Discrete Dynamic Optimization: Six Examples

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Example 1: the neoclassical growth model

- Uhlig (1999), section 4

- Model principles

- Specify the environment explicitly:

1. Preferences

2. Technologies

3. Endowments

4. Information

- State the object of study:

1. The social planner's problem

2. The competition equilibrium

3. The game

- The environment:

- Preferences: the representative agent experiences utility according to

$$U = E_t \left[\sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\eta} - 1}{1 - \eta} \right]$$

- $0 < \beta < 1$

- $\eta > 0$

- Absolute risk aversion: $-\frac{u''(c)}{u'(c)}$

- Relative risk aversion: $-\frac{cu''(c)}{u'(c)}$

- η is the coefficient of relative risk aversion

- Technologies: we assume a Cobb-Douglas production function

$$Y_t = Z_t K_t^{\rho} N_t^{1-\rho}$$

- Budget constraint:

$$K_{t+1} = I_t + (1 - \delta) K_t$$

$$C_t + I_t = Y_t$$

- or equivalently

$$C_t + K_{t+1} = Z_t K_t^{\rho} N_t^{1-\rho} + (1 - \delta) K_t$$

- $0 < \rho < 1$
- $0 < \delta < 1$
- Z_t, the total factor productivity, is exogenously evolving according to

$$\log Z_t = (1 - \psi) \log \bar{Z} + \psi \log Z_{t-1} + \epsilon_t$$

$$\epsilon_t \sim i.i.d.\mathcal{N}\left(0; \sigma^2\right)$$

- $0 < \psi < 1$

- Endowment:

- $N_t = 1$

- K_0

- Information: C_t, N_t, and K_{t+1} need to be chosen based on all information $\mathcal{I}_t$ up to time t.

- The social planner's problem

- The objective of the social planner is to maximize the utility of the representative agent subject to feasibility,

$$\max_{(C_t,K_t)_{t=0}^{\infty}} E_t \left[\sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\eta} - 1}{1-\eta} \right]$$

$$s.t. \qquad K_0, Z_0$$

$$C_t + K_{t+1} = Z_t K_t^{\rho} + (1-\delta) K_t$$

$$\log Z_t = (1 - \psi) \log \bar{Z} + \psi \log Z_{t-1} + \epsilon_t$$

$$\epsilon_t \sim i.i.d.\mathcal{N}\left(0; \sigma^2\right)$$

- To solve it, form the Lagrangian:

$$L = \max_{(C_t, K_{t+1})_{t=0}^{\infty}} E_t \sum_{t=0}^{\infty} \left[\beta^t \frac{C_t^{1-\eta} - 1}{1 - \eta} + \beta^t \lambda_t \left(Z_t K_t^{\rho} + (1 - \delta) K_t - C_t - K_{t+1}\right)\right]$$

- The first order conditions are:

$$\frac{\partial L}{\partial \lambda_t} : 0 = C_t + K_{t+1} - Z_t K_t^{\rho} - (1 - \delta) K_t$$

$$\frac{\partial L}{\partial C_t} : 0 = C_t^{-\eta} - \lambda_t$$

$$\frac{\partial L}{\partial K_{t+1}} : 0 = -\lambda_t + \beta E_t \left[\lambda_{t+1} \left(\rho Z_{t+1} K_{t+1}^{\rho - 1} + (1 - \delta) \right) \right]$$

- Notice that K_{t+1} appears in period t and period $t+1$.

- The first order conditions are often also called Euler equations.

- One also obtains the transversality condition.

$$0 = \lim_{T\to\infty} E_0 \left[\beta^T C_T^{-\eta} K_{T+1} \right]$$

- The transversality condition is a limiting Kuhn-Tucker condition.

- It means that in net present value terms, the agent should not have capital left over at infinity.

