An Up-to-date DSGE Model

Dr. Tai-kuang Ho*

*Associate Professor. Department of Quantitative Finance, National Tsing Hua University, No. 101, Section 2, Kuang-Fu Road, Hsinchu, Taiwan 30013, Tel: +886-3-571-5131, ext. 62136, Fax: +886-3-562-1823, E-mail: tkho@mx.nthu.edu.tw. This note draws heavily on lecture notes of Jesus Fernandez-Villaverde and Stephanie Schmitt-Grohe.

1 Households

- Utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \frac{c_t^{1-\sigma}}{1-\sigma} + \frac{\upsilon}{1-\xi} \left(\frac{m_t}{p_t} \right)^{1-\xi} - \psi \frac{l_t^{1+\gamma}}{1+\gamma} \right\}$$

- Money in the utility function (MIU)

- Budget constraint

$$c_t + x_t + \frac{m_t}{p_t} + \frac{b_{t+1}}{p_t} = w_t l_t + r_t k_t + \frac{m_{t-1}}{p_t} + R_t \frac{b_t}{p_t} + T_t + \Pi_t$$

- R_t :gross nominal interest rate

- Law of motion for capital

$$k_{t+1} = (1-\delta)\,k_t + x_t$$

- Decision variables:

- $c_t, m_t, l_t, b_{t+1}, k_{t+1}\,(x_t)$

- x_t :investment

- nominal variables: m_t, b_{t+1}

- Lagrangian

$$
\begin{aligned}
\mathcal{L} \;=\; & \sum_{t=0}^{\infty} \beta^t \left\{ \frac{c_t^{1-\sigma}}{1-\sigma} + \frac{\upsilon}{1-\xi} \left(\frac{m_t}{p_t} \right)^{1-\xi} - \psi \frac{l_t^{1+\gamma}}{1+\gamma} \right\} \\
& + \sum_{t=0}^{\infty} \lambda_t \beta^t \left[\begin{array}{c} w_t l_t + r_t k_t + \frac{m_{t-1}}{p_t} + R_t \frac{b_t}{p_t} + T_t + \Pi_t \\ -c_t + (1-\delta) k_t - k_{t+1} - \frac{m_t}{p_t} - \frac{b_{t+1}}{p_t} \end{array} \right]
\end{aligned}
$$

- First-order conditions

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0$$

$$c_t^{-\sigma} = \lambda_t$$

$$\frac{\partial \mathcal{L}}{\partial m_t} = 0$$

$$\beta^t \frac{\upsilon}{1} \left(\frac{m_t}{p_t} \right)^{-\xi} \frac{1}{p_t} - \lambda_t \beta^t \frac{1}{p_t} + \lambda_{t+1} \beta^{t+1} \frac{1}{p_{t+1}} = 0$$

$$\frac{\upsilon}{1}\left(\frac{m_t}{p_t}\right)^{-\xi}\frac{1}{p_t} = \lambda_t\frac{1}{p_t} - \lambda_{t+1}\beta\frac{1}{p_{t+1}}$$

$$\frac{\upsilon}{1}\left(\frac{m_t}{p_t}\right)^{-\xi}\frac{1}{p_t} = \frac{c_t^{-\sigma}}{p_t} - \beta\frac{c_{t+1}^{-\sigma}}{p_{t+1}}$$

$$\frac{\upsilon}{1}\left(\frac{m_t}{p_t}\right)^{-\xi} = c_t^{-\sigma} - \beta c_{t+1}^{-\sigma}\frac{p_t}{p_{t+1}}$$

$$\frac{\partial\mathcal{L}}{\partial l_t} = 0$$

$$-\beta^t \psi \frac{l_t^{\gamma}}{1} + \lambda_t \beta^t w_t = 0$$

$$\psi l_t^{\gamma} = c_t^{-\sigma} w_t$$

$$\frac{\partial \mathcal{L}}{\partial b_{t+1}} = 0$$

$$\lambda_t \beta^t \frac{-1}{p_t} + \lambda_{t+1} \beta^{t+1} \frac{R_{t+1}}{p_{t+1}} = 0$$

$$\frac{\lambda_t}{p_t} = \lambda_{t+1}\beta\frac{R_{t+1}}{p_{t+1}}$$

$$\frac{c_t^{-\sigma}}{p_t} = \beta R_{t+1}\frac{c_{t+1}^{-\sigma}}{p_{t+1}}$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = 0$$

$$-\lambda_t\beta^t + \lambda_{t+1}\beta^{t+1}\left(r_{t+1} + 1 - \delta\right) = 0$$

$$\lambda_t = \lambda_{t+1}\beta\left(r_{t+1} + 1 - \delta\right)$$

$$c_t^{-\sigma} = c_{t+1}^{-\sigma}\beta\left(r_{t+1} + 1 - \delta\right)$$

- A summary of first-order conditions

$$c_t^{-\sigma} = \beta E_t\left[c_{t+1}^{-\sigma}\left(r_{t+1} + 1 - \delta\right)\right]$$

$$c_t^{-\sigma} = \beta E_t\left[c_{t+1}^{-\sigma} R_{t+1}\frac{p_t}{p_{t+1}}\right]$$

$$\psi l_t^{\gamma} = c_t^{-\sigma} w_t$$

$$\upsilon \left(\frac{m_t}{p_t} \right)^{-\xi} = c_t^{-\sigma} - E_t \left(\beta c_{t+1}^{-\sigma} \frac{p_t}{p_{t+1}} \right)$$

