Derivation and Log-linearization of Otsu (2007)'s Small Open Economy Model

Chen-chiao Kuo*

*Otsu, Keisuke (2007), "A Neoclassical Analysis of the Asian Crisis: Business Cycle Accounting of a Small Open Economy," IMES Discussion Paper Series No. 07-E-16, Bank of Japan.

1 The benchmark prototype economy

- The representative consumer depends on an expected intertemporal discounted utility function that includes utility from consumption and disutility from labor:

$$\max U = E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$$

- The utility is:

$$u(c_t, l_t) = \log(c_t - \chi l_t^{\nu})$$

- where parameters $\chi(> 0)$ and $\nu(> 1)$ represent the level and curvature of the utility cost of labor respectively.

- subject to the budget constraint:

$$\frac{w_t}{\tau_{lt}} l_t + r_t k_t + t_t + \frac{\Gamma d_{t+1}}{R \tau_{dt}} = c_t + \tau_{xt} x_t + d_t + \Phi(d_{t+1})$$

- The law of motion for capital:

$$\Gamma k_{t+1} = x_t + (1 - \delta) k_t$$

- where define $\Gamma = (1+\gamma)(1+n)$, γ is the growth rate of labor augmenting technical progress and n is the population growth rate.

- The debt adjustment cost function, $\Phi(d_{t+1})$, as $\frac{\phi(d_{t+1}-d)^2}{2}$, where d is the steady state foreign debt.

- Solution of consumers' maximization problem:

$$\max_{\{c_t, l_t, k_{t+1}, d_{t+1}\}} \; U = E_0 \sum_{t=0}^{\infty} \beta^t \log\left(c_t - \chi l_t^{\nu}\right)$$

$$\frac{w_t}{\tau_{lt}} l_t + r_t k_t + t_t + \frac{\Gamma d_{t+1}}{R\tau_{dt}} = c_t + \tau_{xt} \left[\Gamma k_{t+1} - (1-\delta) k_t \right] + d_t + \Phi(d_{t+1})$$

- The Lagrangian

$$\begin{aligned} \mathcal{L} &= \sum_{t=0}^{\infty} \beta^t \log(c_t - \chi l_t^{\nu}) + \\ & \sum_{t=0}^{\infty} \lambda_t \beta^t \left\{ \begin{array}{c} \frac{w_t}{\tau_{lt}} l_t + r_t k_t + t_t + \frac{\Gamma d_{t+1}}{R\tau_{dt}} - c_t \\ -\tau_{xt} \left[\Gamma k_{t+1} - (1-\delta) k_t \right] - d_t - \Phi\left(d_{t+1}\right) \end{array} \right\} \end{aligned}$$

- First-order-condition of the model show in Equation (A1) to (A4) can be simplified to the below four equations.

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t u_{ct} - \lambda_t \beta^t = 0$$

$$\frac{\partial \mathcal{L}}{\partial l_t} = \beta^t u_{lt} + \lambda_t \beta^t \frac{w_t}{\tau_{lt}} = 0$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = -\lambda_t \beta^t \tau_{xt} \Gamma + \lambda_{t+1} \beta^{t+1} [\gamma_{t+1} + \tau_{xt+1}(1-\delta)] = 0$$

$$\frac{\partial \mathcal{L}}{\partial d_{t+1}} = \lambda_t \beta^t \left[\frac{\Gamma}{R \tau_{dt}} - \Phi^{'}(d_{t+1}) \right] - \lambda_{t+1} \beta^{t+1} = 0$$

$$\frac{1}{c_t - \chi l_t^{\nu}} = \lambda_t \tag{A1}$$

$$-\frac{\chi \nu l_t^{\nu-1}}{c_t - \chi l_t^{\nu}} = -\lambda_t \frac{w_t}{\tau_{lt}}$$

- Use Equation (A1) substitute λ_t, and we obtain:

$$\chi \nu l_t^{\nu-1} = \frac{w_t}{\tau_{lt}} \tag{A2}$$

$$\lambda_t \tau_{xt} \Gamma = \lambda_{t+1} \beta [\gamma_{t+1} + \tau_{xt+1}(1-\delta)]$$

$$\frac{1}{c_t - \chi l_t^{\nu}} \tau_{xt} \Gamma = \beta E_t u_{ct+1} [\gamma_{t+1} + \tau_{xt+1}(1-\delta)] \tag{A3}$$

$$\lambda_t \left[\frac{\Gamma}{R\tau_{dt}} - \Phi^{'}(d_{t+1}) \right] = \lambda_{t+1}\beta$$

$$\frac{1}{c_t - \chi l_t^{\nu}} \left[\frac{\Gamma}{R\tau_{dt}} - \Phi^{'}(d_{t+1}) \right] = \beta E_t u_{ct+1} \quad \text{(A4)}$$

- The firm produces a final good (output y with a constant returns to scale) from capital and labor using a Cobb-Douglas production function.

$$y_t = z_t k_t^{\theta} l_t^{1-\theta}$$

- The firm's profit maximization problem is:

$$\Pi = y_t - w_t l_t - r_t k_t$$

- Solution of firms' maximization problem:

$$\max_{\{k_t, l_t\}} \Pi = z_t k_t^{\theta} l_t^{1-\theta} - w_t l_t - r_t k_t$$

$$\frac{\partial \Pi}{\partial k_t} = \theta z_t k_t^{\theta-1} l_t^{1-\theta} - r_t = 0$$

$$\frac{\partial \Pi}{\partial l_t} = (1-\theta) z_t k_t^{\theta} l_t^{-\theta} - w_t = 0$$

$$\theta z_t k_t^{\theta-1} l_t^{1-\theta} = r_t$$

$$\theta \frac{y_t}{k_t} = r_t \tag{A5}$$

$$(1-\theta) z_t k_t^{\theta} l_t^{-\theta} = w_t$$

$$(1-\theta) \frac{y_t}{l_t} = w_t \tag{A6}$$

- Substitute (A6) into (A2), and we get the labor wedge.

