Derivation and Log-linearization of Chari, Kehoe, and McGrattan (2007)'s Closed Economy Model

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*Chari, V. V., Patrick J. Kehoe, and Ellen R. McGrattan (2007), "Business cycle accounting," *Econometrica*, 75 (3), pp. 781-836.

1 The benchmark prototype economy

- The benchmark prototype economy that we employ in our accounting procedure is a stochastic model.

- The economy has four exogenous stochastic variables: the efficiency wedge z_t, the labor wedge $1 - \tau_{lt}$, the investment wedge $\frac{1}{1+\tau_{xt}}$, and the government consumption wedge $g_{t.}$

- c_t : per capita consumption

- l_t : per capita labor

- k_t : per capita capital stock
- x_t : per capita investment
- w_t : wage rate
- r_t : rental rate on capital
- N_t : population
- T_t : per capita lump-sum transfers

- y_t : per capita output
- Z_t : productivity $\left(Z_t = z_t (1+\gamma)^t\right)$
- β : discount factor
- δ : depreciation rate of capital
- $1+\gamma$: the rate of labor-augmenting technical process
- $1+\gamma_n$: the population growth rate

- The representative consumer maximize expected utility subject to the budget constraint and the capital accumulation law:

$$E_t \sum_{t=0}^{\infty} \beta^t U\left(c_t, l_t\right) N_t$$

$$c_t + \left(1 + \tau_{xt}\right) x_t = \left(1 - \tau_{lt}\right) w_t l_t + r_t k_t + T_t$$

$$\left(1 + \gamma_n\right) k_{t+1} = \left(1 - \delta\right) k_t + x_t$$

- Solution of consumers' maximization problem:

$$\max_{\{c_t, l_t, k_{t+1}\}} E_t \sum_{t=0}^{\infty} \beta^t U(c_t, l_t) N_t$$

$$c_t + (1+\tau_{xt}) \left[(1+\gamma_n) k_{t+1} - (1-\delta) k_t\right] = (1-\tau_{lt}) w_t l_t + r_t k_t + T_t$$

- The Lagrangian:

$$
\begin{aligned}
\mathcal{L} \;&=\; \sum_{t=0}^{\infty} \beta^t U\left(c_t, l_t\right) N_t + \\
&\sum_{t=0}^{\infty} \lambda_t \beta^t \{ w_t l_t \left(1 - \tau_{lt}\right) + r_t k_t + T_t - c_t - \left(1 + \tau_{xt}\right) \left[\left(1 + \gamma_n\right) k_{t+1} - \left(1 - \delta\right) k_t\right] \}
\end{aligned}
$$

$$
\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t U_{ct} N_t - \beta^t \lambda_t = 0
$$

$$
\frac{\partial \mathcal{L}}{\partial l_t} = \beta^t U_{lt} N_t + \beta^t \lambda_t \left(1 - \tau_{lt}\right) w_t = 0
$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = -\beta^t \lambda_t \left(1 + \tau_{xt}\right)\left(1 + \gamma_n\right) + E_t \beta^{t+1} \lambda_{t+1} \left[r_{t+1} + \left(1 + \tau_{xt+1}\right)\left(1 - \delta\right)\right] = 0$$

$$U_{ct} N_t = \lambda_t$$

$$\frac{U_{lt}}{U_{ct}} = -\left(1 - \tau_{lt}\right) w_t \tag{A1}$$

$$\lambda_t \left(1 + \tau_{xt}\right)\left(1 + \gamma_n\right) = \beta E_t \lambda_{t+1} \left[r_{t+1} + \left(1 - \delta\right)\left(1 + \tau_{xt+1}\right)\right]$$

$$U_{ct}N_t\left(1+\tau_{xt}\right)\left(1+\gamma_n\right)=\beta E_tU_{ct+1}N_{t+1}\left[r_{t+1}+\left(1-\delta\right)\left(1+\tau_{xt+1}\right)\right]$$

$$U_{ct}\left(1+\tau_{xt}\right)=\beta E_tU_{ct+1}\left\{r_{t+1}+\left(1-\delta\right)\left(1+\tau_{xt+1}\right)\right\} \tag{A2}$$