2 Government

- The government sets the nominal interest rates according to the Taylor rule:

$$\frac{R_{t+1}}{R} = \left(\frac{R_t}{R}\right)^{\gamma_R} \left(\frac{\pi_t}{\pi}\right)^{\gamma_\pi} \left(\frac{y_t}{y}\right)^{\gamma_y} e^{\varphi_t}$$

$$\varphi_t \sim \mathcal{N}\left(0, \sigma_\varphi\right)$$

- π :target level of inflation
- y :the steady state output
- R :steady state gross return of capital
- $\varphi_t \sim \mathcal{N}\left(0, \sigma_\varphi\right)$

- Lump-sum transfers:

$$T_t = \frac{m_t}{p_t} - \frac{m_{t-1}}{p_t} + \frac{b_{t+1}}{p_t} - R_t \frac{b_t}{p_t}$$

3 Monopolistic competition

- Competitive behavior of final good producer

- Continuum of immediate good producers with market power

- Alternative formulation: continuum of goods in the utility function

3.1 Final good producer

- Production function

$$y_t = \left(\int_0^1 y_{it}^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}}$$

- $\varepsilon > 1$ and controls the elasticity of substitution

- Final good producer is perfectly competitive and maximizes profits, taking as given all intermediate goods prices p_{it} and the final good price p_t

- The maximization problem of the final good producer is:

$$\max_{y_{it}} p_t y_t - \int_0^1 p_{it} y_{it} di$$

- First order conditions are:

$$p_t \frac{\varepsilon}{\varepsilon - 1} \left(\int_0^1 y_{it}^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}-1} \frac{\varepsilon - 1}{\varepsilon} y_{it}^{\frac{\varepsilon-1}{\varepsilon}-1} - p_{it} = 0, \qquad \forall i$$

$$p_t y_t^{\frac{1}{\varepsilon}} y_{it}^{\frac{-1}{\varepsilon}} = p_{it}$$

$$p_t y_t^{\frac{1}{\varepsilon}} y_{jt}^{\frac{-1}{\varepsilon}} = p_{jt}$$

$$\frac{p_{it}}{p_{jt}} = \left(\frac{y_{it}}{y_{jt}}\right)^{-\frac{1}{\varepsilon}}$$

$$p_{jt} = \left(\frac{y_{it}}{y_{jt}}\right)^{\frac{1}{\varepsilon}} p_{it}$$

$$p_{jt}y_{jt} = p_{it}y_{it}^{\frac{1}{\varepsilon}}y_{jt}^{\frac{-1}{\varepsilon}+1} = p_{it}y_{it}^{\frac{1}{\varepsilon}}y_{jt}^{\frac{\varepsilon-1}{\varepsilon}}$$

- Integrating out:

$$\int_0^1 p_{jt} y_{jt} dj = p_{it} y_{it}^{\frac{1}{\varepsilon}} \underbrace{\int_0^1 y_{jt}^{\frac{\varepsilon-1}{\varepsilon}} dj}_{y_t^{\frac{\varepsilon-1}{\varepsilon}}} = p_{it} y_{it}^{\frac{1}{\varepsilon}} y_t^{\frac{\varepsilon-1}{\varepsilon}}$$

- Because of perfect competition, the profits of final good producer are zero:

$$p_t y_t = \int_0^1 p_{it} y_{it} di$$

$$p_t y_t = p_{it} y_{it}^{\frac{1}{\varepsilon}} y_t^{\frac{\varepsilon-1}{\varepsilon}} \Rightarrow p_t = p_{it} y_{it}^{\frac{1}{\varepsilon}} y_t^{-\frac{1}{\varepsilon}}$$

$$y_{it} = \left(\frac{p_{it}}{p_t}\right)^{-\varepsilon} y_t, \qquad \forall i$$

- Deduction of price level

$$p_t y_t = \int_0^1 p_{it} y_{it} di$$

$$y_{it} = \left(\frac{p_{it}}{p_t}\right)^{-\varepsilon} y_t$$

$$p_t y_t = \int_0^1 p_{it} \left(\frac{p_{it}}{p_t} \right)^{-\varepsilon} y_t di \Rightarrow p_t^{1-\varepsilon} = \int_0^1 p_{it}^{1-\varepsilon} di$$

$$p_t = \left(\int_0^1 p_{it}^{1-\varepsilon} di \right)^{\frac{1}{1-\varepsilon}}$$