- Rewrite the necessary conditions:

$$C_t = Z_t K_t^{\rho} + (1 - \delta) K_t - K_{t+1} \tag{1}$$

$$R_t = \rho Z_t K_t^{\rho - 1} + (1 - \delta) \tag{2}$$

$$1 = E_t \left[\beta \left(\frac{C_t}{C_{t+1}} \right)^{\eta} R_{t+1} \right] \tag{3}$$

$$\log Z_t = (1 - \psi) \log \bar{Z} + \psi \log Z_{t-1} + \epsilon_t, \qquad \epsilon_t \sim i.i.d.\mathcal{N}\left(0; \sigma^2\right) \tag{4}$$

- Equation accounting.

- To solve for the steady state, dropping the time indices and yield

$$\bar{C} = \bar{Z}\bar{K}^{\rho} + (1 - \delta)\bar{K} - \bar{K}$$

$$\bar{R} = \rho\bar{Z}\bar{K}^{\rho-1} + (1 - \delta)$$

$$1 = \beta\bar{R}$$

- From the welfare theorems, the solution to the competitive equilibrium yields the same allocation as the solution to the social planner's problem.

- The case for competitive equilibrium is provided in Uhlig (1999) section 4.4.

Example 2: Hansen's real business cycle model

- Uhlig (1999), section 4

- Hansen's real business cycle model

- The model is an extension of the stochastic neoclassical growth model.

- The main difference is to endogenize the labour supply.

- The social planner solves the problem of the representative agent:

$$\max E_t \sum_{t=1}^{\infty} \beta^t \left(\frac{C_t^{1-\eta} - 1}{1-\eta} - AN_t \right)$$

$$s.t.$$

$$C_t + I_t = Y_t$$

$$K_{t+1} = I_t + (1-\delta) K_t$$

$$Y_t = Z_t K_t^{\rho} N_t^{1-\rho}$$

$$\log Z_t = (1-\psi)\log \bar{Z} + \psi \log Z_{t-1} + \epsilon_t, \qquad \epsilon_t \sim i.i.d.\mathcal{N}\left(0;\sigma^2\right)$$

- Hansen only consider the case $\eta = 1$, so that the objective function is:

$$E_t \sum_{t=1}^{\infty} \beta^t \left(\log C_t - A N_t\right)$$

- Proof:

$$
\begin{aligned}
& \lim_{\eta \longrightarrow 1} \frac{C_t^{1-\eta} - 1}{1-\eta} \\
= & \lim_{\eta \longrightarrow 1} \frac{e^{(1-\eta)\ln C_t} - 1}{1-\eta} = \lim_{\eta \longrightarrow 1} \frac{\partial \left[e^{(1-\eta)\ln C_t} - 1 \right] / \partial \eta}{\partial \left[1-\eta \right] / \partial \eta} \\
= & \lim_{\eta \longrightarrow 1} \frac{e^{(1-\eta)\ln C_t} \cdot (-\ln C_t)}{-1} = \frac{e^{0} \cdot (-\ln C_t)}{-1} = \ln C_t
\end{aligned}
$$

- To solve it, form the Lagrangian:

$$L = \max_{(C_t, N_t, K_{t+1})_{t=0}^{\infty}} E_t \sum_{t=0}^{\infty} \begin{bmatrix} \beta^t \left(\frac{C_t^{1-\eta} - 1}{1-\eta} - A N_t \right) \\ + \beta^t \lambda_t \left(Z_t K_t^{\rho} N_t^{1-\rho} + (1-\delta) K_t - C_t - K_{t+1} \right) \end{bmatrix}$$

- The first order conditions are:

$$\frac{\partial L}{\partial C_t} = C_t^{-\eta} - \lambda_t = 0$$

$$\frac{\partial L}{\partial N_t} = -A + \lambda_t (1-\rho) Z_t K_t^{\rho} N_t^{-\rho} = 0$$

$$\frac{\partial L}{\partial K_{t+1}} = -\lambda_t + \beta E_t \lambda_{t+1} \left(\rho Z_{t+1} K_{t+1}^{\rho-1} N_{t+1}^{1-\rho} + (1-\delta) \right) = 0$$

