpha$, we see that $u=\alpha t$ for some $0<t<1$. It follows that

$$H_{0,-1}(u,\alpha)=H_{0,-1}(\alpha t,\alpha)=\frac{3}{4}\alpha^8(1-t^4)^2>0 \text{ for } 0<u<\alpha. \tag{1.8}$$

We assert that

$$H_{1,-1}(\alpha t,\alpha)=\frac{1}{1024}\alpha^8(t^2-1)^2V_1(t,t,\alpha^2)>0 \ \text{ for } 0<\alpha\le 1 \text{ and } 0<t<1. \tag{1.9}$$

where

$$\begin{aligned}V_1(t,s,u) \ \equiv \ & (t^{16}+2t^{14}-2s^{12}-6s^{10}+6t^6+2t^4-2s^2-1)u^6+(10t^{14}+10t^{12}\\ & -30s^{10}-30s^8+30t^6+30t^4-10s^2-10)u^5+(20t^{12}-40s^{10}\\ & -100s^8+80t^6+140t^4-40s^2-60)u^4+(-80s^{10}-240s^8+160t^6\\ & +480t^4-80s^2-240)u^3+(-784s^8-928s^6+960t^4+928t^2-176)u^2\\ & +(-1760s^6-608s^4+1888t^2+480)u-320s^4+2176t^2+1216.\end{aligned}$$

It follows that $H_{1,-1}(u,\alpha)>0$ for $0<u<\alpha$ and $0<\alpha\le 1$. So by Lemma 1.2 and (1.8), we see that $H_{\lambda,-1}(u,\alpha)>0$ for $0<u<\alpha\le 1$ and $0<\lambda\le 1$. It implies that (4.27) holds for $(\alpha,\lambda)\in\Theta_4$.

Next, we prove assertion (1.9). We easily compute that $H_{1,-1}(\alpha t,\alpha)=\frac{1}{1024}\alpha^8(t^2-1)^2V_1(t,t,\alpha^2)$. It is sufficient to prove that $V_1(t,t,\alpha^2)>0$ for $0<\alpha\le 1$ and $0<t<1$. We let $\{t_i\}_{i=0}^5$ be a uniform partition of $[0,1]$ and let

$$V_{1,i}(u)\equiv V_1(t_i,t_{i+1},u) \ \text{ for } i=0,1,2,3,4.$$

We compute and find that

$$sgn\left(s_{V_{1,i}}(0)\right)=\begin{cases}\{1,1,-1,-1,1,1,-1\} & \text{for } i=0,1,2,3,\\ \{1,-1,-1,-1,1,1,-1\} & \text{for } i=4,\end{cases} \tag{1.10}$$

and

$$sgn\left(s_{V_{1,i}}(1)\right)=\begin{cases}\{1,-1,-1,1,1,1,-1\} & \text{for } i=0,1,2,\\ \{1,-1,-1,-1,1,1,-1\} & \text{for } i=3,4.\end{cases} \tag{1.11}$$

The verification of these computations can be found in [4]. By (1.10) and (1.11), we obtain that, for $i=0,1,2,3,4$,

$$V_{1,i}(0)>0,\ \ V_{1,i}(1)>0 \ \text{ and } \ \sigma_{V_{1,i}}(0)-\sigma_{V_{1,i}}(1)=3-3=0.$$

So by Theorem 1.1, we obtain that $V_{1,i}(u)>0$ for $0\le u\le 1$ and $i=0,1,2,3,4$. So we see that, for $0<\alpha<1$, $t_i<t\le t_{i+1}$ and $i=0,1,2,3,4$,

$$V_1(t,t,\alpha^2)>V_1(t_i,t_{i+1},\alpha^2)=V_{1,i}(\alpha^2)>0.$$

It follows that (1.9) holds.