3.2 Intermediate good producers

- Continuum of intermediate goods producers

- No entry ; no exit

- Each intermediate good producer i has a production function:

$$y_{it} = A_t k_{it}^{\alpha} l_{it}^{1-\alpha}$$

- A_t follows the AR(1) process:

$$\log A_t = \rho \log A_{t-1} + z_t$$

$$z_t \sim N(0, \sigma_z)$$

- Intermediate goods producers solve a two-stage problem

- First, given w_t and r_t, they rent l_{it} and k_{it} in perfectly competitive factor markets in order to minimize real cost:

$$\min_{l_{it}, k_{it}} \{w_t l_{it} + r_t k_{it}\}$$

$$st. \qquad y_{it} = A_t k_{it}^{\alpha} l_{it}^{1-\alpha}$$

- Solution

$$\mathcal{L} = (w_t l_{it} + r_t k_{it}) + \varrho \left(y_{it} - A_t k_{it}^{\alpha} l_{it}^{1-\alpha} \right)$$

$$\frac{\partial \mathcal{L}}{\partial l_{it}} = w_t - \varrho (1 - \alpha) A_t k_{it}^{\alpha} l_{it}^{-\alpha} = 0$$

$$\frac{\partial \mathcal{L}}{\partial k_{it}} = r_t - \varrho (\alpha) A_t k_{it}^{\alpha-1} l_{it}^{1-\alpha} = 0$$

$$w_t = \varrho (1 - \alpha) A_t k_{it}^{\alpha} l_{it}^{-\alpha}$$

$$r_t = \varrho \alpha A_t k_{it}^{\alpha-1} l_{it}^{1-\alpha}$$

$$k_{it} = \frac{\alpha}{1-\alpha}\frac{w_t}{r_t}l_{it}$$

$$\frac{k_{it}}{l_{it}} = \frac{\alpha}{1-\alpha}\frac{w_t}{r_t}$$

- Capital-labour ratio is the same for all firms

- The real cost of optimal production is:

$$w_t l_{it} + r_t k_{it} = w_t l_{it} + \frac{\alpha}{1-\alpha} w_t l_{it} = \frac{1}{1-\alpha} w_t l_{it}$$

- The real marginal cost mc_t is the optimal cost of producing one unit of good

$$
\begin{aligned}
A_t k_{it}^{\alpha} l_{it}^{1-\alpha} &= A_t \left(\frac{\alpha}{1-\alpha} \frac{w_t}{r_t} l_{it} \right)^{\alpha} l_{it}^{1-\alpha} \\
&= A_t \left(\frac{\alpha}{1-\alpha} \frac{w_t}{r_t} \right)^{\alpha} l_{it} = 1
\end{aligned}
$$

$$
l_{it} = \frac{1}{A_t} \left(\frac{\alpha}{1-\alpha} \frac{w_t}{r_t} \right)^{-\alpha}
$$

$$
\begin{aligned}
mc_t &= \frac{1}{1-\alpha} w_t l_{it} = \left(\frac{1}{1-\alpha}\right) w_t \frac{1}{A_t} \left(\frac{\alpha}{1-\alpha} \frac{w_t}{r_t}\right)^{-\alpha} \\
&= \left(\frac{1}{1-\alpha}\right)^{1-\alpha} \left(\frac{1}{\alpha}\right)^{\alpha} \frac{1}{A_t} w_t^{1-\alpha} r_t^{\alpha}
\end{aligned}
$$

- The marginal cost is the same for all firms
- The marginal cost above is in real term
- There is a more convenient way to derive real marginal cost, which is exactly ϱ.

$$
w_t = \varrho \left(1-\alpha\right) A_t k_{it}^{\alpha} l_{it}^{-\alpha}
$$

$$k_{it} = \frac{\alpha}{1-\alpha}\frac{w_t}{r_t} l_{it}$$

$$k_{it}^{\alpha} = \left(\frac{\alpha}{1-\alpha}\right)^{\alpha} \left(\frac{w_t}{r_t}\right)^{\alpha} l_{it}^{\alpha}$$

$$\frac{1}{k_{it}^{\alpha} l_{it}^{-\alpha}} = \left(\frac{\alpha}{1-\alpha}\right)^{-\alpha} \left(\frac{w_t}{r_t}\right)^{-\alpha}$$

$$\begin{aligned} \varrho &= \frac{1}{1-\alpha} w_t \frac{1}{A_t} \frac{1}{k_{it}^{\alpha} l_{it}^{-\alpha}} = \frac{1}{1-\alpha} w_t \frac{1}{A_t} \left(\frac{\alpha}{1-\alpha}\right)^{-\alpha} \left(\frac{w_t}{r_t}\right)^{-\alpha} \\ &= \left(\frac{1}{1-\alpha}\right)^{1-\alpha} \left(\frac{1}{\alpha}\right)^{\alpha} \frac{1}{A_t} w_t^{1-\alpha} r_t^{\alpha} \end{aligned}$$

- Remind

$$k_t = \frac{\alpha}{1-\alpha}\frac{w_t}{r_t}l_t$$

- The second part of the firm's problem is to choose price that maximizes discounted real profits

$$\max_{p_{it}} \left\{ \left(\frac{p_{it}}{p_t} - mc_t \right) y^*_{it} \right\}$$

$$st. \qquad y^*_{it+\tau} = \left(\frac{p_{it}}{p_t} \right)^{-\varepsilon} y_{t+\tau},$$