- After arrangement

$$A = C_t^{-\eta} (1-\rho) \frac{Y_t}{N_t} \tag{1}$$

$$R_t = \rho \frac{Y_t}{K_t} + 1 - \delta \tag{2}$$

$$1 = \beta E_t \left[\left(\frac{C_t}{C_{t+1}} \right)^{\eta} R_{t+1} \right] \tag{3}$$

- The steady state for the real business cycle model above is obtained by dropping the time subscripts and stochastic shocks in the equations above.

$$A = \bar{C}^{-\eta} (1 - \rho) \frac{\bar{Y}}{\bar{N}}$$

$$1 = \beta \bar{R}$$

$$\bar{R} = \rho \frac{\bar{Y}}{\bar{K}} + 1 - \delta$$

- Exercise: Equation accounting.

Example 3: Brock-Mirman growth model I

- Chow (1997), chapter 2

- Consider the Brock-Mirman growth model:

$$\max_{\{c_t\}} E_t \sum_{t=0}^{\infty} \beta^t \ln c_t$$

$$s.t.$$

$$k_{t+1} = k_t^{\alpha} z_t - c_t$$

- The Lagrangian is:

$$L = E_t \sum_{t=0}^{\infty} \left[\beta^t \ln c_t + \beta^t \lambda_t \left(k_t^{\alpha} z_t - c_t - k_{t+1} \right) \right]$$

- The first order conditions are:

$$\frac{\partial L}{\partial c_t} = \left(\beta^t \frac{1}{c_t} - \beta^t \lambda_t\right) = 0$$

$$\frac{\partial L}{\partial k_{t+1}} = -\beta^t \lambda_t + \beta^{t+1} E_t \left(\lambda_{t+1} \alpha k_{t+1}^{\alpha-1} z_{t+1}\right) = 0$$

- The first order conditions can be rearranged as:

$$\frac{1}{c_t} = \lambda_t$$

$$\lambda_t = \alpha\beta E_t \left(\lambda_{t+1} k_{t+1}^{\alpha-1} z_{t+1}\right)$$

- Solve the problem explicitly.
- $k_{T+1} = 0$
- $c_T = k_T^{\alpha} z_T$
- $c_t = d \cdot k_t^{\alpha} z_t$ (guess a solution)

$$
\begin{aligned}
E_t\left(\lambda_{t+1}k_{t+1}^{\alpha-1}z_{t+1}\right) &= E_t\left(\frac{1}{c_{t+1}}k_{t+1}^{\alpha-1}z_{t+1}\right)\\
&= E_t\left(\frac{1}{d\cdot k_{t+1}^{\alpha}z_{t+1}}k_{t+1}^{\alpha-1}z_{t+1}\right)\\
&= E_t\left(\frac{1}{d}\frac{1}{k_{t+1}}\right)=\frac{1}{d}\frac{1}{k_t^{\alpha}z_t-c_t}
\end{aligned}
$$

$$
\lambda_t=\alpha\beta E_t\left(\lambda_{t+1}k_{t+1}^{\alpha-1}z_{t+1}\right)\Leftrightarrow\frac{1}{c_t}=\alpha\beta\frac{1}{d}\frac{1}{k_t^{\alpha}z_t-c_t}
$$

$$
c_t=\frac{d}{\alpha\beta}\left(k_t^{\alpha}z_t-c_t\right)
$$

$$d \cdot k_t^{\alpha} z_t = \frac{d}{\alpha\beta} \left(k_t^{\alpha} z_t - d \cdot k_t^{\alpha} z_t \right)$$

$$1 = \frac{1}{\alpha\beta} (1 - d)$$

$$d = 1 - \alpha\beta$$

$$\therefore c_t = d \cdot k_t^{\alpha} z_t = (1 - \alpha\beta)\, k_t^{\alpha} z_t$$

Example 4: Brock-Mirman growth model II

- Chow (1997), chapter 2

- We shall use dynamic programming to solve the Brock-Mirman growth model.