- The production function is:

$$F\left(k_t, Z_tl_t\right)$$

- The representative firm maximizes profits:

$$F(k_t, Z_t l_t) - w_t l_t - r_t k_t$$

- Solution of firms' maximization problem

$$\max_{\{k_t, l_t\}} \Pi_t \equiv F(k_t, Z_t l_t) - w_t l_t - r_t k_t$$

$$\frac{\partial \Pi_t}{\partial k_t} = F_{kt}(k_t, Z_t l_t) - r_t = 0$$

$$\frac{\partial \Pi_t}{\partial l_t} = Z_t F_{lt}(k_t, Z_t l_t) - w_t = 0$$

$$F_{kt}(k_t, Z_t l_t) = r_t \tag{A3}$$

$$Z_t F_{lt}(k_t, Z_t l_t) = w_t \tag{A4}$$

- Substitute (A4) into (A1) and rearrange

$$\frac{U_{lt}}{U_{ct}} = -\left(1 - \tau_{lt}\right) w_t = -\left(1 - \tau_{lt}\right) Z_t F_{lt}\left(k_t, Z_t l_t\right)$$

$$-\frac{U_{lt}}{U_{ct}} = \left(1 - \tau_{lt}\right) Z_t F_{lt}\left(k_t, Z_t l_t\right)$$

- Substitute (A3) into (A2) and rearrange

$$U_{ct}\left(1+\tau_{xt}\right)=\beta E_t U_{ct+1}\left[r_{t+1}+\left(1-\delta\right)\left(1+\tau_{xt+1}\right)\right]$$

$$U_{ct}\left(1+\tau_{xt}\right)=\beta E_t U_{ct+1}\left[F_{kt+1}\left(k_{t+1},Z_{t+1}l_{t+1}\right)+\left(1-\delta\right)\left(1+\tau_{xt+1}\right)\right]$$

- The government maintains a balanced budget every period:

$$g_t+T_t=\tau_{lt}w_t l_t+\tau_{xt}x_t$$

- Substituting the government budget constraint and profit function into consumer budget constraint to get the resource constraint:

$$c_t + (1 + \tau_{xt})\, x_t = w_t l_t \,(1 - \tau_{lt}) + r_t k_t + \tau_{lt} w_t l_t + \tau_{xt} x_t - g_t$$

$$c_t + x_t + \tau_{xt} x_t = w_t l_t - \tau_{lt} w_t l_t + r_t k_t + \tau_{lt} w_t l_t + \tau_{xt} x_t - g_t$$

$$c_t + x_t + g_t = w_t l_t + r_t k_t$$

$$c_t + x_t + g_t = y_t$$

- The equilibrium conditions of the benchmark prototype economy are:

$$c_t + x_t + g_t = y_t \tag{1}$$

$$y_t = F\left(k_t, Z_t l_t\right) \tag{2}$$

$$-\frac{U_{lt}}{U_{ct}} = (1 - \tau_{lt}) Z_t F_{lt} (k_t, Z_t l_t) \tag{3}$$

$$U_{ct} (1 + \tau_{xt}) = \beta E_t U_{ct+1} \left[F_{kt+1} (k_{t+1}, Z_{t+1} l_{t+1}) + (1 - \delta)(1 + \tau_{xt+1}) \right] \tag{4}$$

- The method assumes that frictions associated with specific economic environment manifest themselves as distortions in the first-order condition and resource constraints in the prototype model.

- These distortions are called wedges.

- The economy with four wedges exactly reproduces the data on output, labor, investment, and government consumption.

2 In explicit functional form

- We substitute the utility function and production function into the equilibrium conditions.

- The equilibrium conditions of the prototype economy are:

$$c_t + x_t + g_t = y_t \tag{1}$$

$$y_t = F\left(k_t, Z_t l_t\right) \tag{2}$$

$$-\frac{U_{lt}}{U_{ct}} = \left(1 - \tau_{lt}\right) Z_t F_{lt}\left(k_t, Z_t l_t\right) \tag{3}$$

$$U_{ct}\left(1 + \tau_{xt}\right) = \beta E_t U_{ct+1}\left[F_{kt+1}\left(k_{t+1}, z_{t+1} l_{t+1}\right) + \left(1 - \delta\right)\left(1 + \tau_{xt+1}\right)\right] \tag{4}$$

- Production function and utility function are:

$$F(k, zl) = k^{\alpha} (zl)^{1-\alpha}$$

$$u(c, l) = \log c + \psi \log(1 - l)$$

- Equilibrium conditions in explicit form:

$$c_t + x_t + g_t = y_t \qquad (1.a)$$

$$y_t = k_t^{\alpha} (Z_t l_t)^{1-\alpha} \tag{2.a}$$

$$\left(\frac{\psi c_t}{1 - l_t}\right) = (1 - \tau_{lt}) Z_t (1 - \alpha) k_t^{\alpha} (Z_t l_t)^{-\alpha} \tag{3.a}$$

$$(1 + \tau_{xt}) \frac{1}{c_t} = \beta E_t \frac{1}{c_{t+1}} \left[\alpha (k_{t+1})^{\alpha-1} (Z_{t+1} l_{t+1})^{1-\alpha} + (1 - \delta)(1 + \tau_{xt+1})\right] \tag{4.a}$$