- First order condition:

$$\frac{1}{p_t}\left(\frac{p_{it}}{p_t}\right)^{-\varepsilon} y_{t+\tau} - \varepsilon\left(\frac{p_{it}}{p_t} - mc_t\right)\left(\frac{p_{it}}{p_t}\right)^{-\varepsilon-1}\frac{1}{p_t} y_{t+\tau} = 0$$

$$1 - \varepsilon\left(\frac{p_{it}}{p_t} - mc_t\right)\left(\frac{p_{it}}{p_t}\right)^{-1} = 0 \Rightarrow$$

$$p_{it} = \varepsilon\left(p_{it} - mc_t \cdot p_t\right) \Rightarrow$$

$$p_{it} = \frac{\varepsilon}{\varepsilon - 1} \cdot mc_t \cdot p_t$$

- Mark-up condition

$$\frac{p_{it}}{p_t} = \frac{\varepsilon}{\varepsilon - 1} \cdot mc_t$$

$$\frac{p_{it}}{p_t} > mc_t$$

- mc_t is real value, so that $p_t \cdot mc_t$ is nominal marginal cost

- So far the presence of monopolistic competition is pretty irrelevant.

- Because what we have is a constant mark-up similar to a tax.

- A solution is to introduce price rigidities.

3.3 Price rigidities

- The basic structural of the economy is as before: a representative household, a monetary authorities, and a perfectly competitive final good producer.

- Now we introduce one additional constraint: the intermediate good producers face the constraint that they can only change prices following a Calvo-type rule.

- Now the second part of the problem of the intermediate good producers is to choose price that maximizes discounted real profits:

$$\max_{p_{it}} E_t \sum_{\tau=0}^{\infty} (\theta_p)^{\tau} v_{t+\tau} \left\{ \left(\frac{p_{it}}{p_{t+\tau}} - mc_{t+\tau} \right) y^*_{it+\tau} \right\}$$

$$st. \qquad y^*_{it+\tau} = (\frac{p_{it}}{p_{t+\tau}})^{-\varepsilon} y_{t+\tau}$$

- Calvo price setting
- Degree of price rigidity θ_p
- v_t is the marginal value of a dollar to the household, which is treated as exogenous by the firm.

- There are complete markets in securities, so that the marginal value is constant across households.

- Again the problem

$$y^*_{it+\tau} = \left(\frac{p_{it}}{p_{t+\tau}}\right)^{-\varepsilon} y_{t+\tau}$$

$$\begin{aligned} & \max_{p_{it}} E_t \sum_{\tau=0}^{\infty} (\theta_p)^\tau v_{t+\tau} \left\{ \left(\frac{p_{it}}{p_{t+\tau}} - mc_{t+\tau} \right) y^*_{it+\tau} \right\} \\ = \quad & \max_{p_{it}} E_t \sum_{\tau=0}^{\infty} (\theta_p)^\tau v_{t+\tau} \left\{ \left((\frac{p_{it}}{p_{t+\tau}})^{1-\varepsilon} - (\frac{p_{it}}{p_{t+\tau}})^{-\varepsilon} \cdot mc_{t+\tau} \right) y_{it+\tau} \right\} \end{aligned}$$

- First order condition

- p_{it}^*

$$E_t \sum_{\tau=0}^{\infty} (\theta_p)^\tau v_{t+\tau} \left\{ \left((1-\varepsilon)(\frac{p_{it}^*}{p_{t+\tau}})^{1-\varepsilon} p_{it}^{*-1} + \varepsilon(\frac{p_{it}^*}{p_{t+\tau}})^{-\varepsilon} p_{it}^{*-1} \cdot mc_{t+\tau} \right) y_{it+\tau} \right\} = 0$$

$$E_t \sum_{\tau=0}^{\infty} (\theta_p)^\tau v_{t+\tau} \left\{ \left((\varepsilon-1)\frac{p_{it}^*}{p_{t+\tau}} p_{it}^{*-1} - \varepsilon p_{it}^{*-1} \cdot mc_{t+\tau} \right) y_{it+\tau}^* \right\} = 0$$

$$E_t \sum_{\tau=0}^{\infty} (\theta_p)^\tau v_{t+\tau} \left\{ \left(\frac{p_{it}^*}{p_{t+\tau}} - \frac{\varepsilon}{\varepsilon-1} \cdot mc_{t+\tau} \right) y_{it+\tau}^* \right\} = 0$$

- Firms choose p_{it}^* so as to set a weighted average of the difference between marginal revenue and marginal cost to zero.

- All firms that re-optimize will charge the same price.

- If $\theta_p = 0$

$$v_t \left\{ \left(\frac{p_{it}^*}{p_t} - \frac{\varepsilon}{\varepsilon - 1} \cdot mc_t \right) y_{it} \right\} = 0$$

$$p_{it}^* = \frac{\varepsilon}{\varepsilon - 1} mc_t \cdot p_t$$

- At this point, much of the literature using the Calvo-Yun price setting proceeds to linearize the above equation around a deterministic steady state with zero inflation.

- This strategy yields the famous linear New Keynesian Phillips curve involving inflation and marginal costs of the form

$$\hat{\pi}_t = \beta E_t \hat{\pi}_{t+1} + \frac{(1-\alpha)(1-\alpha\beta)}{\alpha} \widehat{mc}_t$$

- Galí, Jordi and Tammaso Monacelli (2005), "Monetary Policy and Exchange Rate Volatility in a Small Open Economy," *Review of Economic Studies*, 72, pp. 707-734.