- Consider the Brock-Mirman growth model:

$$\max_{\{c_t\}} E_t \sum_{t=0}^{\infty} \beta^t \ln c_t$$

$$s.t.$$

$$k_{t+1} = k_t^\alpha z_t - c_t$$

- Control variable: c_t

- State variables: k_t, z_t

- The Bellman equation is:

$$V(k_t, z_t) = \max_{c_t} \left[\ln c_t + \beta E_t V(k_{t+1}, z_{t+1})\right]$$

- We conjecture value function takes the form:

$$V(k_t, z_t) = a + b \ln k_t + c \ln z_t$$

- a, b, c are coefficients to be determined.
- $E_t V(k_{t+1}, z_{t+1})$

$$
\begin{aligned}
V(k_{t+1}, z_{t+1}) &= a + b \ln k_{t+1} + c \ln z_{t+1} \\
&= a + b \ln (k_t^\alpha z_t - c_t) + c \ln z_{t+1}
\end{aligned}
$$

$$
\because E_t \ln z_{t+1} = 0
$$

$$
\therefore E_t V(k_{t+1}, z_{t+1}) = a + b \ln (k_t^\alpha z_t - c_t)
$$

- It follows

$$
\begin{aligned}
V(k_t, z_t) &= \max_{c_t} \left[\ln c_t + \beta E_t V(k_{t+1}, z_{t+1})\right] \\
&= \max_{c_t} \left\{\ln c_t + \beta \left[a + b \ln \left(k_t^{\alpha} z_t - c_t\right)\right]\right\} \\
&\equiv \max_{c_t} \{\}
\end{aligned}
$$

- First order condition.

$$
\frac{\partial \{\}}{\partial c_t} = \frac{1}{c_t} - \frac{\beta b}{k_t^{\alpha} z_t - c_t} = 0
$$

$$\Rightarrow c_t = \frac{k_t^{\alpha} z_t}{1 + \beta b}$$

- Substitute this optimal $c_t = \frac{k_t^{\alpha} z_t}{1+\beta b}$ into $\max_{c_t} \{\}$ and equate the expression to the value function, we can solve the coefficients a, b, c, which after some algebraic manipulation are:

$$a = (1 - \beta)^{-1} \left[\ln(1 - \alpha\beta) + \beta\alpha (1 - \alpha\beta)^{-1} \ln(\alpha\beta) \right]$$

$$b = \alpha (1 - \alpha\beta)^{-1}$$

$$c = (1 - \alpha\beta)^{-1}$$

Example 5: the money-in-the-utility function (MIU) model

- Walsh (2003), chapter 2

- Utility function, labour supply, and production function.

$$u\left(c_t, m_t, l_t\right) = \frac{C_t^{1-\Phi}}{1-\Phi} + \Psi \frac{l_t^{1-\eta}}{1-\eta}$$

$$C_t \equiv \left[a c_t^{1-b} + (1-a) m_t^{1-b} \right]^{\frac{1}{1-b}}$$

$$n_t = 1 - l_t$$

$$y_t = e^{z_t} k_t^{\alpha} n_t^{1-\alpha} = f(k_t, n_t, z_t)$$

$$z_t = \rho z_{t-1} + e_t$$

- Two state variables.

$$a_t \equiv \tau_t + \frac{(1 + i_{t-1})\, b_{t-1}}{1 + \pi_t} + \frac{m_{t-1}}{1 + \pi_t}$$

$$k_t$$

- The objective function is:

$$E_t \sum_{t=0}^{\infty} \beta^t u\left(c_t, m_{t,} 1 - n_t\right)$$

$$s.t.$$

$$f(k_t, n_t, z_t) + (1 - \delta) k_t + a_t = c_t + k_{t+1} + m_t + b_t$$

- The Bellman equation is:

$$V\left(a_t, k_t\right) = \max\left[u\left(c_t, m_{t,} 1 - n_t\right) + \beta E_t V\left(a_{t+1}, k_{t+1}\right)\right]$$