- $\widetilde{v}_t = \frac{v_t}{(1+\gamma)^t}$

- We assume that g_t fluctuates around a trend of $(1+\gamma)^t$
- Equilibrium conditions in detrended per-capita form:

$$\tilde{c}_t + \tilde{x}_t + \tilde{g}_t = \tilde{y}_t \tag{1.b}$$

$$\tilde{y}_t = \tilde{k}_t^{\alpha} \left(\tilde{Z}_t l_t\right)^{1-\alpha} \tag{2.b}$$

$$\left(\frac{\psi \widetilde{c}_t}{1-l_t}\right) = (1-\tau_{lt}) \widetilde{Z}_t (1-\alpha) \widetilde{k}_t^{\alpha} \left(\widetilde{Z}_t l_t\right)^{-\alpha} \tag{3.b}$$

$$(1+\tau_{xt})(1+\gamma)\frac{1}{\tilde{c}_t} = \beta E_t \frac{1}{\tilde{c}_{t+1}} \left[\alpha \left(\widetilde{k}_{t+1} \right)^{\alpha-1} \left(\widetilde{Z}_{t+1} l_{t+1} \right)^{1-\alpha} + (1-\delta)(1+\tau_{xt+1}) \right] \tag{4.b}$$

- Endogenous variables are: $\widetilde{c}_t$, $\widetilde{x}_t$, $\widetilde{y}_t$, $\widetilde{k}_t$, l_t.

- We need the capital accumulation law to close the model.

$$(1+\gamma_n) k_{t+1} = (1-\delta) k_t + x_t \tag{5}$$

$$(1+\gamma_n)(1+\gamma)\tilde{k}_{t+1} = (1-\delta)\tilde{k}_t + \tilde{x}_t \tag{5.b}$$

3 Steady state of the prototype model

- Variables without time index are steady state values.

$$(1+\tau_x)(1+\gamma) = \beta\left[\alpha\left(\widetilde{k}\right)^{\alpha-1}\left(\widetilde{Z}l\right)^{1-\alpha} + (1-\delta)(1+\tau_x)\right]$$

$$\frac{(1+\tau_x)(1+\gamma)}{\beta} - (1-\delta)(1+\tau_x) = \alpha\left(\widetilde{k}\right)^{\alpha-1}\left(\widetilde{Z}l\right)^{1-\alpha}$$

$$\frac{1}{\alpha\widetilde{Z}^{1-\alpha}}\left[\frac{(1+\tau_x)(1+\gamma)}{\beta} - (1-\delta)(1+\tau_x)\right] = \left(\frac{\widetilde{k}}{l}\right)^{\alpha-1}$$

$$\frac{\widetilde{k}}{l} = \left\{\frac{1}{\alpha\widetilde{Z}^{1-\alpha}}\left[\frac{(1+\tau_x)(1+\gamma)}{\beta} - (1-\delta)(1+\tau_x)\right]\right\}^{\frac{1}{\alpha-1}} \qquad \text{(ss1)}$$

- We obtain the steady-state of $\frac{\widetilde{k}}{l}$.

$$(1+\gamma_n)(1+\gamma)\tilde{k} = (1-\delta)\tilde{k} + \tilde{x}$$

$$\left[(1+\gamma_n)(1+\gamma) - (1-\delta)\right]\tilde{k} = \tilde{x}$$

$$\tilde{y} = \tilde{k}^{\alpha}\left(\tilde{Z}l\right)^{1-\alpha} = \tilde{Z}^{1-\alpha}\left(\frac{\widetilde{k}}{l}\right)^{\alpha-1}\widetilde{k} = \tilde{Z}^{1-\alpha}\left(\frac{\widetilde{k}}{l}\right)^{\alpha} l$$

$$\tilde{c} + \tilde{x} + \tilde{g} = \tilde{y}$$

$$\tilde{c} = \tilde{y} - \tilde{x} - \tilde{g} = \tilde{Z}^{1-\alpha} \left(\frac{\widetilde{k}}{l} \right)^{\alpha} l - \left[(1 + \gamma_n) (1 + \gamma) - (1 - \delta) \right] \left(\frac{\tilde{k}}{l} \right) l - \tilde{g} \quad \text{(ss2)}$$

$$\left(\frac{\psi \widetilde{c}}{1 - l} \right) = (1 - \tau_l) \widetilde{Z} (1 - \alpha) \widetilde{k}^{\alpha} \left(\widetilde{Z} l \right)^{-\alpha}$$

$$\widetilde{c} = \frac{(1 - l)}{\psi} (1 - \tau_l) \widetilde{Z}^{1-\alpha} (1 - \alpha) \left(\frac{\widetilde{k}}{l} \right)^{\alpha} \quad \text{(ss3)}$$