Case 5. Assume that $(\alpha,\lambda)\in\Theta_5=(0,0.75]\times(1,1.6]$. In this case, $\rho(\alpha,\lambda)=-1$. By (1.9), we have that

$$H_{1,-1}(u,\alpha)>0 \ \text{ for } 0<u<\alpha\le 0.75. \tag{1.12}$$

We assert that

$$H_{1.6,-1}(\alpha t,\alpha)=\frac{16}{3125}\alpha^8\left(1-t^2\right)^2V_2(t,t,\alpha^2)>0 \ \text{ for } 0<\alpha\le 0.75 \text{ and } 0<t<1. \tag{1.13}$$

where

$$\begin{aligned} V_2(t,s,u) \equiv\ & (2t^{16}+4t^{14}-4s^{12}-12s^{10}+12t^6+4t^4-4s^2-2)u^6+(20t^{14}+20t^{12} \\ & -60s^{10}-60s^8+60t^6+60t^4-20s^2-20)u^5+(55t^{12}-50s^{10}-215s^8 \\ & +100t^6+265t^4-50s^2-105)u^4+(-40s^{10}-360s^8+80t^6+720t^4 \\ & -40s^2-360)u^3+(-740s^8-920s^6+1200t^4+920t^2-460)u^2 \\ & +(-1636s^6-292s^4+1892t^2+36)u-625s^4+1550t^2+575. \end{aligned}$$

It follows that $H_{1.6,-1}(u,\alpha)>0$ for $0<u<\alpha$ and $0<\alpha\le 0.75$. So by Lemma 1.2 and (1.12), we see that $H_{\lambda,-1}(u,\alpha)>0$ for $0<u<\alpha\le 0.75$ and $1<\lambda\le 1.6$. It implies that (4.27) holds for $(\alpha,\lambda)\in\Theta_5$.

Next, we prove assertion (1.13). We easily compute that $H_{1.6,-1}(\alpha t,\alpha)=\frac{16}{3125}\alpha^8(1-t^2)^2 V_2(t,t,\alpha^2)$. It is sufficient to prove that $V_2(t,t,\alpha^2)>0$ for $0<\alpha\le 0.75$ and $0<t<1$. We let $\{t_i\}_{i=0}^5$ be a uniform partition of $[0,1]$ and let

$$V_{2,i}(u)\equiv V_2(t_i,t_{i+1},u) \ \text{ for } i=0,1,2,3,4.$$

We compute and find that

$$sgn\left(s_{V_{2,i}}(0)\right)=\begin{cases}\{1,1,-1,-1,1,1,-1\} & \text{for } i=0,1,2,3,\\ \{1,-1,-1,-1,1,1,-1\} & \text{for } i=4,\end{cases} \tag{1.14}$$

and

$$sgn\left(s_{V_{2,i}}([0.75]^2)\right)=\begin{cases}\{1,-1,-1,1,1,1,-1\} & \text{for } i=0,1,2,\\ \{1,-1,-1,-1,1,1,-1\} & \text{for } i=3,4.\end{cases} \tag{1.15}$$

The verification of these computations can be found in [4]. By (1.14) and (1.15), we obtain that, for $i=0,1,2,3,4$,

$$V_{2,i}(0)>0,\ V_{2,i}([0.75]^2)>0 \ \text{ and } \sigma_{V_{2,i}}(0)-\sigma_{V_{2,i}}([0.75]^2)=3-3=0.$$

So by Theorem 1.1, we obtain that $V_{2,i}(u)>0$ for $0\le u\le[0.75]^2$ and $i=0,1,2,3,4$. So we see that, for $0<\alpha<0.75$, $t_i<t\le t_{i+1}$ and $i=0,1,2,3,4$,

$$V_2(t,t,\alpha^2)>V_2(t_i,t_{i+1},\alpha^2)=V_{2,i}(\alpha^2)>0.$$

It follows that (1.13) holds.