- Use the definition of a_t to substitute for a_{t+1}.

$$a_{t+1} \equiv \tau_{t+1} + \frac{\left(1 + i_t\right) b_t}{1 + \pi_{t+1}} + \frac{m_t}{1 + \pi_{t+1}}$$

- Use the budget constraint to eliminate k_{t+1}.

$$k_{t+1} = f(k_t, n_t, z_t) + (1-\delta) k_t + a_t - c_t - m_t - b_t$$

- Rewrite the Bellman equation as:

$$\begin{aligned}
V(a_t, k_t) &= \max \left[u\left(c_t, m_{t,} 1 - n_t\right) + \beta E_t V(a_{t+1}, k_{t+1}) \right] \\
&= \max \left[\begin{array}{c} u\left(c_t, m_{t,} 1 - n_t\right) \\ +\beta E_t V \left(\begin{array}{c} \tau_{t+1} + \frac{(1+i_t) b_t}{1+\pi_{t+1}} + \frac{m_t}{1+\pi_{t+1}}, f(k_t, n_t, z_t) \\ + (1-\delta) k_t + a_t - c_t - m_t - b_t \end{array} \right) \end{array} \right] \\
&= \max_{c_t, b_t, m_t, n_t} \{\}
\end{aligned}$$

- The first order conditions are:

$$\frac{\partial \{\}}{\partial c_t} : u_c\left(c_t, m_{t,} 1 - n_t\right) - \beta E_t V_k\left(a_{t+1}, k_{t+1}\right) = 0$$

$$\frac{\partial \{\}}{\partial b_t} : \beta E_t\left(\frac{1 + i_t}{1 + \pi_{t+1}}\right) V_a\left(a_{t+1}, k_{t+1}\right) - \beta E_t V_k\left(a_{t+1}, k_{t+1}\right) = 0$$

$$\frac{\partial \{\}}{\partial m_t} : u_m\left(c_t, m_{t,} 1 - n_t\right) + \beta E_t\left(\frac{1}{1 + \pi_{t+1}}\right) V_a\left(a_{t+1}, k_{t+1}\right) - \beta E_t V_k\left(a_{t+1}, k_{t+1}\right) = 0$$

$$\frac{\partial \{\}}{\partial n_t} : -u_n\left(c_t, m_{t,} 1 - n_t\right) + \beta E_t V_k\left(a_{t+1}, k_{t+1}\right) f_n\left(k_t, n_t, z_t\right) = 0$$

$$\frac{\partial V\left(a_t, k_t\right)}{\partial a_t} = \frac{\partial \left\{\right\}}{\partial a_t} : V_a\left(a_t, k_t\right) = \beta E_t V_k\left(a_{t+1}, k_{t+1}\right)$$

$$\frac{\partial V\left(a_t, k_t\right)}{\partial k_t} = \frac{\partial \left\{\right\}}{\partial k_t} : V_k\left(a_t, k_t\right) = \beta E_t V_k\left(a_{t+1}, k_{t+1}\right)\left[f_k\left(k_t, n_t, z_t\right) + 1 - \delta\right]$$

- Resources constrains.

$$y_t + (1 - \delta)\, k_t = c_t + k_{t+1}$$

- We can eliminate the partial differentiations of value function in the F.O.C. and then simplify the equilibrium conditions into 9 equations with 9 endogenous variables to be determined.

- Equilibrium conditions include F.O.C., definitions, and resources constraints.