$$\chi \nu l_t^{\nu-1} = \frac{w_t}{\tau_{lt}} = (1-\theta) \frac{y_t}{l_t} \frac{1}{\tau_{lt}}$$

$$\chi \nu l_t^{\nu-1} = (1-\theta)\frac{y_t}{l_t}\frac{1}{\tau_{lt}}$$

- Substitute (A5) into (A3), and we get the investment wedge.

$$\tau_{xt}\Gamma\frac{1}{c_t - \chi l_t^{\nu}} = \beta E_t\left[u_{ct+1}\left(\theta\frac{y_{t+1}}{k_{t+1}} + \tau_{xt+1}(1-\delta)\right)\right]$$

- Substitute $\Phi'(d_{t+1}) = \phi(d_{t+1} - d)$ into (A4), and we get the foreign debt wedge.

$$\frac{1}{c_t - \chi l_t^{\nu}} \left[\frac{\Gamma}{R} \frac{1}{\tau_{dt}} - \phi(d_{t+1} - d) \right] = \beta E_t[u_{ct+1}]$$

- The government collects tax revenues at date t, and the government budget constraint:

$$t_t = (1 - \frac{1}{\tau_{lt}}) w_t l_t + (\tau_{xt} - 1) x_t$$

- Substitute the government budget constraint into the consumer budget constraint and we can get the economy's resource constraint at last:

$$\frac{w_t}{\tau_{lt}} l_t + r_t k_t + (1 - \frac{1}{\tau_{lt}}) w_t l_t + (\tau_{xt} - 1) x_t + \frac{\Gamma d_{t+1}}{R \tau_{dt}} = c_t + \tau_{xt} x_t + d_t + \Phi(d_{t+1})$$

$$r_t k_t + w_t l_t - x_t + \frac{\Gamma d_{t+1}}{R \tau_{dt}} = c_t + d_t + \Phi(d_{t+1})$$

$$r_t k_t + w_t l_t = x_t + c_t + d_t - \frac{\Gamma d_{t+1}}{R \tau_{dt}} + \Phi(d_{t+1})$$

$$y_t = c_t + x_t + d_t - \frac{\Gamma d_{t+1}}{R \tau_{dt}} + \frac{\phi (d_{t+1} - d)^2}{2}$$

- where the trade balance is:

$$tb_t = d_t - \frac{\Gamma d_{t+1}}{R\tau_{dt}} + \frac{\phi(d_{t+1} - d)^2}{2}$$

- The equilibrium conditions in the detrended per-capita form:

$$y_t = c_t + x_t + d_t - \frac{\Gamma d_{t+1}}{R\tau_{dt}} + \frac{\phi(d_{t+1} - d)^2}{2} \tag{1}$$

$$y_t = z_t k_t^{\theta} l_t^{1-\theta} \tag{2}$$

$$\chi \nu l_t^{\nu-1} = (1-\theta)\frac{y_t}{l_t}\frac{1}{\tau_{lt}} \tag{3}$$

$$\tau_{xt}\Gamma\frac{1}{c_t-\chi l_t^{\nu}}=\beta E_t\left[u_{ct+1}\left(\theta\frac{y_{t+1}}{k_{t+1}}+\tau_{xt+1}(1-\delta)\right)\right] \tag{4}$$

$$\frac{1}{c_t-\chi l_t^{\nu}}\left[\frac{\Gamma}{R}\frac{1}{\tau_{dt}}-\phi(d_{t+1}-d)\right]=\beta E_t[u_{ct+1}] \tag{5}$$

- Endogenous variables are : c_t ,x_t ,y_t ,k_t ,d_t ,l_t.

- Exogenous variables are: z_t, τ_{dt}, τ_{xt}, and τ_{lt}.

- We need the capital accumulation law to close the model.

$$\Gamma k_{t+1} = x_t + (1-\delta)k_t \tag{6}$$

2 Steady state of the prototype model

- From equation(5), the calculation process is below:

$$\frac{1}{c_t - \chi l_t^{\nu}}\left[\frac{\Gamma}{R}\frac{1}{\tau_{dt}} - \phi(d_{t+1} - d)\right] = \beta E_t[u_{ct+1}]$$

$$\frac{1}{c-\chi l^{\nu}}[\frac{\Gamma}{R}\frac{1}{\tau_d}-\phi(d-d)]=\beta u_c$$

$$\frac{\Gamma}{\beta}\frac{1}{\tau_d}=R \tag{ss1}$$

- From equation(4), the calculation process is below:

$$\tau_{xt}\Gamma\frac{1}{c_t-\chi l_t^{\nu}}=\beta E_t\left[u_{ct+1}\left(\theta\frac{y_{t+1}}{k_{t+1}}+\tau_{xt+1}(1-\delta)\right)\right]$$

$$\tau_x\Gamma=\beta\left[\theta z\left(\frac{k}{l}\right)^{\theta-1}+\tau_x(1-\delta)\right]$$

$$\frac{\tau_x \Gamma}{\beta} = z\theta \left(\frac{k}{l}\right)^{\theta-1} + \tau_x(1-\delta)$$

$$\left(\frac{k}{l}\right)^{\theta-1} = \frac{1}{z\theta}\left[\frac{\tau_x \Gamma}{\beta} - \tau_x(1-\delta)\right]$$

$$\frac{k}{l} = \left\{\frac{1}{z\theta}\left[\frac{\tau_x \Gamma}{\beta} - \tau_x(1-\delta)\right]\right\}^{\frac{1}{\theta-1}} \qquad \text{(ss2)}$$

- From equation(6), the calculation process is below:

$$\Gamma k_{t+1} = x_t + (1-\delta)k_t$$

$$\Gamma k = x + (1-\delta)k$$

$$x = [\Gamma - (1-\delta)]k \tag{ss3}$$

- From equation(2), the calculation process is below:

$$y_t = z_t k_t^{\theta} l_t^{1-\theta}$$

$$y = z k^{\theta} l^{1-\theta} = z \left(\frac{k}{l}\right)^{\theta-1} k = z \left(\frac{k}{l}\right)^{\theta} l$$