- Monacelli, Tommaso (2005), "Monetary Policy in a Low Pass-Through Environment," *Journal of Money, Crdit, and Banking*, 37 (6), pp. 1047-1066.

- We do not follow this strategy because we do not want to restrict attention to the case of log-linear dynamics around a zero inflation steady state.

- For this purpose, we have to define two more variables.

$$E_t \sum_{\tau=0}^{\infty} (\theta_p)^\tau v_{t+\tau} \left\{ \left(\frac{p_{it}^*}{p_{t+\tau}} - \frac{\varepsilon}{\varepsilon - 1} \cdot mc_{t+\tau} \right) y_{it+\tau}^* \right\} = 0$$

$$E_t \sum_{\tau=0}^{\infty} (\theta_p)^\tau v_{t+\tau} \left\{ \left((\frac{p_{it}^*}{p_{t+\tau}})^{1-\varepsilon} - \frac{\varepsilon}{\varepsilon - 1} (\frac{p_{it}^*}{p_{t+\tau}})^{-\varepsilon} \cdot mc_{t+\tau} \right) y_{t+\tau} \right\} = 0$$

- Rearrange

$$E_t \sum_{\tau=0}^{\infty} (\theta_p)^\tau v_{t+\tau} \left(\frac{p_{it}^*}{p_{t+\tau}} \right)^{1-\varepsilon} y_{t+\tau}$$

$$= E_t \sum_{\tau=0}^{\infty} (\theta_p)^\tau v_{t+\tau} \frac{\varepsilon}{\varepsilon - 1} mc_{t+\tau} \left(\frac{p_{it}^*}{p_{t+\tau}} \right)^{-\varepsilon} y_{t+\tau}$$

$$x_t^1 \equiv E_t \sum_{\tau=0}^{\infty} (\theta_p)^\tau v_{t+\tau} \left(\frac{p_{it}^*}{p_{t+\tau}} \right)^{1-\varepsilon} y_{t+\tau}$$

$$x_t^2 \equiv E_t \sum_{\tau=0}^{\infty} (\theta_p)^\tau v_{t+\tau} \frac{\varepsilon}{\varepsilon - 1} mc_{t+\tau} \left(\frac{p_{it}^*}{p_{t+\tau}} \right)^{-\varepsilon} y_{t+\tau}$$

- auxiliary equation 1

$$x_t^1 = x_t^2$$

$$\begin{aligned}
x_t^1 &\equiv E_t \sum_{\tau=0}^{\infty} (\theta_p)^\tau \, v_{t+\tau} \left(\frac{p_{it}^*}{p_{t+\tau}} \right)^{1-\varepsilon} y_{t+\tau} \\
&= (\theta_p)^0 \, v_t \left(\frac{p_{it}^*}{p_t} \right)^{1-\varepsilon} y_t + E_t \sum_{\tau=1}^{\infty} (\theta_p)^\tau \, v_{t+\tau} \left(\frac{p_{it}^*}{p_{t+\tau}} \right)^{1-\varepsilon} y_{t+\tau} \\
&= \left(\frac{p_{it}^*}{p_t} \right)^{1-\varepsilon} y_t + E_t \sum_{\tau=1}^{\infty} (\theta_p)^\tau \, v_{t+\tau} \left(\frac{p_{it}^*}{p_{t+\tau}} \right)^{1-\varepsilon} y_{t+\tau}
\end{aligned}$$

$$x_{t+1}^1 \equiv E_{t+1} \sum_{\tau=0}^{\infty} (\theta_p)^\tau \, v_{t+1+\tau} \left(\frac{p_{it+1}^*}{p_{t+1+\tau}} \right)^{1-\varepsilon} y_{t+1+\tau}$$

$$\tau - 1 = \tau^*$$

$$\tau = \tau^* + 1$$

$$\tau = 1 \Leftrightarrow \tau^* = 0$$

$$
\begin{aligned}
& E_t \sum_{\tau=1}^{\infty} (\theta_p)^{\tau} v_{t+\tau} \left(\frac{p_{it}^*}{p_{t+\tau}}\right)^{1-\varepsilon} y_{t+\tau} \\
= \; & \theta_p E_t \sum_{\tau=1}^{\infty} (\theta_p)^{\tau-1} v_{t+\tau} \left(\frac{p_{it}^*}{p_{t+\tau}}\right)^{1-\varepsilon} y_{t+\tau} \\
= \; & \theta_p E_t \sum_{\tau^*=0}^{\infty} (\theta_p)^{\tau^*} v_{t+\tau^*+1} \left(\frac{p_{it}^*}{p_{t+\tau^*+1}}\right)^{1-\varepsilon} y_{t+\tau^*+1} \\
= \; & \theta_p E_t \left(\frac{p_{it}^*}{p_{it+1}^*}\right)^{1-\varepsilon} \sum_{\tau^*=0}^{\infty} (\theta_p)^{\tau^*} v_{t+\tau^*+1} \left(\frac{p_{it+1}^*}{p_{t+\tau^*+1}}\right)^{1-\varepsilon} y_{t+\tau^*+1} \\
= \; & \theta_p E_t \left(\frac{p_{it}^*}{p_{it+1}^*}\right)^{1-\varepsilon} v_{t+1} \left\{ E_{t+1} \sum_{\tau^*=0}^{\infty} (\theta_p)^{\tau^*} v_{t+\tau^*+1} \left(\frac{p_{it+1}^*}{p_{t+\tau^*+1}}\right)^{1-\varepsilon} y_{t+\tau^*+1} \right\} \\
= \; & \theta_p E_t \left(\frac{p_{it}^*}{p_{it+1}^*}\right)^{1-\varepsilon} v_{t+1} x_{t+1}^1
\end{aligned}
$$