- $y_t, c_t, k_{t+1}, m_t, n_t, R_t, \pi_t, z_t, u_t$

$$u_c\left(c_t, m_{t,} 1 - n_t\right) = u_m\left(c_t, m_{t,} 1 - n_t\right) + \beta E_t\left[\frac{u_c\left(c_{t+1}, m_{t+1,} 1 - n_{t+1}\right)}{1 + \pi_{t+1}}\right] \tag{1}$$

$$u_l\left(c_t, m_{t,}1 - n_t\right) = u_c\left(c_t, m_{t,}1 - n_t\right) f_n\left(k_t, n_t, z_t\right) \tag{2}$$

$$u_c\left(c_t, m_{t,}1 - n_t\right) = \beta E_t R_t u_c\left(c_{t+1}, m_{t+1,}1 - n_{t+1}\right) \tag{3}$$

$$R_t = 1 - \delta + f_k\left(k_t, n_t, z_t\right) \tag{4}$$

$$y_t + \left(1 - \delta\right) k_t = c_t + k_{t+1} \tag{5}$$

$$y_t = f\left(k_t, n_t, z_t\right) \tag{6}$$

$$m_t = \left(\frac{1+\theta_t}{1+\pi_t}\right) m_{t-1} \tag{7}$$

$$z_t = \rho z_{t-1} + e_t \tag{8}$$

$$u_t = \gamma u_{t-1} + \phi z_{t-1} + \varphi_t \tag{9}$$

- Exercises: use Lagrangian to solve the MIU model.

$$
\begin{aligned}
L \;=\; & \sum_{t=0}^{\infty} \beta^{t} u\left(c_t, m_{t,} 1 - n_t\right) \\
& + \sum_{t=0}^{\infty} \beta^{t+1} E_t \lambda_{t+1} \left[\begin{array}{c} f\left(k_t, n_t, z_t\right) + \left(1-\delta\right) k_t + \tau_t + \frac{(1+i_{t-1})b_{t-1}}{1+\pi_t} + \frac{m_{t-1}}{1+\pi_t} \\ -c_t - k_{t+1} - m_t - b_t \end{array} \right]
\end{aligned}
$$

- $\frac{\partial L}{\partial c_t}, \frac{\partial L}{\partial m_t}, \frac{\partial L}{\partial n_t}, \frac{\partial L}{\partial k_{t+1}}, \frac{\partial L}{\partial b_t}$

- Show that combing the F.O.C. of the Lagrangian method gives us the same equations as by using the Bellman equation.

Example 6: the cash-in-advance (CIA) model

- Walsh (2003), chapter 3, a certainty case

- Utility function:

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

- CIA constraint:

$$P_t c_t \leq M_{t-1} + T_t$$

- CIA constraint in real terms:

$$c_t \leq \frac{m_{t-1}}{\Pi_t} + \tau_t$$

$$\Pi_t \equiv \frac{P_t}{P_{t-1}}, \qquad \tau_t \equiv \frac{T_t}{P_t}, \qquad I_t = 1 + i_t$$

- The budget constraint in real terms:

$$f(k_t) + (1-\delta)k_t + \tau_t + \frac{m_{t-1} + I_{t-1}b_{t-1}}{\Pi_t} \geq c_t + k_{t+1} + m_t + b_t$$

$$w_t \equiv f(k_t) + (1-\delta)k_t + \tau_t + \frac{m_{t-1} + I_{t-1}b_{t-1}}{\Pi_t}$$

- The budget constraint can be rewritten as:

$$w_t \geq c_t + k_{t+1} + m_t + b_t$$

- Solve the problem
- Two state variables

$$w_t$$

$$m_{t-1}$$

- The Bellman equation is:

$$V\left(w_t, m_{t-1}\right) = \max\left[u\left(c_t\right) + \beta V\left(w_{t+1}, m_t\right)\right]$$

$$s.t.$$

$$w_t \geq c_t + k_{t+1} + m_t + b_t$$

$$c_t \leq \frac{m_{t-1}}{\Pi_t} + \tau_t$$

$$w_t = f(k_t) + (1-\delta) k_t + \tau_t + \frac{m_{t-1} + I_{t-1} b_{t-1}}{\Pi_t}$$