- From equation(1), the calculation process is below:

$$y_t = c_t + x_t + d_t - \frac{\Gamma d_{t+1}}{R\tau_{dt}} + \frac{\phi(d_{t+1} - d)^2}{2}$$

$$y = c + x + d - \frac{\Gamma d}{R\tau_d} + \frac{\phi(d - d)^2}{2}$$

$$c = y - x - d + \frac{\Gamma d}{R\tau_d} - \frac{\phi(d - d)^2}{2} = z\left(\frac{k}{l}\right)^{\theta-1} k - [\Gamma - (1-\delta)]k - d + \frac{\Gamma d}{R\tau_d}$$

$$c = z\left(\frac{k}{l}\right)^{\theta} l - [\Gamma - (1-\delta)]\left(\frac{k}{l}\right) l - d + \frac{\Gamma d}{R\tau_d}$$

$$c = z\left(\frac{k}{l}\right)^{\theta} l - [\Gamma - (1-\delta)]\left(\frac{k}{l}\right) l - d + \beta d$$

$$c = z\left(\frac{k}{l}\right)^{\theta} l - [\Gamma - (1-\delta)]\left(\frac{k}{l}\right) l + (\beta - 1)d \tag{ss4}$$

- From equation(3), the calculation process is below:

$$\chi \nu l_t^{\nu-1} = (1-\theta)\frac{y_t}{l_t}\frac{1}{\tau_{lt}}$$

$$\chi \nu l^{\nu-1} = (1-\theta)\frac{y}{l}\frac{1}{\tau_l}$$

$$\chi \nu l^{\nu} = (1-\theta)z\left(\frac{k}{l}\right)^{\theta}\frac{1}{\tau_l}$$

$$l^{\nu} = (1-\theta)\frac{z}{\chi\nu}\left(\frac{k}{l}\right)^{\theta}\frac{1}{\tau_l}$$

$$l = \left[(1-\theta)\frac{z}{\chi\nu}\left(\frac{k}{l}\right)^{\theta}\frac{1}{\tau_l}\right]^{\frac{1}{\nu}} \qquad \text{(ss5)}$$

- We obtain $\frac{k}{l}$ first, and in order obtain l, then k, then c, then y, then x.

$$k = \frac{k}{l}l$$

$$c = z\left(\frac{k}{l}\right)^{\theta} l - [\Gamma - (1-\delta)]\left(\frac{k}{l}\right) l + d(\beta - 1)$$

$$y = z\left(\frac{k}{l}\right)^{\theta} l$$

$$x = [\Gamma - (1-\delta)]k$$

$$d = d$$

3 Log-linearization

- $\hat{c}_t = \log c_t - \log c$

- $\hat{x}_t = \log x_t - \log x$

- $\hat{y}_t = \log y_t - \log y$

- $\hat{k}_t = \log k_t - \log k$

- $\hat{l}_t = \log l_t - \log l$

- $\hat{z}_t = \log z_t - \log z$

- $\hat{d}_t = \left(\frac{d_t}{d} - 1\right)$

- $\hat{\tau}_{dt} = \log \tau_{dt} - \log \tau_d$

- $\hat{\tau}_{xt} = \log \tau_{xt} - \log \tau_x$

- $\hat{\tau}_{lt} = \log \tau_{lt} - \log \tau_l$

- Equilibrium conditions in the detrended per-capita form:

$$y_t = c_t + x_t + d_t - \frac{\Gamma d_{t+1}}{R\tau_{dt}} + \frac{\phi(d_{t+1} - d)^2}{2} \tag{1}$$

$$y_t = z_t k_t^{\theta} l_t^{1-\theta} \tag{2}$$

$$\chi \nu l_t^{\nu-1} = (1-\theta)\frac{y_t}{l_t}\frac{1}{\tau_{lt}} \tag{3}$$

$$\tau_{xt}\Gamma\frac{1}{c_t - \chi l_t^{\nu}} = \beta E_t\left[u_{ct+1}\left(\theta\frac{y_{t+1}}{k_{t+1}} + \tau_{xt+1}(1-\delta)\right)\right] \tag{4}$$

$$\frac{1}{c_t - \chi l_t^{\nu}}\left[\frac{\Gamma}{R}\frac{1}{\tau_{dt}} - \phi(d_{t+1} - d)\right] = \beta E_t[u_{ct+1}] \tag{5}$$

$$\Gamma k_{t+1} = x_t + (1-\delta)k_t \tag{6}$$

- From equation (1)

$$y_t = c_t + x_t + d_t - \frac{\Gamma d_{t+1}}{R\tau_{dt}} + \frac{\phi(d_{t+1} - d)^2}{2}$$

$$ye^{\hat{y}_t} = ce^{\hat{c}_t} + xe^{\hat{x}_t} + \left(\hat{d}_t + 1\right) d - \frac{\Gamma \left[\left(\hat{d}_{t+1} + 1\right) d\right]}{R\tau_d e^{\hat{\tau}_{dt}}} + \frac{\phi \left[\left(\hat{d}_{t+1} + 1\right) d - d\right]^2}{2}$$

$$y(1+\hat{y}_t) = c\left(1 + \hat{c}_t\right) + x(1+\hat{x}_t) + \left(\hat{d}_t + 1\right) d - \frac{\Gamma \left(\hat{d}_{t+1} d + d\right)}{R\tau_d(1 + \hat{\tau}_{dt})} + \frac{\phi \left(\left(\hat{d}_{t+1} d + d\right) - d\right)^2}{2}$$

$$y + y\hat{y}_t = c + c\hat{c}_t + x + x\hat{x}_t + d\hat{d}_t + d - \frac{\Gamma \hat{d}_{t+1} d + \Gamma d}{R\tau_d(1 + \hat{\tau}_{dt})} + \frac{\phi \left(\hat{d}_{t+1} d\right)^2}{2}$$

