Solving Linear Rational Expectation Models

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1 The problem

- Approaches to solve linear rational expectation models include Sims (2002), Anderson and Moore (1985), Binder and Pesaran (1994), King and Watson (1998), Klein (2000), and Uhlig (1999).

- A recent view is Anderson (2008), who compares the accuracy and computational speed of alternative approaches to solving linear rational expectations models.

- Martin Uribe's *Lectures in Open Economy Macroeconomics*, Appendix of Chapter 4, provides a very clear explanation of the linear solution method to dynamic general equilibrium models.

Fabrice Collard's lecture note. The latter contains many typos and I have tried my best to make they right in this document.

- Martin Uribe's Lectures are available from his website.
- McCandless, George (2008), *The ABCs of RBCs: An Introduction to Dynamic Macroeconomic Models*, Harvard University Press.
- This book provides an detailing introduction to solving dynamic stochastic general equilibrium model.
- It is very practical for beginners as the author explains the deduction step by step, and the book includes many examples and solutions that facilitate learning.
- The author uses first-order approximation to the model, and adopts Uhlig's toolkits and related computer programs to solve the log-linearized model.

- Klein (2000) uses a complex generalized Schur decomposition to solve linear rational expectation models.

- Why generalized Schur decomposition?

- First, it treats infinite and finite unstable eigenvalues in a unified way.

- Second, Schur decomposition is computationally more preferable.

- Setting the stage

- Measurement equation, which describe variables of interest, such as output or gross interest rate.

$$N_y Y_t = N_x X_t + N_z Z_t \tag{1}$$

- Endogenous variables

$$M_{x0} E_t X_{t+1} + M_{y0} E_t Y_{t+1} + M_{z0} E_t Z_{t+1} = M_{x1} X_t + M_{y1} Y_t + M_{z1} Z_t \tag{2}$$

- Exogenous shocks (forcing variables)

$$Z_t = \Phi Z_{t-1} + \Psi \epsilon_t \tag{3}$$

- Dimension of variables

$$\begin{aligned} Y_t &: n_y \times 1 \\ X_t &: n_x \times 1 \\ Z_t &: n_z \times 1 \end{aligned}$$

- Predetermined variables X_t^b : n_b

- Jump (control) variables X_t^f : n_f

- $n_x = n_b + n_f$

$$X_t = \binom{X_t^b}{X_t^f}$$

- Dimension of matrices

N_y	N_x	N_z
$(n_y \times n_y)$	$(n_y \times n_x)$	$(n_y \times n_z)$
M_{x0}	M_{y0}	M_{z0}
$(n_x \times n_x)$	$(n_x \times n_y)$	$(n_x \times n_z)$
M_{x1}	M_{y1}	M_{z1}
$(n_x \times n_x)$	$(n_x \times n_y)$	$(n_x \times n_z)$
$\mathbf{\Phi}$	$\mathbf{\Psi}$	
$(n_z \times n_z)$	$(n_z \times n_e)$	

- N_y is invertible, which means that the variable of interest is uniquely defined.

- All eigenvalues of Φ lies within the unit circle

- $\epsilon_t \sim \mathcal{N}(0, \Sigma)$

- Transforming the problem

$$Y_t = N_y^{-1} N_x X_t + N_y^{-1} N_z Z_t$$

$$E_t Z_{t+1} = \Phi Z_t$$

- Substitute and rewrite equation (2) as:

$$AE_tX_{t+1} = BX_t + CZ_t$$

$$A = M_{x0} + M_{y0}N_y^{-1}N_x$$

$$B = M_{x1} + M_{y1}N_y^{-1}N_x$$

$$C = M_{z1} + M_{y1}N_y^{-1}N_z - \left(M_{z0} + M_{y0}N_y^{-1}N_z\right)\Phi$$

- This system comes from the linearization of the individual optimization conditions and market clearing conditions in a dynamic equilibrium model.

- The matrix A is allowed to be singular.

- A singular matrix A implies that static (intra-temporal) equilibrium conditions are included among the dynamic relationships.

2 Generalized Schur decomposition

- The idea of Klein's approach is to use complex generalized Schur decomposition to reduce the system into an unstable and a stable block of equations.

- The stable solution is found by solving the unstable block forward and the stable block backward.

Definition of predetermined or backward-looking variables: a process k is called backward-looking if the prediction error $\epsilon_{t+1} \equiv k_{t+1} - E_t k_{t+1}$ is an exogenous martingale difference process ($E_t \epsilon_{t+1} = 0$) and k_0 is exogenous given.

