Log-linearization of Equilibrium Conditions

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1 Principles

- The principle of log-linearization is to use Taylor expansion around the steady state to replace all equations by approximations.

- The approximations are linear functions in the log-deviations of the variables.

- X_t :vector of variables

- $\bar{X}$:steady state

- x_t :vector of log-deviations

$$x_t = \log X_t - \log \bar{X}$$

- The vector $100 \cdot x_t$ tell us, by how much the variables differ from their steady state levels in period t in percent.

- Write the necessary equations characterizing the equilibrium as:

$$1 = f\left(x_t, x_{t-1}\right)$$
$$1 = E_t\left[g\left(x_{t+1}, x_t\right)\right]$$

- Equilibrium implies

$$f(0,0) = 1$$
$$g(0,0) = 1$$

- Taking first-order approximation around $(x_t, x_{t-1}) = (0, 0)$ yields

$$0 \approx f_1 x_t + f_2 x_{t-1}$$
$$0 \approx E_t \left[g_1 x_{t+1} + g_2 x_t \right]$$

- One obtains a linear system in x_t and x_{t-1} in the deterministic equations and x_{t+1} and x_t in the expectational equations.

2 Techniques

- There is no need to differentiate the functions f and g explicitly.

- The following are the techniques.

$$X_t = \bar{X} e^{x_t} \approx \bar{X} (1 + x_t)$$

$$e^{x_t + a y_t} \approx 1 + x_t + a y_t$$

$$x_t y_t \approx 0$$

$$E_t\left[ae^{x_{t+1}}\right] \approx E_t\left[ax_{t+1}\right] \qquad \text{up to a constant}$$

- For examples

$$e^{x_t} \approx 1 + x_t$$

$$aX_t \approx a\bar{X}x_t \qquad \text{up to a constant}$$

$$(X_t + a)\, Y_t \approx \bar{X}\bar{Y}x_t + \left(\bar{X} + a\right)\bar{Y}y_t \qquad \text{up to a constant}$$

1. Multiply out everything before log-linearization.

2. Constants drop out of each equation in the end, since they satisfy steady state relationships.

Example 1: the neoclassical growth model

- Lower case letters denote log deviations from respective steady-state values while capital letters denote levels.

- The necessary conditions:

$$C_t = Z_t K_t^{\rho} + (1 - \delta) K_t - K_{t+1} \tag{1}$$

$$R_t = \rho Z_t K_t^{\rho - 1} + (1 - \rho) \tag{2}$$

$$1 = E_t \left[\beta \left(\frac{C_t}{C_{t+1}} \right)^{\eta} R_{t+1} \right] \tag{3}$$

$$\log Z_t = (1 - \psi) \log \bar{Z} + \psi \log Z_{t-1} + \epsilon_t, \qquad \epsilon_t \sim i.i.d. \mathcal{N} \left(0; \sigma^2 \right) \tag{4}$$

- The steady state:

$$\bar{C} = \bar{Z} \bar{K}^{\rho} + (1 - \delta) \bar{K} - \bar{K} = \bar{Z} \bar{K}^{\rho} - \delta \bar{K}$$

$$\bar{R} = \rho \bar{Z} \bar{K}^{\rho-1} + (1 - \delta)$$

$$1 = \beta \bar{R}$$

- Log-linearization of the necessary conditions

$$c_t = \frac{\bar{Y}}{\bar{C}} z_t + \frac{\bar{K}}{\beta \bar{C}} k_t - \frac{\bar{K}}{\bar{C}} k_{t+1}$$

$$r_t = (1 - \beta(1 - \delta))(z_t - (1 - \rho) k_t)$$

$$0 = E_t [\eta (c_t - c_{t+1}) + r_{t+1}]$$

$$z_t = \psi z_{t-1} + \epsilon_t$$

- Details 1.