- auxiliary equation 2

$$x_t^1 = \left(\frac{p_{it}^*}{p_t}\right)^{1-\varepsilon} y_t + \theta_p E_t \left(\frac{p_{it}^*}{p_{it+1}^*}\right)^{1-\varepsilon} v_{t+1} x_{t+1}^1$$

- Now for x_t^2

$$
\begin{aligned}
x_t^2 &\equiv E_t \sum_{\tau=0}^{\infty} (\theta_p)^\tau v_{t+\tau} \frac{\varepsilon}{\varepsilon-1} mc_{t+\tau} \left(\frac{p_{it}^*}{p_{t+\tau}} \right)^{-\varepsilon} y_{t+\tau} \\
&= (\theta_p)^0 v_t \frac{\varepsilon}{\varepsilon-1} mc_t \left(\frac{p_{it}^*}{p_t} \right)^{-\varepsilon} y_t + E_t \sum_{\tau=1}^{\infty} (\theta_p)^\tau v_{t+\tau} \frac{\varepsilon}{\varepsilon-1} mc_{t+\tau} \left(\frac{p_{it}^*}{p_{t+\tau}} \right)^{-\varepsilon} y_{t+\tau} \\
&= \frac{\varepsilon}{\varepsilon-1} mc_t \left(\frac{p_{it}^*}{p_t} \right)^{-\varepsilon} y_t + E_t \sum_{\tau=1}^{\infty} (\theta_p)^\tau v_{t+\tau} \frac{\varepsilon}{\varepsilon-1} mc_{t+\tau} \left(\frac{p_{it}^*}{p_{t+\tau}} \right)^{-\varepsilon} y_{t+\tau}
\end{aligned}
$$

$$
x_{t+1}^2 \equiv E_{t+1} \sum_{\tau=0}^{\infty} (\theta_p)^\tau v_{t+1+\tau} \frac{\varepsilon}{\varepsilon-1} mc_{t+1+\tau} \left(\frac{p_{it+1}^*}{p_{t+1+\tau}} \right)^{-\varepsilon} y_{t+1+\tau}
$$

$$
\begin{aligned}
& E_t \sum_{\tau=1}^{\infty} (\theta_p)^{\tau} v_{t+\tau} \frac{\varepsilon}{\varepsilon-1} mc_{t+\tau} \left(\frac{p_{it}^*}{p_{t+\tau}}\right)^{-\varepsilon} y_{t+\tau} \\
= \;& \theta_p E_t \sum_{\tau=1}^{\infty} (\theta_p)^{\tau-1} v_{t+\tau} \frac{\varepsilon}{\varepsilon-1} mc_{t+\tau} \left(\frac{p_{it}^*}{p_{t+\tau}}\right)^{-\varepsilon} y_{t+\tau} \\
= \;& \theta_p E_t \sum_{\tau=0}^{\infty} (\theta_p)^{\tau} v_{t+1+\tau} \frac{\varepsilon}{\varepsilon-1} mc_{t+1+\tau} \left(\frac{p_{it}^*}{p_{t+1+\tau}}\right)^{-\varepsilon} y_{t+1+\tau} \\
= \;& \theta_p E_t \left(\frac{p_{it}^*}{p_{it+1}^*}\right)^{-\varepsilon} \sum_{\tau=0}^{\infty} (\theta_p)^{\tau} v_{t+1+\tau} \frac{\varepsilon}{\varepsilon-1} mc_{t+1+\tau} \left(\frac{p_{it+1}^*}{p_{t+1+\tau}}\right)^{-\varepsilon} y_{t+1+\tau} \\
= \;& \theta_p E_t \left(\frac{p_{it}^*}{p_{it+1}^*}\right)^{-\varepsilon} v_{t+1} E_{t+1} \left\{ \sum_{\tau=0}^{\infty} (\theta_p)^{\tau} v_{t+1+\tau} \frac{\varepsilon}{\varepsilon-1} mc_{t+1+\tau} \left(\frac{p_{it+1}^*}{p_{t+1+\tau}}\right)^{-\varepsilon} y_{t+1+\tau} \right\} \\
= \;& \theta_p E_t \left(\frac{p_{it}^*}{p_{it+1}^*}\right)^{-\varepsilon} v_{t+1} x_{t+1}^2
\end{aligned}
$$