- The dynamic equation

$$AE_tX_{t+1} = BX_t + CZ_t$$

- Generalized Schur decomposition of the pencil (A, B)

$$S = QAZ$$

$$T = QBZ$$

$$QQ' = ZZ' = I$$

- See my handout for a description of generalized Schur decomposition

- The dynamic equation can be rewritten as

$$A\underbrace{ZZ'}_{I}E_tX_{t+1} = B\underbrace{ZZ'}_{I}X_t + CZ_t$$

$$\omega_t = Z'X_t$$

$$AZE_t\omega_{t+1} = BZ\omega_t + CZ_t$$

$$QAZE_t\omega_{t+1} = QBZ\omega_t + QCZ_t$$

$$R \equiv QC$$

$$SE_t\omega_{t+1} = T\omega_t + RZ_t$$

- Don't confuse Z with Z_t.

- Remark

$$Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}$$

$$\omega_t = Z' X_t = \begin{bmatrix} Z'_{11} & Z'_{21} \\ Z'_{12} & Z'_{22} \end{bmatrix} \begin{bmatrix} X_t^b \\ X_t^f \end{bmatrix} = \begin{bmatrix} Z'_{11} X_t^b + Z'_{21} X_t^f \\ Z'_{12} X_t^b + Z'_{22} X_t^f \end{bmatrix} \equiv \begin{bmatrix} \omega_t^b \\ \omega_t^f \end{bmatrix}$$

$$\omega_t^b = Z'_{11} X_t^b + Z'_{21} X_t^f$$

$$\omega_t^f = Z'_{12} X_t^b + Z'_{22} X_t^f$$

- The generalized eigenvalues of the system are

$$\frac{T_{ii}}{S_{ii}}$$

- We sort the generalized eigenvalues in ascending order.

- n_s stable eigenvalues

- n_u unstable eigenvalues

Blanchard and Kahn condition: if $n_b = n_s$ (and $n_f = n_u$) then the system admits a unique saddle path.

- There are as many predetermined variables as there are stable eigenvalues.

- How likely is that $n_b = n_s$?

- In practice, very likely.

- If the system of equations is derived from a linear-quadratic dynamic optimization problem, we are almost guaranteed that $n_b = n_s$.

- Partition the system

$$Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}$$

$$S = \begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix}$$

$$T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}$$

$$\begin{array}{lcl} Z_{11} & : & n_s \times n_s \\ Z_{12} & : & n_s \times n_f \\ Z_{21} & : & n_f \times n_s \\ Z_{22} & : & n_f \times n_f \end{array}$$

- Rewrite the system as

$$\begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix} \begin{pmatrix} E_t \omega_{t+1}^b \\ E_t \omega_{t+1}^f \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix} \begin{pmatrix} \omega_t^b \\ \omega_t^f \end{pmatrix} + \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} Z_t$$

- S_{11} and T_{22} are invertible by construction.

3 The forward part of the solution

- Look at the unstable part of the system

$$S_{22} E_t \omega^f_{t+1} = T_{22} \omega^f_t + R_2 Z_t$$

$$\omega^f_t = T_{22}^{-1} S_{22} E_t \omega^f_{t+1} - T_{22}^{-1} R_2 Z_t$$

$$\omega^f_t = \lim_{k \to \infty} \left(T_{22}^{-1} S_{22} \right)^k E_t \omega^f_{t+k} - \sum_{k=0}^{\infty} \left(T_{22}^{-1} S_{22} \right)^k T_{22}^{-1} R_2 \Phi^k Z_t$$

$$E_t \omega^f_{t+k} < \infty$$

$$\lim_{k\to\infty}\left(T_{22}^{-1}S_{22}\right)^k E_t\omega_{t+k}^f = 0$$

$$\omega_t^f = -\sum_{k=0}^{\infty}\left(T_{22}^{-1}S_{22}\right)^k T_{22}^{-1}R_2\mathbf{\Phi}^k Z_t = \mathbf{\Gamma} Z_t$$

$$\mathbf{\Gamma} \equiv -\sum_{k=0}^{\infty}\left(T_{22}^{-1}S_{22}\right)^k T_{22}^{-1}R_2\mathbf{\Phi}^k$$