Case 6. Assume that $(\alpha,\lambda)\in\Theta_6=(0,0.71]\times(1.6,2.01]$. In this case, $\rho(\alpha,\lambda)=-1$. By (1.13), we see that

$$H_{1.6,-1}(u,\alpha)>0 \ \text{ for } 0<u<\alpha\le 0.71. \tag{1.16}$$

We assert that

$$H_{2.01,-1}(\alpha t,\alpha)=\frac{3}{1024\times 10^6}\alpha^8\left(t^2-1\right)^2 V_3(t,t,\alpha^2)>0 \ \text{ for } 0<\alpha\le 0.71 \text{ and } 0<t<1. \tag{1.17}$$

where

$$\begin{aligned} V_3(t,t,\alpha^2) \equiv\ & (2706867t^{16} + 5413734t^{14} - 5413734s^{12} - 16241202s^{10} + 16241202t^6 \\ & +5413734t^4 - 5413734s^2 - 2706867)u^6 + (27068670t^{14} + 27068670t^{12} \\ & -81206010s^{10} - 81206010s^8 + 81206010t^6 + 81206010t^4 - 27068670s^2 \\ & -27068670)u^5 + (81340680t^{12} - 53868000s^{10} - 297890040s^8 + 107736000t^6 \\ & +351758040t^4 - 53868000s^2 - 135208680)u^4 + (1077360t^{10} - 432021360s^8 \\ & -2154720s^6 + 864042720t^4 + 1077360t^2 - 432021360)u^3 + (-687146640s^8 \\ & -947658720s^6 + 1292832000t^4 + 947658720t^2 - 605685360)u^2 \\ & +(-1547108256s^6 - 169811232s^4 + 1893587232t^2 - 176667744)u \\ & -689504000s^4 + 1373888000t^2 + 339616000. \end{aligned}$$

It follows that $H_{2.01,-1}(u,\alpha) > 0$ for $0 < u < \alpha$ and $0 < \alpha \le 0.71$. So by Lemma 1.2 and (1.16), we see that $H_{\lambda,-1}(u,\alpha) > 0$ for $0 < u < \alpha \le 0.71$ and $1.6 < \lambda \le 2.01$. It implies that (4.27) holds for $(\alpha,\lambda) \in \Theta_6$.

Next, we prove assertion (1.17). We easily compute that

$$H_{2.01,-1}(\alpha t,\alpha) = \frac{3}{1024 \times 10^6}\alpha^8\left(t^2-1\right)^4 V_3(t,t,\alpha^2).$$

It is sufficient to prove that $V_3(t,t,\alpha^2) > 0$ for $0 < \alpha \le 0.71$ and $0 < t < 1$. We let $\{t_i\}_{i=0}^{8}$ be a uniform partition of $[0,1]$ and let

$$V_{3,i}(u) \equiv V_3(t_i, t_{i+1}, u) \ \text{ for } i = 0,1,\ldots,7.$$

We compute and find that

$$sgn\left(s_{V_{3,i}}(0)\right) = \begin{cases} \{1,-1,-1,-1,1,1,-1\} & \text{for } i = 0,1,2, \\ \{1,1,-1,-1,1,1,-1\} & \text{for } i = 3,4,5,6, \\ \{1,-1,-1,-1,1,-1,-1\} & \text{for } i = 7, \end{cases} \tag{1.18}$$

and

$$sgn\left(s_{V_{3,i}}([0.71]^2)\right) = \begin{cases} \{1,-1,-1,1,1,1,-1\} & \text{for } i = 0,1,2,3,4, \\ \{1,-1,-1,-1,1,1,-1\} & \text{for } i = 5,6, \\ \{1,-1,-1,-1,1,-1,-1\} & \text{for } i = 7. \end{cases} \tag{1.19}$$

The verification of these computations can be found in [4]. By (1.18) and (1.19), we obtain that, for $i = 0,1,\ldots,7$,

$$V_{3,i}(0) > 0,\ V_{3,i}([0.71]^2) > 0 \ \text{ and } \sigma_{V_{3,i}}(0) - \sigma_{V_{3,i}}([0.71]^2) = 3 - 3 = 0.$$

So by Theorem 1.1, we obtain that $V_{3,i}(u) > 0$ for $0 \le u \le [0.71]^2$ and $i = 0,1,\ldots,7$. So we see that, for $0 < \alpha < 0.71$, $t_i < t \le t_{i+1}$ and $i = 0,1,\ldots,7$,

$$V_2(t,t,\alpha^2) > V_2(t_i,t_{i+1},\alpha^2) = V_{3,i}(\alpha^2) > 0.$$

It follows that (1.17) holds.

By above discussions, the proof of assertion (4.27) is complete. ■

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