$$C_t = Z_t K_t^{\rho} + (1 - \delta) K_t - K_{t+1}$$

$$\overline{C} e^{c_t} = \overline{ZK}^{\rho} e^{z_t + \rho k_t} + (1 - \delta) \overline{K} e^{k_t} - \overline{K} e^{k_{t+1}}$$

$$\overline{C} + \overline{C} c_t \approx \overline{ZK}^{\rho} + (1 - \delta) \overline{K} - \overline{K} + \overline{ZK}^{\rho} (z_t + \rho k_t) + (1 - \delta) \overline{K} k_t - \overline{K} k_{t+1}$$

$$\overline{Y} = \overline{ZK}^{\rho}$$

$$\overline{C} = \overline{Y} - \delta \overline{K}$$

$$\overline{C} c_t \approx \overline{ZK}^{\rho} (z_t + \rho k_t) + (1 - \delta) \overline{K} k_t - \overline{K} k_{t+1}$$

$$c_t \approx \frac{\overline{Y}}{\overline{C}} z_t + \frac{\overline{K}}{\overline{C}} \overline{R} k_t - \frac{\overline{K}}{\overline{C}} k_{t+1}$$

- Details 2.

$$R_t = \rho Z_t K_t^{\rho-1} + 1 - \delta$$

$$\overline{R} e^{r_t} = \rho \overline{ZK}^{\rho-1} e^{z_t + (\rho-1) k_t} + 1 - \delta$$

$$\overline{R} + \overline{R} r_t \approx \rho \overline{ZK}^{\rho-1} + 1 - \delta + \rho \overline{ZK}^{\rho-1} \left(z_t + (\rho - 1) \, k_t \right)$$

$$\frac{1}{\beta} = \overline{R} = \rho \overline{ZK}^{\rho} + 1 - \delta$$

$$\overline{R}r_t \approx \rho\overline{ZK}^{\rho-1}\left(z_t + (\rho-1)\,k_t\right)$$

$$r_t \approx \left(1-\beta\left(1-\delta\right)\right)\left(z_t - \left(1-\rho\right)k_t\right)$$

- Details 3.

$$1 = E_t\left[\beta\left(\frac{C_t}{C_{t+1}}\right)^{\eta} R_{t+1}\right]$$

$$1 = E_t \left[\beta \left(\frac{\overline{C} e^{c_t - c_{t-1}}}{\overline{C}} \right)^{\eta} \overline{R} e^{r_{t+1}} \right]$$

$$1 \approx E_t \left[\beta \overline{R} + \beta \overline{R} \left(\eta \left(c_t - c_{t+1} \right) + r_{t+1} \right) \right]$$

$$1 = \beta \overline{R}$$

$$0 \approx E_t \left[\eta \left(c_t - c_{t+1} \right) + r_{t+1} \right]$$

- Details 4.

$$\log Z_t = (1-\psi)\log \overline{Z} + \psi \log Z_{t-1} + \epsilon_t$$

$$\log\left(\overline{Z}e^{z_t}\right) = (1-\psi)\log \overline{Z} + \psi \log\left(\overline{Z}e^{z_t - 1}\right) + \epsilon_t$$

$$z_t = \psi z_{t-1} + \epsilon_t$$

Example 2: Hansen's real business cycle model

- The first order conditions are:

$$C_t + I_t = Y_t \tag{1}$$

$$K_{t+1} = I_t + (1 - \delta) K_t \tag{2}$$

$$Y_t = Z_t K_t^{\rho} N_t^{1-\rho} \tag{3}$$

$$\log Z_t = (1-\psi)\log \bar{Z} + \psi \log Z_{t-1} + \epsilon_t, \qquad \epsilon_t \sim i.i.d.\mathcal{N}\left(0;\sigma^2\right) \tag{4}$$

$$A = C_t^{-\eta}(1-\rho)\frac{Y_t}{N_t} \tag{5}$$

$$1 = \beta E_t\left[\left(\frac{C_t}{C_{t+1}}\right)^{\eta} R_{t+1}\right] \tag{6}$$

$$R_t = \rho\frac{Y_t}{K_t} + 1 - \delta \tag{7}$$

- The steady state for the real business cycle model above is obtained by dropping the time subscripts and stochastic shocks in the equations above.