- To solve the problem, first use the definition of w_t to eliminate w_{t+1} at the RHS of Bellman equation:

$$V(w_t, m_{t-1}) = \max \left[u(c_t) + \beta V \left(f(k_{t+1}) + (1-\delta) k_{t+1} + \tau_{t+1} + \frac{m_t + I_t b_t}{\Pi_{t+1}}, m_t \right) \right]$$

- Secondly, from the Lagrangian:

$$
\begin{aligned}
L &= \max \left[u\left(c_t\right) + \beta V \left(f\left(k_{t+1}\right) + \left(1-\delta\right) k_{t+1} + \tau_{t+1} + \frac{m_t + I_t b_t}{\Pi_{t+1}}, m_t \right) \right] \\
& + \lambda_t \left(w_t - c_t - k_{t+1} - m_t - b_t \right) \\
& + \mu_t \left(\frac{m_{t-1}}{\Pi_t} + \tau_t - c_t \right)
\end{aligned}
$$

- The first order conditions are:

$$
\frac{\partial L}{\partial c_t} : u_c\left(c_t\right) - \lambda_t - \mu_t = 0
$$

$$\frac{\partial L}{\partial k_{t+1}} : \beta V_w \left(w_{t+1}, m_t \right) \left[f_k \left(k_{t+1} \right) + 1 - \delta \right] - \lambda_t = 0$$

$$\frac{\partial L}{\partial b_t} : \beta V_w \left(w_{t+1}, m_t \right) R_t - \lambda_t = 0$$

$$R_t \equiv \frac{I_t}{\Pi_{t+1}}$$

$$\frac{\partial L}{\partial m_t} : \beta V_w \left(w_{t+1}, m_t \right) \frac{1}{\Pi_{t+1}} + \beta V_m \left(w_{t+1}, m_t \right) - \lambda_t = 0$$

$$\frac{1}{\Pi_{t+1}} = R_t - \frac{i_t}{\Pi_{t+1}}$$

- Use envelope theorem to eliminate the partial differentiations of value function in the first order conditions.

- Envelope theorem:

$$\frac{dV}{d\theta} = \frac{\partial L}{\partial \theta} \equiv L_\theta$$

- For details description of envelope theorem, see Dixit, Avinash K. (1990), *Optimization in Economic Theory*, Second Edition, Oxford University Press.

$$V_w\left(w_t, m_{t-1}\right) = L_w = \lambda_t$$

$$V_m\left(w_t, m_{t-1}\right) = L_m = \frac{\mu_t}{\Pi_t}$$

- Rearrange the first order conditions as:

$$u_c\left(c_t\right) - \lambda_t - \mu_t = 0 \tag{1}$$

$$\beta \lambda_{t+1}\left[f_k\left(k_{t+1}\right) + 1 - \delta\right] - \lambda_t = 0 \tag{2}$$

$$\beta \lambda_{t+1} R_t - \lambda_t = 0 \tag{3}$$

$$R_t \equiv \frac{I_t}{\Pi_{t+1}} \tag{4}$$

$$\beta \lambda_{t+1} \frac{1}{\Pi_{t+1}} + \beta \frac{\mu_{t+1}}{\Pi_{t+1}} - \lambda_t = 0 \tag{5}$$

$$\frac{1}{\Pi_{t+1}} = R_t - \frac{i_t}{\Pi_{t+1}} \tag{6}$$

- Exercises: use Lagrangian method to solve the CIA model.

$$
\begin{aligned}
L &= \sum_{t=0}^{\infty} \beta^t u(c_t) \\
&+ \sum_{t=0}^{\infty} \beta^t \lambda_t \left(\begin{array}{c} f(k_t) + (1-\delta) k_t + \tau_t + \frac{m_{t-1} + I_{t-1} b_{t-1}}{\Pi_t} \\ -c_t - k_{t+1} - m_t - b_t \end{array} \right) \\
&+ \sum_{t=0}^{\infty} \beta^t \mu_t \left(\frac{m_{t-1}}{\Pi_t} + \tau_t - c_t \right)
\end{aligned}
$$

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