- where $y = c + x + d - \frac{\Gamma d}{R\tau_d}$ and $\frac{\phi\left(\hat{d}_{t+1} d\right)^2}{2} \approx 0$

$$y + y\hat{y}_t = c + c\hat{c}_t + x + x\hat{x}_t + d\hat{d}_t + d - \frac{\Gamma \hat{d}_{t+1} d + \Gamma d}{R\tau_d(1 + \hat{\tau}_{dt})} - \frac{\Gamma d}{R\tau_d} + \frac{\Gamma d}{R\tau_d}$$

$$\begin{aligned} R\tau_d(1 + \hat{\tau}_{dt}) y\hat{y}_t \quad &= \quad R\tau_d(1 + \hat{\tau}_{dt}) c\hat{c}_t + R\tau_d(1 + \hat{\tau}_{dt}) x\hat{x}_t + R\tau_d(1 + \hat{\tau}_{dt}) d\hat{d}_t - \Gamma \hat{d}_{t+1} d \\ &\quad - \Gamma d + R\tau_d(1 + \hat{\tau}_{dt}) \frac{\Gamma d}{R\tau_d} \end{aligned}$$

$$R\tau_d y\hat{y}_t = R\tau_d c\hat{c}_t + R\tau_d x\hat{x}_t + R\tau_d d\hat{d}_t - \Gamma d\hat{d}_{t+1} - \Gamma d + \Gamma d + \Gamma d\hat{\tau}_{dt}$$

$$\hat{y}_t = \frac{c}{y}\hat{c}_t + \frac{x}{y}\hat{x}_t + \frac{d}{y}\hat{d}_t - \frac{\Gamma d}{R\tau_d y}\hat{d}_{t+1} + \frac{\Gamma d}{R\tau_d y}\hat{\tau}_{dt} \qquad (1.c)$$

- equation (2)

$$y_t = z_t k_t^{\theta} l_t^{1-\theta}$$

$$y e^{\hat{y}_t} = z e^{\hat{z}_t} (k e^{\hat{k}_t})^{\theta} (l e^{\hat{l}_t})^{1-\theta} = (z k^{\theta} l^{1-\theta}) e^{\hat{z}_t + \theta \hat{k}_t + (1-\theta)\hat{l}_t}$$

$$e^{\hat{y}_t} = e^{\hat{z}_t + \theta \hat{k}_t + (1-\theta)\hat{l}_t}$$

$$(1 + \hat{y}_t) = 1 + \hat{z}_t + \theta \hat{k}_t + (1-\theta)\hat{l}_t$$

$$\hat{y}_t = \hat{z}_t + \theta \hat{k}_t + (1-\theta)\hat{l}_t \tag{2.c}$$

- Equation (3)

$$\chi\nu l_t^{\nu-1} = (1-\theta)z_t\left(\frac{k_t}{l_t}\right)^{\theta}\frac{1}{\tau_{lt}}$$

$$\chi\nu\left(le^{\hat{l}_t}\right)^{\nu-1} = (1-\theta)ze^{\hat{z}_t}\left(ke^{\hat{k}_t}\right)^{\theta}\left(le^{\hat{l}_t}\right)^{-\theta}\frac{1}{\tau_l e^{\hat{\tau}_{lt}}}$$

$$\chi\nu l^{\nu-1}e^{(\nu-1)\hat{l}_t} = (1-\theta)ze^{\hat{z}_t}k^{\theta}e^{\theta\hat{k}_t}l^{-\theta}e^{-\theta\hat{l}_t}\frac{1}{\tau_l e^{\hat{\tau}_{lt}}}$$

$$\tau_l\chi\nu l^{\nu-1}e^{(\nu-1)\hat{l}_t+\hat{\tau}_{lt}} = (1-\theta)ze^{\hat{z}_t}k^{\theta}e^{\theta\hat{k}_t}l^{-\theta}e^{-\theta\hat{l}_t}$$

$$\tau_l\chi\nu l^{\nu-1}\left[1+(\nu-1)\,\hat{l}_t+\hat{\tau}_{lt}\right] = (1-\theta)zk^{\theta}l^{-\theta}\left(1+\hat{z}_t+\theta\hat{k}_t-\theta\hat{l}_t\right)$$

$$\tau_l\chi\nu l^{\nu}\left[1+(\nu-1)\,\hat{l}_t+\hat{\tau}_{lt}\right] = (1-\theta)zk^{\theta}l^{1-\theta}\left(1+\hat{z}_t+\theta\hat{k}_t-\theta\hat{l}_t\right)$$

- where $\chi \nu l^{\nu} \tau_l = (1-\theta) y$ and $y = z k^{\theta} l^{1-\theta}$

$$(1-\theta) y \left[1 + (\nu - 1)\, \hat{l}_t + \hat{\tau}_{lt}\right] = (1-\theta) y \left(1 + \hat{z}_t + \theta \hat{k}_t - \theta \hat{l}_t\right)$$

$$1 + (\nu - 1)\, \hat{l}_t + \hat{\tau}_{lt} = \left(1 + \hat{z}_t + \theta \hat{k}_t - \theta \hat{l}_t\right)$$

$$\hat{\tau}_{lt} = \hat{z}_t + \theta \hat{k}_t - \theta \hat{l}_t - (\nu - 1)\, \hat{l}_t$$

$$\hat{\tau}_{lt} = \hat{z}_t + \theta \hat{k}_t + (1 - \theta - \nu)\, \hat{l}_t \qquad (3.c)$$

- equation (4)

$$\tau_{xt}\Gamma\frac{1}{c_t-\chi l_t^{\nu}}=\beta E_t\left[u_{ct+1}\left(\theta\frac{y_{t+1}}{k_{t+1}}+\tau_{xt+1}(1-\delta)\right)\right]$$

$$\tau_x e^{\hat{\tau}_{xt}}\Gamma\frac{1}{ce^{\hat{c}_t}-\chi l^{\nu}e^{\nu\hat{l}_t}}=\beta E_t\left[\frac{1}{ce^{\hat{c}_{t+1}}-\chi l^{\nu}e^{\nu\hat{l}_{t+1}}}\left(\begin{array}{c}\theta y e^{\hat{y}_{t+1}}k^{-1}e^{-\hat{k}_{t+1}}\\+\tau_x e^{\hat{\tau}_{xt+1}}(1-\delta)\end{array}\right)\right]$$