- Use (ss2) and (ss3) to solve l:

$$\tilde{Z}^{1-\alpha}\left(\frac{\widetilde{k}}{l}\right)^{\alpha} l-\left[\left(1+\gamma_{n}\right)(1+\gamma)-(1-\delta)\right]\left(\frac{\tilde{k}}{l}\right) l-\tilde{g}=\frac{(1-l)}{\psi}\left(1-\tau_{l}\right) \widetilde{Z}^{1-\alpha}\left(\frac{\widetilde{k}}{l}\right)^{\alpha}$$

$$\begin{aligned}
& \left\{\tilde{Z}^{1-\alpha}\left(\frac{\widetilde{k}}{l}\right)^{\alpha}-\left[\left(1+\gamma_{n}\right)(1+\gamma)-(1-\delta)\right]\left(\frac{\tilde{k}}{l}\right)\right\} l-\tilde{g} \\
= \quad & \frac{1}{\psi}\left(1-\tau_{l}\right) \widetilde{Z}^{1-\alpha}\left(\frac{\widetilde{k}}{l}\right)^{\alpha}-\frac{1}{\psi}\left(1-\tau_{l}\right) \widetilde{Z}^{1-\alpha}\left(\frac{\widetilde{k}}{l}\right)^{\alpha} l
\end{aligned}$$

$$
\begin{aligned}
&\left\{\tilde{Z}^{1-\alpha}\left(\frac{\widetilde{k}}{l}\right)^{\alpha}-\left[\left(1+\gamma_{n}\right)(1+\gamma)-(1-\delta)\right]\left(\frac{\tilde{k}}{l}\right)+\frac{1}{\psi}\left(1-\tau_{l}\right)\widetilde{Z}^{1-\alpha}\left(\frac{\widetilde{k}}{l}\right)^{\alpha}\right\} l \\
= \;& \tilde{g}+\frac{1}{\psi}\left(1-\tau_{l}\right)\widetilde{Z}^{1-\alpha}\left(\frac{\widetilde{k}}{l}\right)^{\alpha}
\end{aligned}
$$

- The above equation solves the steady-state value of l.

- It follows:

$$
\widetilde{k}=\left(\frac{\widetilde{k}}{l}\right) l
$$

$$\widetilde{c} = \frac{(1-l)}{\psi}(1-\tau_l)\,\widetilde{Z}^{1-\alpha}(1-\alpha)\left(\frac{\widetilde{k}}{l}\right)^{\alpha}$$

$$\tilde{y} = \tilde{Z}^{1-\alpha}\left(\frac{\widetilde{k}}{l}\right)^{\alpha} l$$

$$\tilde{x} = [(1+\gamma_n)(1+\gamma)-(1-\delta)]\,\tilde{k}$$

4 Log-linearization

- We log-linearize the equilibrium conditions around the steady state.

- $\widehat{c}_t = \log \widetilde{c}_t - \log \widetilde{c}$

- $\widehat{x}_t = \log \widetilde{x}_t - \log \widetilde{x}$

- $\widehat{y}_t = \log \widetilde{y}_t - \log \widetilde{y}$

- $\widehat{k}_t = \log \widetilde{k}_t - \log \widetilde{k}$

- $\widehat{l}_t = \log \widetilde{l}_t - \log \widetilde{l}$

- $\widehat{g}_t = \log \widetilde{g}_t - \log \widetilde{g}$

- $\widehat{Z}_t = \log \widetilde{Z}_t - \log \widetilde{Z}$

- $\widehat{\tau}_{lt} = \tau_{lt} - \tau_l$

- $\widehat{\tau}_{xt} = \tau_{xt} - \tau_x$

- Equilibrium conditions in detrended per-capita form:

$$\tilde{c}_t + \tilde{x}_t + \tilde{g}_t = \tilde{y}_t \tag{1.b}$$

$$\tilde{y}_t = \tilde{k}_t^{\alpha} \left(\tilde{Z}_t l_t\right)^{1-\alpha} \tag{2.b}$$

$$\left(\frac{\psi \widetilde{c}_t}{1 - l_t}\right) = (1 - \tau_{lt}) \widetilde{Z}_t (1 - \alpha) \widetilde{k}_t^{\alpha} \left(\widetilde{Z}_t l_t\right)^{-\alpha} \tag{3.b}$$

$$(1 + \tau_{xt}) (1 + \gamma) \frac{1}{\tilde{c}_t} = \beta E_t \frac{1}{\tilde{c}_{t+1}} \left[\alpha \left(\widetilde{k}_{t+1}\right)^{\alpha-1} \left(\widetilde{Z}_{t+1} l_{t+1}\right)^{1-\alpha} + (1 - \delta) (1 + \tau_{xt+1})\right] \tag{4.b}$$

$$(1+\gamma_n)(1+\gamma)\tilde{k}_{t+1} = (1-\delta)\tilde{k}_t + \tilde{x}_t \quad (5.b)$$