- Some matrix algebraic

$$vec(A+B) = vec(A) + vec(B)$$

$$vec(ABC) = \left(C' \otimes A\right) vec(B)$$

$$(AB \otimes CD) = (A \otimes C)(B \otimes D)$$

$$S = \sum_{k=0}^{\infty} A^k B C^k = B + ASC$$

- Back to the problem

$$\Gamma \equiv \sum_{k=0}^{\infty} \underbrace{\left(T_{22}^{-1} S_{22}\right)^{k}} \underbrace{\left(-T_{22}^{-1} R_{2}\right)} \underbrace{\Phi^{k}}$$

$$\Gamma = -T_{22}^{-1} R_{2} + \left(T_{22}^{-1} S_{22}\right) \Gamma \Phi$$

$$vec\left(\Gamma\right) = -vec\left(T_{22}^{-1} R_{2}\right) + vec\left(\left(T_{22}^{-1} S_{22}\right) \Gamma \Phi\right)$$

$$vec(\Gamma) = -\left(I \otimes T_{22}^{-1}\right) vec(R_2) + \left(\Phi' \otimes \left(T_{22}^{-1} S_{22}\right)\right) vec(\Gamma)$$

$$vec(\Gamma) = -\left(I \otimes T_{22}^{-1}\right) vec(R_2) + \left(\Phi' \otimes T_{22}^{-1}\right) (I \otimes (S_{22})) vec(\Gamma)$$

$$(I \otimes T_{22}) vec(\Gamma) = -vec(R_2) + \left(\Phi' \otimes S_{22}\right) vec(\Gamma)$$

$$vec(\Gamma) = \left(\Phi' \otimes S_{22} - I \otimes T_{22}\right)^{-1} vec(R_2)$$

- Important

$$\omega_t^f = \Gamma Z_t$$

- Recall that

$$\omega_t^f = Z_{12}' X_t^b + Z_{22}' X_t^f$$

$$Z_{12}' X_t^b + Z_{22}' X_t^f = \Gamma Z_t$$

- Guess a solution for X_t^f

$$X_t^f = \alpha X_t^b + \beta Z_t$$

$$Z'_{12} X_t^b + Z'_{22} \left(\alpha X_t^b + \beta Z_t \right) = \Gamma Z_t$$

$$Z'_{12} + Z'_{22} \alpha = 0 \tag{4}$$

$$Z'_{22}\beta = \Gamma \tag{5}$$

- α, β?

$$Z'Z = I$$

$$\begin{pmatrix} Z'_{11} & Z'_{21} \\ Z'_{12} & Z'_{22} \end{pmatrix} \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} = I$$

$$\begin{bmatrix} Z'_{11}Z_{11} + Z'_{21}Z_{21} & Z'_{11}Z_{12} + Z'_{21}Z_{22} \\ Z'_{12}Z_{11} + Z'_{22}Z_{21} & Z'_{12}Z_{12} + Z'_{22}Z_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$Z'_{12}Z_{11} + Z'_{22}Z_{21} = 0$$

$$Z'_{12} + Z'_{22}Z_{21}Z_{11}^{-1} = 0$$

- Recall equation (4)

$$Z'_{12} + Z'_{22}\alpha = 0 \tag{4}$$

$$\alpha = Z_{21}Z_{11}^{-1}$$

- Let $\beta = \tilde{\beta}\Gamma$
- Recall equation (5)

$$Z'_{22}\beta = \Gamma \tag{5}$$

$$Z'_{22}\beta = \Gamma$$

$$Z'_{22}\tilde{\beta}\Gamma = \Gamma$$

$$Z'_{12}Z_{11} + Z'_{22}Z_{21} = 0$$

$$Z'_{12} = -Z'_{22}Z_{21}Z_{11}^{-1}$$

$$Z_{12}'Z_{12} + Z_{22}'Z_{22} = I$$

$$Z_{22}'\left(Z_{22} - Z_{21}Z_{11}^{-1}Z_{12}\right) = I$$

$$Z_{22}'\underbrace{\left(Z_{22} - Z_{21}Z_{11}^{-1}Z_{12}\right)}\Gamma = \Gamma$$

$$\tilde{\beta} = \left(Z_{22} - Z_{21}Z_{11}^{-1}Z_{12}\right)$$

$$\beta = \left(Z_{22} - Z_{21}Z_{11}^{-1}Z_{12}\right)\Gamma$$

- We obtain the forward part of the solution

$$X_t^f = Z_{21}Z_{11}^{-1}X_t^b + \left(Z_{22} - Z_{21}Z_{11}^{-1}Z_{12}\right)\Gamma Z_t$$

- or express the solution as

$$X_t^f = F_x X_t^b + F_z Z_t$$

$$F_x \equiv Z_{21} Z_{11}^{-1}$$

$$F_z \equiv \left(Z_{22} - Z_{21} Z_{11}^{-1} Z_{12} \right) \Gamma$$

$$vec\left(\Gamma\right) = \left(\Phi' \otimes S_{22} - I \otimes T_{22} \right)^{-1} vec\left(R_2\right)$$