- Ignore E_t for a moment

$$\begin{aligned}\tau_x e^{\hat{\tau}_{xt}}\Gamma\left(ce^{\hat{c}_{t+1}}-\chi l^{\nu}e^{\nu\hat{l}_{t+1}}\right) &= \beta\left(\theta y e^{\hat{y}_{t+1}}k^{-1}e^{-\hat{k}_{t+1}}+\tau_x e^{\hat{\tau}_{xt+1}}(1-\delta)\right)\\ &\quad \left(ce^{\hat{c}_t}-\chi l^{\nu}e^{\nu\hat{l}_t}\right)\end{aligned}$$

- LHS

$$\tau_x e^{\hat{\tau}_{xt}} \Gamma c e^{\hat{c}_{t+1}} - \tau_x e^{\hat{\tau}_{xt}} \Gamma \chi l^{\nu} e^{\nu \hat{l}_{t+1}}$$

$$= \tau_x \Gamma c e^{\hat{\tau}_{xt} + \hat{c}_{t+1}} - \tau_x \Gamma \chi l^{\nu} e^{\hat{\tau}_{xt} + \nu \hat{l}_{t+1}}$$

$$= \tau_x \Gamma c \left(1 + \hat{\tau}_{xt} + \hat{c}_{t+1}\right) - \tau_x \Gamma \chi l^{\nu} \left(1 + \hat{\tau}_{xt} + \nu \hat{l}_{t+1}\right)$$

$$= \tau_x \Gamma c + \tau_x \Gamma c \hat{\tau}_{xt} + \tau_x \Gamma c \hat{c}_{t+1} - \tau_x \Gamma \chi l^{\nu} - \tau_x \Gamma \chi l^{\nu} \hat{\tau}_{xt} - \tau_x \Gamma \chi l^{\nu} \nu \hat{l}_{t+1}$$

$$= \left(c - \chi l^{\nu}\right) \tau_x \Gamma \hat{\tau}_{xt} + \left(c - \chi l^{\nu}\right) \Gamma \tau_x + \tau_x \Gamma c \hat{c}_{t+1} - \tau_x \Gamma \chi l^{\nu} \nu \hat{l}_{t+1}$$

- RHS

$$\beta\left(\theta y e^{\hat{y}_{t+1}} k^{-1} e^{-\hat{k}_{t+1}} + \tau_x e^{\hat{\tau}_{xt+1}}(1-\delta)\right)\left(c e^{\hat{c}_t} - \chi l^{\nu} e^{\nu \hat{l}_t}\right)$$

$$= \beta\left[\begin{array}{c} \left(\theta y e^{\hat{y}_{t+1}} k^{-1} e^{-\hat{k}_{t+1}} c e^{\hat{c}_t} - \theta y e^{\hat{y}_{t+1}} k^{-1} e^{-\hat{k}_{t+1}} \chi l^{\nu} e^{\nu \hat{l}_t}\right) \\ + \left(\tau_x e^{\hat{\tau}_{xt+1}}(1-\delta) c e^{\hat{c}_t} - \tau_x e^{\hat{\tau}_{xt+1}}(1-\delta)\chi l^{\nu} e^{\nu \hat{l}_t}\right) \end{array}\right]$$

$$= \beta\left[\begin{array}{c} \left(c\theta \frac{y}{k} e^{\hat{y}_{t+1} - \hat{k}_{t+1} + \hat{c}_t} - \chi l^{\nu} \theta \frac{y}{k} e^{\hat{y}_{t+1} - \hat{k}_{t+1} + \nu \hat{l}_t}\right) \\ + \left((1-\delta) c \tau_x e^{\hat{\tau}_{xt+1} + \hat{c}_t} - (1-\delta)\chi l^{\nu} \tau_x e^{\hat{\tau}_{xt+1} + \nu \hat{l}_t}\right) \end{array}\right]$$

$$= \beta\left[\begin{array}{c} c\theta \frac{y}{k}\left(1 + \hat{y}_{t+1} - \hat{k}_{t+1} + \hat{c}_t\right) - \chi l^{\nu} \theta \frac{y}{k}\left(1 + \hat{y}_{t+1} - \hat{k}_{t+1} + \nu \hat{l}_t\right) \\ +(1-\delta) c \tau_x \left(1 + \hat{\tau}_{xt+1} + \hat{c}_t\right) - (1-\delta)\chi l^{\nu} \tau_x \left(1 + \hat{\tau}_{xt+1} + \nu \hat{l}_t\right) \end{array}\right]$$

$$
= \beta \left[\begin{array}{c} c\theta\frac{y}{k}\left(1+\hat{y}_{t+1}-\hat{k}_{t+1}\right)+c\theta\frac{y}{k}\hat{c}_t-\chi l^{\nu}\theta\frac{y}{k}\left(1+\hat{y}_{t+1}-\hat{k}_{t+1}\right) \\ -\chi l^{\nu}\theta\frac{y}{k}\nu\hat{l}_t+(1-\delta)c\tau_x\left(1+\hat{\tau}_{xt+1}\right)+(1-\delta)c\tau_x\hat{c}_t \\ -(1-\delta)\chi l^{\nu}\tau_x\left(1+\hat{\tau}_{xt+1}\right)-(1-\delta)\tau_x\chi l^{\nu}\nu\hat{l}_t \end{array} \right]
$$

$$
= \left[\begin{array}{c} \beta\left(c-\chi l^{\nu}\right)\theta\frac{y}{k}\left(1+\hat{y}_{t+1}-\hat{k}_{t+1}\right)+\beta\left[\theta\frac{y}{k}+(1-\delta)\tau_x\right]c\hat{c}_t \\ -\beta(1-\delta)\left(c-\chi l^{\nu}\right)\tau_x\left(1+\hat{\tau}_{xt+1}\right)-\beta\left[\theta\frac{y}{k}+(1-\delta)\tau_x\right]\chi l^{\nu}\nu\hat{l}_t \end{array} \right]
$$