- Equation (1.b)

$$\tilde{c}_t + \tilde{x}_t + \tilde{g}_t = \tilde{y}_t$$

$$\tilde{c}e^{\hat{c}_t} + \tilde{x}e^{\hat{x}_t} + \tilde{g}e^{\hat{g}_t} = \tilde{y}e^{\widehat{y}_t}$$

$$\widetilde{c}\left(1+\hat{c}_t\right)+\widetilde{x}\left(1+\hat{x}_t\right)+\widetilde{g}\left(1+\hat{g}_t\right)=\widetilde{y}\left(1+\hat{y}_t\right)$$

$$\widetilde{c}\hat{c}_t+\widetilde{x}\hat{x}_t+\widetilde{g}\hat{g}_t=\widetilde{y}\hat{y}_t$$

$$\frac{\widetilde{c}}{\widetilde{y}}\hat{c}_t+\frac{\widetilde{x}}{\widetilde{y}}\hat{x}_t+\frac{\widetilde{g}}{\widetilde{y}}\hat{g}_t=\hat{y}_t \tag{1.c}$$

- Equation (2.b)

$$\tilde{y}_t = \tilde{k}_t^{\alpha} \left(\tilde{Z}_t l_t \right)^{1-\alpha}$$

$$\tilde{y} e^{\hat{y}_t} = \left(\tilde{k} e^{\hat{k}_t} \right)^{\alpha} \left(\tilde{Z} e^{\hat{Z}_t} l e^{\hat{l}_t} \right)^{1-\alpha}$$

$$\tilde{y} e^{\hat{y}_t} = \left(\tilde{k} \right)^{\alpha} \left(\tilde{Z} l \right)^{1-\alpha} e^{\alpha \hat{k}_t + (1-\alpha)\hat{Z}_t + (1-\alpha)\hat{l}_t}$$

$$e^{\hat{y}_t} = e^{\alpha \hat{k}_t + (1-\alpha)\hat{Z}_t + (1-\alpha)\hat{l}_t}$$

$$1 + \hat{y}_t = 1 + \alpha \hat{k}_t + (1 - \alpha) \hat{Z}_t + (1 - \alpha) \hat{l}_t$$

$$\hat{y}_t = \alpha \hat{k}_t + (1 - \alpha) \hat{Z}_t + (1 - \alpha) \hat{l}_t \tag{2.c}$$

- Equation (3.b)

$$\left(\frac{\psi \widetilde{c}_t}{1 - l_t} \right) = (1 - \tau_{lt}) \widetilde{Z}_t (1 - \alpha) \widetilde{k}_t^{\alpha} \left(\widetilde{Z}_t l_t \right)^{-\alpha}$$

$$\psi \widetilde{c}_t = (1 - \tau_{lt}) \widetilde{Z}_t (1 - \alpha) \widetilde{k}_t^{\alpha} \left(\widetilde{Z}_t l_t \right)^{-\alpha} (1 - l_t)$$

$$\psi \widetilde{c} = (1 - \tau_l) \widetilde{Z} (1 - \alpha) \widetilde{k}^{\alpha} \left(\widetilde{Z} l \right)^{-\alpha} (1 - l)$$

$$\frac{1}{(1 - \tau_t)(1 - l)} = \frac{1}{\psi \widetilde{c}} \widetilde{Z} (1 - \alpha) \widetilde{k}^{\alpha} \left(\widetilde{Z} l \right)^{-\alpha}$$

$$\psi \widetilde{c} e^{\hat{c}_t} = (1 - \hat{\tau}_{lt} - \tau_l) \widetilde{Z} e^{\hat{Z}_t} (1 - \alpha) \left(\widetilde{k} e^{\hat{k}_t} \right)^{\alpha} \left(\widetilde{Z} e^{\hat{Z}_t} l e^{\hat{l}_t} \right)^{-\alpha} \left(1 - l e^{\hat{l}_t} \right)$$

$$\psi\widetilde{c}e^{\hat{c}_t} = (1-\hat{\tau}_{lt}-\tau_l)\left[\widetilde{Z}(1-\alpha)\widetilde{k}^{\alpha}(Zl)^{-\alpha}\right]e^{\alpha\hat{k}_t+(1-\alpha)\hat{Z}_t-\alpha\hat{l}_t}\left(1-le^{\hat{l}_t}\right)$$