- auxiliary equation 3

$$x_t^2 = \frac{\varepsilon}{\varepsilon - 1} mc_t \left(\frac{p_{it}^*}{p_t}\right)^{-\varepsilon} y_t + \theta_p E_t \left(\frac{p_{it}^*}{p_{it+1}^*}\right)^{-\varepsilon} v_{t+1} x_{t+1}^2$$

- Price level

- The price index evolves according to

$$p_t = \left[\theta_p p_{t-1}^{1-\varepsilon} + (1 - \theta_p) p_t^{*1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}$$

- Deduction

$$\begin{aligned} p_t^{1-\varepsilon} &= \int_0^1 p_{it}^{1-\varepsilon} di \\ &= \int_0^{\theta_p} p_{it-1}^{1-\varepsilon} di + \int_{\theta_p}^1 p_{it}^{*1-\varepsilon} di \\ &= \theta_p p_{t-1}^{1-\varepsilon} + (1-\theta_p)\, p_t^{*1-\varepsilon} \end{aligned}$$

$$p_t = \left[\theta_p p_{t-1}^{1-\varepsilon} + (1-\theta_p) p_t^{*1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}$$

$$\pi_t = \frac{p_t}{p_{t-1}} = \left[\theta_p + (1-\theta_p)\pi_t^{*1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}$$

$$\pi_t^* = \frac{p_t^*}{p_{t-1}}$$

4 Aggregation

- To derive an expression for aggregate output

- Remember

$$\frac{k_{it}}{l_{it}} = \frac{\alpha}{1-\alpha}\frac{w_t}{r_t}$$

- It follows

$$\frac{k_t}{l_t} = \frac{\alpha}{1-\alpha}\frac{w_t}{r_t}$$

$$\frac{k_{it}}{l_{it}} = \frac{k_t}{l_t}$$

- Production function of intermediate good firm

$$y_{it} = A_t k_{it}^{\alpha} l_{it}^{1-\alpha}$$

$$y_{it} = A_t(\frac{k_{it}}{l_{it}})^\alpha l_{it} = A_t(\frac{k_t}{l_t})^\alpha l_{it}$$

- The demand function for the firm is

$$y_{it} = (\frac{p_{it}}{p_t})^{-\varepsilon} y_t, \qquad \forall i$$

$$(\frac{p_{it}}{p_t})^{-\varepsilon} y_t = A_t(\frac{k_t}{l_t})^\alpha l_{it}$$

$$y_t \int_0^1 (\frac{p_{it}}{p_t})^{-\varepsilon} di = A_t(\frac{k_t}{l_t})^\alpha \int_0^1 l_{it} di = A_t k_t^\alpha l_t^{1-\alpha}$$

$$y_t = \frac{A_t}{s_t} k_t^{\alpha} l_t^{1-\alpha}$$

$$s_t = \int_0^1 (\frac{p_{it}}{p_t})^{-\varepsilon} di = \frac{j_t^{-\varepsilon}}{p_t^{-\varepsilon}} = j_t^{-\varepsilon} p_t^{\varepsilon}$$

$$j_t = (\int_0^1 p_{it}^{-\varepsilon} di)^{\frac{-1}{\varepsilon}}$$

- j_t?
- Measure of price dispersion

- An inefficient variable

- $p_{it} = p_{jt} = p_t^*$

$$s_t = \int_0^1 \left(\frac{p_{it}}{p_t}\right)^{-\varepsilon} di = (1-\theta_p)\left(\frac{p_t^*}{p_t}\right)^{-\varepsilon} + \int_0^{\theta_p} \left(\frac{p_{it-1}}{p_t}\right)^{-\varepsilon} di$$

$$s_{t-1} = \int_0^1 \left(\frac{p_{it-1}}{p_{t-1}}\right)^{-\varepsilon} di$$

$$\int_0^{\theta_p} \left(\frac{p_{it-1}}{p_t}\right)^{-\varepsilon} di = \left(\frac{p_{t-1}}{p_t}\right)^{-\varepsilon} \int_0^{\theta_p} \left(\frac{p_{it-1}}{p_{t-1}}\right)^{-\varepsilon} di = \left(\frac{p_{t-1}}{p_t}\right)^{-\varepsilon} \theta_p s_{t-1}$$

$$
\begin{aligned}
s_t &= (1-\theta_p)\left(\frac{p_t^*}{p_t}\right)^{-\varepsilon} + \left(\frac{p_{t-1}}{p_t}\right)^{-\varepsilon}\theta_p s_{t-1} \\
&= (1-\theta_p)\left(\frac{p_t^*}{p_t}\right)^{-\varepsilon} + \theta_p \pi_t^{\varepsilon} s_{t-1}
\end{aligned}
$$

5 Discount factor

- $v_{t+\tau}$ is the pricing kernel used to value random date $t+\tau$ payoffs.