$$
= \left[\begin{array}{c} \beta\left(c-\chi l^{\nu}\right)\theta\frac{y}{k}\left(1+\hat{y}_{t+1}-\hat{k}_{t+1}\right)+\ \tau_x\Gamma c\hat{c}_t \\ -\beta(1-\delta)\left(c-\chi l^{\nu}\right)\tau_x\left(1+\hat{\tau}_{xt+1}\right)-\tau_x\Gamma\chi l^{\nu}\nu\hat{l}_t \end{array} \right]
$$

$$
= \beta\left(c-\chi l^{\nu}\right)\left[\theta\frac{y}{k}\left(1+\hat{y}_{t+1}-\hat{k}_{t+1}\right)+(1-\delta)\tau_x\left(1+\hat{\tau}_{xt+1}\right)\right]+\tau_x\Gamma c\hat{c}_t-\tau_x\Gamma\chi l^{\nu}\nu\hat{l}_t
$$

$$
= \beta\left(c-\chi l^{\nu}\right)\ \left[\begin{array}{c} \theta\frac{y}{k}+\theta\frac{y}{k}\hat{y}_{t+1}-\theta\frac{y}{k}\hat{k}_{t+1} \\ +(1-\delta)\tau_x+(1-\delta)\tau_x\hat{\tau}_{xt+1} \end{array} \right]+\tau_x\Gamma c\hat{c}_t-\tau_x\Gamma\chi l^{\nu}\nu\hat{l}_t
$$

$$= (c - \chi l^{\nu})\,\tau_x \Gamma + \beta\,(c - \chi l^{\nu})\,\left[\begin{array}{c} \theta \frac{y}{k}\hat{y}_{t+1} - \theta \frac{y}{k}\hat{k}_{t+1} \\ +(1-\delta)\tau_x \hat{\tau}_{xt+1} \end{array}\right] + \tau_x \Gamma c \hat{c}_t - \tau_x \Gamma \chi l^{\nu} \nu \hat{l}_t$$

$$= (c - \chi l^{\nu})\,\Gamma \tau_x + \beta \left(\begin{array}{c} (c - \chi l^{\nu})\,\theta \frac{y}{k}\hat{y}_{t+1} - (c - \chi l^{\nu})\,\theta \frac{y}{k}\hat{k}_{t+1} \\ +\,(c - \chi l^{\nu})\,(1-\delta)\tau_x \hat{\tau}_{xt+1} + \frac{\tau_x \Gamma}{\beta} c \hat{c}_t - \frac{\tau_x \Gamma}{\beta} \chi l^{\nu} \nu \hat{l}_t \end{array}\right)$$

- where $\tau_x \Gamma = \beta \left[\theta \frac{y}{k} + \tau_x (1-\delta)\right]$ and $\frac{\tau_x \Gamma}{\beta} = \theta \frac{y}{k} + \tau_x (1-\delta)$

- LHS=RHS

  $$(c - \chi l^{\nu})\,\tau_x \Gamma \hat{\tau}_{xt} + (c - \chi l^{\nu})\,\Gamma \tau_x + \tau_x \Gamma c \hat{c}_{t+1} - \tau_x \Gamma \chi l^{\nu} \nu \hat{l}_{t+1}$$

  $$= (c - \chi l^{\nu})\,\Gamma \tau_x + \beta \left(\begin{array}{c} (c - \chi l^{\nu})\,\theta \frac{y}{k}\hat{y}_{t+1} - (c - \chi l^{\nu})\,\theta \frac{y}{k}\hat{k}_{t+1} \\ +\,(c - \chi l^{\nu})\,(1-\delta)\tau_x \hat{\tau}_{xt+1} + \frac{\tau_x \Gamma}{\beta} c \hat{c}_t - \frac{\tau_x \Gamma}{\beta} \chi l^{\nu} \nu \hat{l}_t \end{array}\right)$$

$$
\begin{aligned}
\left(c-\chi l^{\nu}\right)\Gamma\tau_x\hat{\tau}_{xt} &= \beta\left(\begin{array}{c}\left(c-\chi l^{\nu}\right)\theta\frac{y}{k}\hat{y}_{t+1}-\left(c-\chi l^{\nu}\right)\theta\frac{y}{k}\hat{k}_{t+1}\\ +\left(c-\chi l^{\nu}\right)(1-\delta)\tau_x\hat{\tau}_{xt+1}+\frac{\tau_x\Gamma}{\beta}c\hat{c}_t-\frac{\tau_x\Gamma}{\beta}\chi l^{\nu}\nu\hat{l}_t\end{array}\right)\\
&\quad -\tau_x\Gamma c\hat{c}_{t+1}+\tau_x\Gamma\chi l^{\nu}\nu\hat{l}_{t+1}
\end{aligned}
$$

$$
\begin{aligned}
\Gamma\tau_x\hat{\tau}_{xt} &= \beta\left(\begin{array}{c}\theta\frac{y}{k}\hat{y}_{t+1}-\theta\frac{y}{k}\hat{k}_{t+1}+(1-\delta)\tau_x\hat{\tau}_{xt+1}\\ +\frac{\tau_x\Gamma}{\beta(c-\chi l^{\nu})}c\hat{c}_t-\frac{\tau_x\Gamma}{\beta(c-\chi l^{\nu})}\chi l^{\nu}\nu\hat{l}_t\end{array}\right)\\
&\quad -\frac{\tau_x\Gamma c}{\left(c-\chi l^{\nu}\right)}\hat{c}_{t+1}+\frac{\tau_x\Gamma\chi l^{\nu}\nu}{\left(c-\chi l^{\nu}\right)}\hat{l}_{t+1}
\end{aligned}
$$

$$
\begin{aligned}
\hat{\tau}_{xt} \quad = \quad & \frac{\beta\theta}{\Gamma\tau_x}\frac{y}{k}\hat{y}_{t+1} - \frac{\beta\theta}{\Gamma\tau_x}\frac{y}{k}\hat{k}_{t+1} + \frac{\beta(1-\delta)}{\Gamma}\hat{\tau}_{xt+1} + \frac{c}{(c-\chi l^{\nu})}\hat{c}_t - \frac{\chi l^{\nu}\nu}{(c-\chi l^{\nu})}\hat{l}_t \\
& -\frac{c}{(c-\chi l^{\nu})}\hat{c}_{t+1} + \frac{\chi l^{\nu}\nu}{(c-\chi l^{\nu})}\hat{l}_{t+1}
\end{aligned}
$$