$$e^{\hat{c}_t} = (1-\hat{\tau}_{lt}-\tau_l)\left[\frac{1}{\psi\widetilde{c}}\widetilde{Z}(1-\alpha)\widetilde{k}^{\alpha}\left(\widetilde{Z}l\right)^{-\alpha}\right]e^{\alpha\hat{k}_t+(1-\alpha)\hat{Z}_t-\alpha\hat{l}_t}\left(1-le^{\hat{l}_t}\right)$$

$$e^{\hat{c}_t} = \frac{(1-\hat{\tau}_{lt}-\tau_l)}{(1-\tau_t)(1-l)}e^{\alpha\hat{k}_t+(1-\alpha)\hat{Z}_t-\alpha\hat{l}_t}\left(1-le^{\hat{l}_t}\right)$$

$$1+\hat{c}_t = \frac{(1-\hat{\tau}_{lt}-\tau_l)}{(1-\tau_t)(1-l)}\left(1+\alpha\hat{k}_t+(1-\alpha)\hat{Z}_t-\alpha\hat{l}_t\right)\left[1-l\left(1+\hat{l}_t\right)\right]$$

$$\left(1-\tau_t\right)\left(1-l\right)\left(1+\hat{c}_t\right)=\left(1-\hat{\tau}_{lt}-\tau_l\right)\left(1+\alpha\hat{k}_t+\left(1-\alpha\right)\hat{Z}_t-\alpha\hat{l}_t\right)\left[1-l-l\hat{l}_t\right]$$

$$
\begin{aligned}
(1-\tau_t)(1-l)(1+\hat{c}_t) &= (1-\hat{\tau}_{lt}-\tau_l)\left(1+\alpha\hat{k}_t+(1-\alpha)\hat{Z}_t-\alpha\hat{l}_t\right)\left[1-l-l\hat{l}_t\right] \\
&= (1-\hat{\tau}_{lt}-\tau_l)\begin{pmatrix} 1+\alpha\hat{k}_t+(1-\alpha)\hat{Z}_t-\alpha\hat{l}_t-l-\alpha l\hat{k}_t \\ -(1-\alpha)l\hat{Z}_t+\alpha l\hat{l}_t-l\hat{l}_t-l\hat{l}_t\alpha\hat{k}_t \\ -l\hat{l}_t(1-\alpha)\hat{Z}_t+l\hat{l}_t\alpha\hat{l}_t \end{pmatrix} \\
&= (1-\hat{\tau}_{lt}-\tau_l)\begin{pmatrix} 1+\alpha\hat{k}_t+(1-\alpha)\hat{Z}_t-\alpha\hat{l}_t-l-\alpha l\hat{k}_t \\ -(1-\alpha)l\hat{Z}_t+\alpha l\hat{l}_t-l\hat{l}_t \end{pmatrix} \\
&= [(1-\tau_l)-\hat{\tau}_{lt}]\left[(1-l)\left(1+\alpha\hat{k}_t+(1-\alpha)\hat{Z}_t-\alpha\hat{l}_t\right)-l\hat{l}_t\right] \\
&= (1-\tau_l)(1-l)\left(1+\alpha\hat{k}_t+(1-\alpha)\hat{Z}_t-\alpha\hat{l}_t\right)-(1-\tau_l)l\hat{l}_t \\
&\quad -\hat{\tau}_{lt}(1-l)\left(1+\alpha\hat{k}_t+(1-\alpha)\hat{Z}_t-\alpha\hat{l}_t\right)+\hat{\tau}_{lt}l\hat{l}_t \\
&= (1-\tau_l)(1-l)\left(1+\alpha\hat{k}_t+(1-\alpha)\hat{Z}_t-\alpha\hat{l}_t\right) \\
&\quad -(1-\tau_l)l\hat{l}_t-\hat{\tau}_{lt}(1-l)
\end{aligned}
$$

$$
\begin{aligned}
(1-\tau_l)(1-l)(1+\hat{c}_t) \quad = \quad & (1-\tau_l)(1-l)\left(1+\alpha\hat{k}_t+(1-\alpha)\hat{Z}_t-\alpha\hat{l}_t\right) \\
& -(1-\tau_l)\,l\hat{l}_t-\hat{\tau}_{lt}(1-l)
\end{aligned}
$$

$$
(1+\hat{c}_t)=\left(1+\alpha\hat{k}_t+(1-\alpha)\hat{Z}_t-\alpha\hat{l}_t\right)-\frac{1}{(1-l)}l\hat{l}_t-\frac{1}{(1-\tau_l)}\hat{\tau}_{lt}
$$

$$
\hat{c}_t=\alpha\hat{k}_t+(1-\alpha)\hat{Z}_t-\alpha\hat{l}_t-\frac{1}{(1-l)}l\hat{l}_t-\frac{1}{(1-\tau_l)}\hat{\tau}_{lt}
$$

$$
\hat{c}_t=\alpha\hat{k}_t+(1-\alpha)\hat{Z}_t-\left[\alpha+\frac{1}{(1-l)}l\right]\hat{l}_t-\frac{1}{(1-\tau_l)}\hat{\tau}_{lt} \qquad \text{(3.c)}
$$