- Since firms are assumed to be owned by the representative household, it is assumed that firms value future payoffs according to the household's inter-temporal marginal rate of substitution in consumption (Bergin, Paul R. (2003), "Putting the 'New

Open Economy Macroeconomics' to a test," *Journal of International Economics*, 60 (1), pp. 3-34).

$$v_{t+\tau} = \beta^{\tau} \frac{U'_{c,t+\tau}}{U'_{c,t}} = \beta^{\tau} \frac{\lambda_{t+\tau}}{\lambda_t}$$

- Remarks

$$v_t = \beta^0 \frac{U'_{c,t}}{U'_{c,t}} = 1$$

$$
\begin{aligned}
v_{t+1} &= \beta\frac{\lambda_{t+1}}{\lambda_t} = \frac{1}{\frac{R_{t+1}}{\pi_{t+1}}} \\
&= \text{inverse of gross real interest rate}
\end{aligned}
$$

6 Summarizing the equilibrium conditions

- Endogenous variables (14)

- $c_t, r_t, R_t, \pi_t, l_t, w_t, mc_t, x_t^1, x_t^2, \tilde{p}_t, y_t, k_t, s_t, x_t$

- Exogenous variables (1)

- A_t

- We have a total of 15 equilibrium conditions.

- We can include more variables of interest such as m_t and v_{t+1}.

- The first order conditions of the household

$$c_t^{-\sigma} = \beta E_t \left\{ c_{t+1}^{-\sigma} (r_{t+1} + 1 - \delta) \right\} \tag{1}$$

$$c_t^{-\sigma} = \beta E_t \left\{ c_{t+1}^{-\sigma} \frac{R_{t+1}}{\pi_{t+1}} \right\} \tag{2}$$

$$\psi l_t^{\gamma} = c_t^{-\sigma} w_t \tag{3}$$

$$\pi_t = \frac{p_t}{p_{t-1}}$$

- Pricing decisions by firms

$$mc_t = \left(\frac{1}{1-\alpha}\right)^{1-\alpha} \left(\frac{1}{\alpha}\right)^{\alpha} \frac{1}{A_t} w_t^{1-\alpha} r_t^{\alpha} \tag{4}$$

$$x_t^1 = x_t^2 \tag{5}$$

$$\tilde{p}_t \equiv \frac{p_t^*}{p_t}$$

$$x_t^1 = \left(\frac{p_t^*}{p_t}\right)^{1-\varepsilon} y_t + \theta_p E_t \left(\frac{p_t^*}{p_{t+1}^*}\right)^{1-\varepsilon} \frac{\pi_{t+1}}{R_{t+1}} x_{t+1}^1$$

$$x_t^1 = (\tilde{p}_t)^{1-\varepsilon} y_t + \theta_p E_t \left(\frac{\tilde{p}_t}{\tilde{p}_{t+1}}\right)^{1-\varepsilon} \frac{\pi_{t+1}^{\varepsilon}}{R_{t+1}} x_{t+1}^1 \tag{6}$$

$$x_t^2 = \frac{\varepsilon}{\varepsilon - 1} mc_t (\tilde{p}_t)^{-\varepsilon} y_t + \theta_p E_t \left(\frac{\tilde{p}_t}{\tilde{p}_{t+1}}\right)^{-\varepsilon} \frac{\pi_{t+1}^{1+\varepsilon}}{R_{t+1}} x_{t+1}^2 \tag{7}$$

- Production function

$$y_t = \frac{A_t}{s_t} k_t^{\alpha} l_t^{1-\alpha} \tag{8}$$

$$s_t = (1 - \theta_p) \left(\frac{p_t^*}{p_t} \right)^{-\varepsilon} + \theta_p \left(\frac{p_{t-1}}{p_t} \right)^{-\varepsilon} s_{t-1}$$

$$s_t = (1 - \theta_p) (\tilde{p}_t)^{-\varepsilon} + \theta_p \pi_t^{\varepsilon} s_{t-1} \tag{9}$$

- Aggregate conditions

$$c_t + x_t = y_t \tag{10}$$

$$k_{t+1} = (1 - \delta)\, k_t + x_t \tag{11}$$

$$k_t = \frac{\alpha}{1 - \alpha} \frac{w_t}{r_t} l_t \tag{12}$$

$$p_t^{1-\varepsilon} = \theta_p p_{t-1}^{1-\varepsilon} + (1 - \theta_p) p_t^{*1-\varepsilon}$$

$$p_t = \left[\theta_p p_{t-1}^{1-\varepsilon} + (1-\theta_p) p_t^{*1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}$$

$$1 = \theta_p \pi_t^{-1+\varepsilon} + (1-\theta_p)\,\tilde{p}_t^{1-\varepsilon} \tag{13}$$

$$\pi_t = \frac{p_t}{p_{t-1}} = \left[\theta_p + (1-\theta_p)\left(\frac{p_t^*}{p_{t-1}}\right)^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}$$

- Taylor rule

$$\frac{R_{t+1}}{R} = \left(\frac{R_t}{R}\right)^{\gamma_R} \left(\frac{\pi_t}{\pi}\right)^{\gamma_\pi} \left(\frac{y_t}{y}\right)^{\gamma_y} e^{\varphi_t} \tag{14}$$

$$\varphi_t \sim \mathcal{N}\left(0, \sigma_\varphi\right)$$

- Technological shocks

$$\log\left(A_t\right) = \rho \log\left(A_{t-1}\right) + z_t \tag{15}$$

$$z_t \sim \mathcal{N}\left(0, \sigma_z\right)$$