- Add back expectation

$$
\begin{aligned}
\hat{\tau}_{xt} \quad = \quad & \frac{\beta\theta}{\Gamma\tau_x}\frac{y}{k}E_t\left(\hat{y}_{t+1} - \hat{k}_{t+1}\right) + \frac{c}{(c-\chi l^{\nu})}\left(\hat{c}_t - E_t\hat{c}_{t+1}\right) \\
& -\frac{\chi l^{\nu}\nu}{(c-\chi l^{\nu})}\left(\hat{l}_t - E_t\hat{l}_{t+1}\right) + \frac{\beta}{\Gamma}(1-\delta)E_t\hat{\tau}_{xt+1}
\end{aligned} \qquad \text{(4.c)}
$$

- equation (5)

$$\frac{1}{c_t - \chi l_t^{\nu}} \left[\frac{\Gamma}{R} \frac{1}{\tau_{dt}} - \phi(d_{t+1} - d) \right] = \beta E_t [u_{ct+1}]$$

$$\frac{1}{ce^{\hat{c}_t} - \chi l^{\nu} e^{\nu \hat{l}_t}} \left[\frac{\Gamma}{R} \frac{1}{\tau_d e^{\hat{\tau}_{dt}}} - \phi \left(\left(\hat{d}_{t+1} + 1 \right) d - d \right) \right] = \beta E_t \frac{1}{ce^{\hat{c}_{t+1}} - \chi l^{\nu} e^{\nu \hat{l}_{t+1}}}$$

- Ignore E_t for the moment.

$$\left(ce^{\hat{c}_{t+1}} - \chi l^{\nu} e^{\nu \hat{l}_{t+1}}\right)\left[\frac{\Gamma}{R}\frac{1}{\tau_d e^{\hat{\tau}_{dt}}} - \phi\left(\hat{d}_{t+1}d + d - d\right)\right] = \beta\left(ce^{\hat{c}_t} - \chi l^{\nu} e^{\nu \hat{l}_t}\right)$$

$$ce^{\hat{c}_{t+1}}\frac{\Gamma}{R}\frac{1}{\tau_d e^{\hat{\tau}_{dt}}} - ce^{\hat{c}_{t+1}}\phi d\hat{d}_{t+1} + \chi l^{\nu} e^{\nu \hat{l}_{t+1}}\frac{\Gamma}{R}\frac{1}{\tau_d e^{\hat{\tau}_{dt}}} + \chi l^{\nu} e^{\nu \hat{l}_{t+1}}\phi d\hat{d}_{t+1} = \beta\left(ce^{\hat{c}_t} - \chi l^{\nu} e^{\nu \hat{l}_t}\right)$$

- LHS

$$ce^{\hat{c}_{t+1}}\frac{\Gamma}{R}\frac{1}{\tau_d e^{\hat{\tau}_{dt}}} - ce^{\hat{c}_{t+1}}\phi d\hat{d}_{t+1} + \chi l^{\nu} e^{\nu \hat{l}_{t+1}}\frac{\Gamma}{R}\frac{1}{\tau_d e^{\hat{\tau}_{dt}}} + \chi l^{\nu} e^{\nu \hat{l}_{t+1}}\phi d\hat{d}_{t+1}$$

$$= \frac{c}{\tau_d}e^{\hat{c}_{t+1} - \hat{\tau}_{dt}}\frac{\Gamma}{R} - ce^{\hat{c}_{t+1}}\phi d\hat{d}_{t+1} + \frac{\chi l^{\nu}}{\tau_d}e^{\nu \hat{l}_{t+1} - \hat{\tau}_{dt}}\frac{\Gamma}{R} + \chi l^{\nu} e^{\nu \hat{l}_{t+1}}\phi d\hat{d}_{t+1}$$

$$= \left[\begin{array}{c} \frac{c}{\tau_d}\left(1 + \hat{c}_{t+1} - \hat{\tau}_{dt}\right)\frac{\Gamma}{R} - c\left(1 + \hat{c}_{t+1}\right)\phi d\hat{d}_{t+1} \\ -\frac{\chi l^{\nu}}{\tau_d}\left(1 + \nu \hat{l}_{t+1} - \hat{\tau}_{dt}\right)\frac{\Gamma}{R} + \chi l^{\nu}\left(1 + \nu \hat{l}_{t+1}\right)\phi d\hat{d}_{t+1} \end{array}\right]$$

$$= \left[\begin{array}{c} \frac{c}{\tau_d}\left(1+\hat{c}_{t+1}\right)\frac{\Gamma}{R} - \frac{c}{\tau_d}\hat{\tau}_{dt}\frac{\Gamma}{R} - c\left(1+\hat{c}_{t+1}\right)\phi\left(\hat{d}_{t+1}\right) \\ -\frac{\chi l^{\nu}}{\tau_d}\left(1+\nu\hat{l}_{t+1}\right)\frac{\Gamma}{R} + \frac{\chi l^{\nu}}{\tau_d}\frac{\Gamma}{R}\hat{\tau}_{dt} + \chi l^{\nu}\left(1+\nu\hat{l}_{t+1}\right)\phi d\hat{d}_{t+1} \end{array} \right]$$

where $\frac{\Gamma}{R}\frac{1}{\tau_d} = \beta$ or $\frac{R}{\Gamma}\tau_d = \frac{1}{\beta}$

$$= \beta c\left(1+\hat{c}_{t+1}\right) - \beta c\hat{\tau}_{dt} - c\phi d\hat{d}_{t+1} - \beta\chi l^{\nu}\left(1+\nu\hat{l}_{t+1}\right) + \beta\chi l^{\nu}\hat{\tau}_{dt} + \chi l^{\nu}\phi d\hat{d}_{t+1}$$

$$= \beta\left[c + c\hat{c}_{t+1} - c\hat{\tau}_{dt} - \frac{1}{\beta}c\phi d\hat{d}_{t+1} - \chi l^{\nu} - \chi l^{\nu}\nu\hat{l}_{t+1} + \chi l^{\nu}\hat{\tau}_{dt} + \frac{1}{\beta}\chi l^{\nu}\phi d\hat{d}_{t+1}\right]$$

$$= \beta\left[\left(c-\chi l^{\nu}\right) + c\hat{c}_{t+1} - \left(c-\chi l^{\nu}\right)\hat{\tau}_{dt} - \frac{1}{\beta}\left(c-\chi l^{\nu}\right)\phi d\hat{d}_{t+1} - \chi l^{\nu}\nu\hat{l}_{t+1}\right]$$