$$A = \bar{C}^{-\eta}(1-\rho)\frac{\bar{Y}}{\bar{N}}$$

$$1 = \beta\bar{R}$$

$$\bar{R} = \rho\frac{\bar{Y}}{\bar{K}} + 1 - \delta$$

- Log-linearization of the necessary conditions

$$\bar{C}c_t + \bar{I}i_t = \bar{Y}y_t$$

$$\bar{K}k_{t+1} = \bar{I}i_t + (1-\delta)\,\bar{K}k_t$$

$$y_t = z_t + \rho k_t + (1-\rho)\,n_t$$

$$z_t = \psi z_{t-1} + \epsilon_t$$

$$0 = -\eta c_t + y_t - n_t$$

$$0 = E_t \left[\eta \left(c_t - c_{t+1} \right) + r_{t+1} \right]$$

$$\bar{R} r_t = \rho \frac{\bar{Y}}{\bar{K}} \left(y_t - k_t \right)$$

- Details 1.

$$C_t + I_t = Y_t$$

$$\bar{C}e^{c_t} + \bar{I}e^{i_t} = \bar{Y}e^{y_t}$$

$$\bar{C}\left(1 + c_t\right) + \bar{I}\left(1 + i_t\right) = \bar{Y}\left(1 + y_t\right)$$

$$\bar{C}c_t + \bar{I}i_t = \bar{Y}y_t$$

- Details 2.

$$K_{t+1} = I_t + (1 - \delta) K_t$$

$$\bar{K}e^{k_{t+1}} = \bar{I}e^{i_t} + (1 - \delta) \bar{K}e^{k_t}$$

$$\bar{K}\left(1+k_{t+1}\right)=\bar{I}\left(1+i_t\right)+\left(1-\delta\right)\bar{K}\left(1+k_t\right)$$

$$\bar{K}k_{t+1}=\bar{I}i_t+\left(1-\delta\right)\bar{K}k_t$$

- Details 3.

$$Y_t=Z_tK_t^{\rho}N_t^{1-\rho}$$

$$\bar{Y}e^{y_t} = \bar{Z}\bar{K}^{\rho}\bar{N}^{1-\rho} \cdot e^{z_t+\rho k_t+(1-\rho)n_t}$$

$$e^{y_t} = e^{z_t+\rho k_t+(1-\rho)n_t}$$

$$y_t = z_t + \rho k_t + (1-\rho)\, n_t$$

- Details 4.

$$\log Z_t = (1-\psi)\log \bar{Z} + \psi \log Z_{t-1} + \epsilon_t$$

$$\log\left(\bar{Z}e^{z_t}\right) = (1-\psi)\log \bar{Z} + \psi \log\left(\bar{Z}e^{z_t-1}\right) + \epsilon_t$$

$$z_t = \psi z_{t-1} + \epsilon_t$$

- Details 5.

$$A = C_t^{-\eta} (1 - \rho) \frac{Y_t}{N_t}$$

$$A = \bar{C} e^{-\eta c_t} (1 - \rho) \frac{\bar{Y}}{\bar{N}} e^{y_t - n_t}$$

$$1 = e^{-\eta c_t} e^{y_t - n_t}$$

$$0 = -\eta c_t + y_t - n_t$$

- Details 6.

$$1 = \beta E_t \left[\left(\frac{C_t}{C_{t+1}} \right)^{\eta} R_{t+1} \right]$$

$$1 = \beta E_t \left[\left(\frac{\bar{C}}{\bar{C}} e^{c_t - c_{t+1}} \right)^{\eta} \bar{R} e^{r_{t+1}} \right]$$

$$1 = E_t \left[e^{\eta (c_t - c_{t+1})} e^{r_{t+1}} \right]$$

$$0 = E_t \left[\eta \left(c_t - c_{t+1} \right) + r_{t+1} \right]$$

- Details 7.

$$R_t = \rho \frac{Y_t}{K_t} + 1 - \delta$$

$$\bar{R} e^{r_t} = \rho \frac{\bar{Y}}{\bar{K}} e^{y_t - k_t} + 1 - \delta$$

$$\bar{R}\left(1+r_t\right)=\rho\frac{\bar{Y}}{\bar{K}}\left(1+y_t-k_t\right)+1-\delta$$

$$\bar{R}r_t=\rho\frac{\bar{Y}}{\bar{K}}\left(y_t-k_t\right)$$

References

[1] Uhlig, Harald (1999), "A Toolkit for Analyzing Nonlinear Dynamic Stochastic Models Easily," in Ramon Marimon and Andrew Scott (eds.), *Computational Methods for the Study of Dynamic Economies*, Oxford University Press.