- Equation (4.b)

$$(1+\tau_{xt})(1+\gamma)\frac{1}{\tilde{c}_t} = \beta E_t \frac{1}{\tilde{c}_{t+1}}\left[\alpha\left(\widetilde{k}_{t+1}\right)^{\alpha-1}\left(\widetilde{Z}_{t+1}l_{t+1}\right)^{1-\alpha} + (1-\delta)(1+\tau_{xt+1})\right]$$

$$(1+\tau_x)(1+\gamma)\frac{1}{\tilde{c}} = \beta\frac{1}{\tilde{c}}\left[\alpha\left(\widetilde{k}\right)^{\alpha-1}\left(\widetilde{Z}l\right)^{1-\alpha} + (1-\delta)(1+\tau_x)\right]$$

$$(1+\tau_x)(1+\gamma) = \beta\left[\alpha\left(\widetilde{k}\right)^{\alpha-1}\left(\widetilde{Z}l\right)^{1-\alpha} + (1-\delta)(1+\tau_x)\right]$$

$$(1+\tau_x)(1+\gamma)=\beta\left[\alpha\frac{\tilde{y}}{\tilde{k}}+(1-\delta)(1+\tau_x)\right]$$

$$\frac{(1+\tau_x)(1+\gamma)}{\beta}=\left[\alpha\frac{\tilde{y}}{\tilde{k}}+(1-\delta)(1+\tau_x)\right]$$

$$(1+\hat{\tau}_{xt}+\tau_x)(1+\gamma)\frac{1}{\tilde{c}}e^{-\hat{c}_t}=\beta E_t\frac{1}{\tilde{c}}e^{-\hat{c}_{t+1}}\left[\alpha\frac{\tilde{y}e^{\hat{y}_{t+1}}}{\tilde{k}e^{\hat{k}_{t+1}}}+(1-\delta)(1+\hat{\tau}_{xt+1}+\tau_x)\right]$$

$$(1+\hat{\tau}_{xt}+\tau_x)(1+\gamma)e^{-\hat{c}_t}=\beta E_te^{-\hat{c}_{t+1}}\left[\left(\alpha\frac{\tilde{y}}{\tilde{k}}\right)e^{\hat{y}_{t+1}-\hat{k}_{t+1}}+(1-\delta)(1+\hat{\tau}_{xt+1}+\tau_x)\right]$$

- Ignore E_t for a moment

$$(1+\hat{\tau}_{xt}+\tau_x)(1+\gamma)\,e^{\hat{c}_{t+1}-\hat{c}_t}=\beta\left[\alpha\frac{\widetilde{y}}{\widetilde{k}}e^{\hat{y}_{t+1}-\hat{k}_{t+1}}+(1-\delta)(1+\hat{\tau}_{xt+1}+\tau_x)\right]$$

$$(1+\hat{\tau}_{xt}+\tau_x)(1+\gamma)(1+\hat{c}_{t+1}-\hat{c}_t)=\beta\left[\begin{array}{c}\alpha\frac{\widetilde{y}}{\widetilde{k}}(1+\hat{y}_{t+1}-\hat{k}_{t+1})\\+(1-\delta)(1+\hat{\tau}_{xt+1}+\tau_x)\end{array}\right]$$

$$(1+\hat{\tau}_{xt}+\tau_x)(1+\hat{c}_{t+1}-\hat{c}_t)\frac{(1+\gamma)}{\beta}=\left[\begin{array}{c}\alpha\frac{\widetilde{y}}{\widetilde{k}}(1+\hat{y}_{t+1}-\hat{k}_{t+1})\\+(1-\delta)(1+\hat{\tau}_{xt+1}+\tau_x)\end{array}\right]$$