- RHS

$$\beta\left(ce^{\hat{c}_t}-\chi l^{\nu}e^{\nu\hat{l}_t}\right)$$

$$=\beta\left[c\left(1+\hat{c}_t\right)-\chi l^{\nu}\left(1+\nu\hat{l}_t\right)\right]$$

$$=\beta\left(c+c\hat{c}_t-\chi l^{\nu}-\chi l^{\nu}\nu\hat{l}_t\right)$$

- LHS=RHS

$$\beta\left[\begin{array}{c}\left(c-\chi l^{\nu}\right)+c\hat{c}_{t+1}-\left(c-\chi l^{\nu}\right)\hat{\tau}_{dt}\\-\frac{1}{\beta}\left(c-\chi l^{\nu}\right)\phi d\hat{d}_{t+1}-\chi l^{\nu}\nu\hat{l}_{t+1}\end{array}\right]=\beta\left(c+c\hat{c}_t-\chi l^{\nu}-\chi l^{\nu}\nu\hat{l}_t\right)$$

$$c\hat{c}_{t+1}-\left(c-\chi l^{\nu}\right)\hat{\tau}_{dt}-\frac{1}{\beta}\left(c-\chi l^{\nu}\right)\phi d\hat{d}_{t+1}-\chi l^{\nu}\nu\hat{l}_{t+1}=c\hat{c}_t-\chi l^{\nu}\nu\hat{l}_t$$

$$\left(c-\chi l^{\nu}\right)\hat{\tau}_{dt}=c\hat{c}_{t+1}-c\hat{c}_{t}-\chi l^{\nu}\nu\hat{l}_{t+1}+\chi l^{\nu}\nu\hat{l}_{t}-\tfrac{1}{\beta}\left(c-\chi l^{\nu}\right)\phi d\hat{d}_{t+1}$$

$$\hat{\tau}_{dt}=\tfrac{c}{\left(c-\chi l^{\nu}\right)}\left(\hat{c}_{t+1}-\hat{c}_{t}\right)-\tfrac{\chi l^{\nu}\nu}{\left(c-\chi l^{\nu}\right)}\left(\hat{l}_{t+1}-\hat{l}_{t}\right)-\tfrac{1}{\beta}\phi d\hat{d}_{t+1}$$

- Add back expectation.

$$\hat{\tau}_{dt}=\frac{c}{\left(c-\chi l^{\nu}\right)}\left(E_{t}\hat{c}_{t+1}-\hat{c}_{t}\right)-\frac{\chi l^{\nu}\nu}{\left(c-\chi l^{\nu}\right)}\left(E_{t}\hat{l}_{t+1}-\hat{l}_{t}\right)-\frac{1}{\beta}\phi dE_{t}\hat{d}_{t+1}\quad(5.c)$$

- equation (6)

$$\begin{aligned}
\Gamma k_{t+1} &= x_t + (1-\delta)k_t \\
\Gamma k e^{\hat{k}_{t+1}} &= x e^{\hat{x}_t} + (1-\delta)k e^{\hat{k}_t} \\
\Gamma k(1+\hat{k}_{t+1}) &= x(1+\hat{x}_t) + (1-\delta)k(1+\hat{k}_t) \\
\Gamma k\hat{k}_{t+1} &= x\hat{x}_t + (1-\delta)k\hat{k}_t
\end{aligned}$$

$$\Gamma \hat{k}_{t+1} = \frac{x}{k}\hat{x}_t + (1-\delta)\hat{k}_t \tag{6.c}$$

- We will solve the model by the linearizing the equations that characterize the competitive equilibrium around the steady state.

- Here again the log-linearized model :

$$\hat{y}_t = \frac{c}{y}\hat{c}_t + \frac{x}{y}\hat{x}_t + \frac{d}{y}\hat{d}_t - \frac{\Gamma d}{R\tau_d y}\hat{d}_{t+1} + \frac{\Gamma d}{R\tau_d y}\hat{\tau}_{dt} \tag{1.c}$$

$$\hat{y}_t = \hat{z}_t + \theta\hat{k}_t + (1-\theta)\hat{l}_t \tag{2.c}$$

$$\hat{\tau}_{lt} = \hat{z}_t + \theta\hat{k}_t + (1-\theta-\nu)\,\hat{l}_t \tag{3.c}$$

$$\begin{aligned}\hat{\tau}_{xt} \;&=\; \frac{\beta\theta}{\Gamma\tau_x}\frac{y}{k}E_t\left(\hat{y}_{t+1} - \hat{k}_{t+1}\right) + \frac{c}{(c-\chi l^{\nu})}\left(\hat{c}_t - E_t\hat{c}_{t+1}\right) \\ &\quad -\frac{\chi l^{\nu}\nu}{(c-\chi l^{\nu})}\left(\hat{l}_t - E_t\hat{l}_{t+1}\right) + \frac{\beta}{\Gamma}(1-\delta)E_t\hat{\tau}_{xt+1}\end{aligned} \tag{4.c}$$

$$\hat{\tau}_{dt} = \frac{c}{(c-\chi l^{\nu})}\left(E_t\hat{c}_{t+1} - \hat{c}_t\right) - \frac{\chi l^{\nu}\nu}{(c-\chi l^{\nu})}\left(E_t\hat{l}_{t+1} - \hat{l}_t\right) - \frac{1}{\beta}\phi d E_t\hat{d}_{t+1} \tag{5.c}$$

$$\Gamma \hat{k}_{t+1} = \frac{x}{k} \hat{x}_t + (1 - \delta) \hat{k}_t \tag{6.c}$$