- LHS

$$
\begin{aligned}
& (1+\hat{\tau}_{xt}+\tau_x)\,(1+\hat{c}_{t+1}-\hat{c}_t)\frac{(1+\gamma)}{\beta} \\
= \;& [1+(\hat{c}_{t+1}-\hat{c}_t)]\,[(1+\tau_x)+\hat{\tau}_{xt}]\,\frac{(1+\gamma)}{\beta} \\
= \;& (1+\tau_x)\,\frac{(1+\gamma)}{\beta}+\hat{\tau}_{xt}\frac{(1+\gamma)}{\beta}+(\hat{c}_{t+1}-\hat{c}_t)\,(1+\tau_x)\,\frac{(1+\gamma)}{\beta} \\
& +(\hat{c}_{t+1}-\hat{c}_t)\,\hat{\tau}_{xt}\frac{(1+\gamma)}{\beta} \\
= \;& (1+\tau_x)\,\frac{(1+\gamma)}{\beta}+\hat{\tau}_{xt}\frac{(1+\gamma)}{\beta}+(\hat{c}_{t+1}-\hat{c}_t)\,(1+\tau_x)\,\frac{(1+\gamma)}{\beta}
\end{aligned}
$$

- RHS

$$
\begin{aligned}
& \left[\alpha\frac{\widetilde{y}}{\widetilde{k}}(1+\hat{y}_{t+1}-\hat{k}_{t+1})+(1-\delta)\left(1+\hat{\tau}_{xt+1}+\tau_x\right)\right] \\
= & \left[\alpha\frac{\widetilde{y}}{\widetilde{k}}+\alpha\frac{\widetilde{y}}{\widetilde{k}}\left(\hat{y}_{t+1}-\hat{k}_{t+1}\right)\right]+(1-\delta)\left(1+\tau_x\right)+(1-\delta)\hat{\tau}_{xt+1}
\end{aligned}
$$

- LHS=RHS

$$
\begin{aligned}
&(1+\tau_x)\frac{(1+\gamma)}{\beta}+\hat{\tau}_{xt}\frac{(1+\gamma)}{\beta}+(\hat{c}_{t+1}-\hat{c}_t)(1+\tau_x)\frac{(1+\gamma)}{\beta} \\
= &\left[\alpha\frac{\widetilde{y}}{\widetilde{k}}+\alpha\frac{\widetilde{y}}{\widetilde{k}}\left(\hat{y}_{t+1}-\hat{k}_{t+1}\right)\right]+(1-\delta)(1+\tau_x)+(1-\delta)\hat{\tau}_{xt+1}
\end{aligned}
$$

$$
\hat{\tau}_{xt}\frac{(1+\gamma)}{\beta}+(\hat{c}_{t+1}-\hat{c}_t)(1+\tau_x)\frac{(1+\gamma)}{\beta}=\left[\alpha\frac{\widetilde{y}}{\widetilde{k}}\left(\hat{y}_{t+1}-\hat{k}_{t+1}\right)+(1-\delta)\hat{\tau}_{xt+1}\right]
$$

- Add back expectation

$$\hat{\tau}_{xt}\frac{(1+\gamma)}{\beta}+E_t\left(\hat{c}_{t+1}-\hat{c}_t\right)(1+\tau_x)\frac{(1+\gamma)}{\beta}=E_t\left[\alpha\frac{\widetilde{y}}{\widetilde{k}}\left(\hat{y}_{t+1}-\hat{k}_{t+1}\right)+(1-\delta)\hat{\tau}_{xt+1}\right]$$

$$\begin{aligned} & \hat{\tau}_{xt}\frac{(1+\gamma)}{\beta}+(1+\tau_x)\frac{(1+\gamma)}{\beta}E_t\hat{c}_{t+1}-(1+\tau_x)\frac{(1+\gamma)}{\beta}\hat{c}_t && \text{(4.c)} \\ = \quad & E_t\left[\alpha\frac{\widetilde{y}}{\widetilde{k}}\left(\hat{y}_{t+1}-\hat{k}_{t+1}\right)+(1-\delta)\hat{\tau}_{xt+1}\right] && \text{(5)} \end{aligned}$$

- Equation (5.b)

$$(1+\gamma_n)(1+\gamma)\tilde{k}_{t+1}=(1-\delta)\tilde{k}_t+\tilde{x}_t$$

$$(1+\gamma_n)(1+\gamma)\tilde{k}e^{\hat{k}_{t+1}} = (1-\delta)\tilde{k}e^{\hat{k}_t} + \tilde{x}e^{\hat{x}_t}$$

$$(1+\gamma_n)(1+\gamma)\tilde{k}\left(1+\hat{k}_{t+1}\right) = (1-\delta)\tilde{k}\left(1+\hat{k}_t\right) + \tilde{x}(1+\hat{x}_t)$$

$$(1+\gamma_n)(1+\gamma)\tilde{k}\hat{k}_{t+1} = (1-\delta)\tilde{k}\hat{k}_t + \tilde{x}\hat{x}_t$$

$$(1+\gamma_n)(1+\gamma)\hat{k}_{t+1} = (1-\delta)\hat{k}_t + \frac{\tilde{x}}{\tilde{k}}\hat{x}_t \tag{5.c}$$