4 The backward part of the solution

- The upper part of the transformed system is

$$S_{11}E_t\omega_{t+1}^b + S_{12}E_t\omega_{t+1}^f = T_{12}\omega_t^f + T_{11}\omega_t^b + R_1 Z_t$$

$$E_t\omega_{t+1}^b = S_{11}^{-1}T_{11}\omega_t^b + S_{11}^{-1}T_{12}\omega_t^f - S_{11}^{-1}S_{12}E_t\omega_{t+1}^f + S_{11}^{-1}R_1 Z_t$$

- Recall that

$$\omega_t = Z' X_t \Leftarrow \omega_t^b = Z'_{11} X_t^b + Z'_{21} X_t^f$$

$$X_t^f = Z_{21} Z_{11}^{-1} X_t^b + \left(Z_{22} - Z_{21} Z_{11}^{-1} Z_{12} \right) \Gamma Z_t$$

- Substitute and rearrange,

$$\omega_t^b = \left(Z'_{11} + Z'_{21} Z_{21} Z_{11}^{-1} \right) X_t^b + Z'_{21} \left(Z_{22} - Z_{21} Z_{11}^{-1} Z_{12} \right) \Gamma Z_t$$

$$Z_{11}'Z_{11} + Z_{21}'Z_{21} = I \Rightarrow Z_{11}' + Z_{21}'Z_{21}Z_{11}^{-1} = Z_{11}^{-1}$$

$$Z_{21}'Z_{22} + Z_{11}'Z_{12} = 0 \Rightarrow Z_{21}'Z_{22} = -Z_{11}'Z_{12}$$

$$\begin{aligned} Z_{21}'\left(Z_{22} - Z_{21}Z_{11}^{-1}Z_{12}\right) &= Z_{21}'Z_{22} - Z_{21}'Z_{21}Z_{11}^{-1}Z_{12} \\ &= -\left(Z_{11}' + Z_{21}'Z_{21}Z_{11}^{-1}\right)Z_{12} \\ &= -Z_{11}'Z_{12} \end{aligned}$$

$$\omega_t^b = Z_{11}^{-1} X_t^b - Z_{11}^{-1} Z_{12} \Gamma Z_t$$

$$Z_{11}^{-1} X_t^b = \omega_t^b + Z_{11}^{-1} Z_{12} \Gamma Z_t$$

$$X_t^b = Z_{11}\omega_t^b + Z_{11} Z_{11}^{-1} Z_{12} \Gamma Z_t = Z_{11}\omega_t^b + Z_{12} \underbrace{\Gamma Z_t}_{=\omega_t^f} = Z_{11}\omega_t^b + Z_{12}\omega_t^f$$

- X_t^b are predetermined variables

$$X_{t+1}^b - E_t X_{t+1}^b = 0$$

- Klein (2000) assumes $X_{t+1}^b - E_t X_{t+1}^b = \xi_{t+1}$

$$X_t^b = Z_{11}\omega_t^b + Z_{12}\omega_t^f$$

$$X_{t+1}^b - E_t X_{t+1}^b = 0$$

$$Z_{11}\left(\omega^b_{t+1} - E_t\omega^b_{t+1}\right) + Z_{12}\left(\omega^f_{t+1} - E_t\omega^f_{t+1}\right) = 0$$

$$\omega^b_{t+1} = E_t\omega^b_{t+1} - Z_{11}^{-1}Z_{12}\left(\omega^f_{t+1} - E_t\omega^f_{t+1}\right)$$

- $\omega^f_{t+1} - E_t\omega^f_{t+1} = \Gamma\Psi\epsilon_{t+1} \qquad \left(\because \omega^f_t = \Gamma Z_t\right)$

$$\omega^b_{t+1} = E_t\omega^b_{t+1} - Z_{11}^{-1}Z_{12}\Gamma\Psi\epsilon_{t+1}$$

- Recall that

$$E_t\omega_{t+1}^b = S_{11}^{-1}T_{11}\omega_t^b + S_{11}^{-1}T_{12}\omega_t^f - S_{11}^{-1}S_{12}E_t\omega_{t+1}^f + S_{11}^{-1}R_1Z_t$$

$$\omega_{t+1}^b = S_{11}^{-1}T_{11}\omega_t^b + S_{11}^{-1}\left(T_{12}\Gamma - S_{12}\Gamma\Phi + R_1\right)Z_t - Z_{11}^{-1}Z_{12}\Gamma\Psi\epsilon_{t+1}$$

$$\omega_t^b = Z_{11}^{-1}X_t^b - Z_{11}^{-1}Z_{12}\Gamma Z_t$$

$$\omega^b_{t+1} = Z_{11}^{-1} X^b_{t+1} - Z_{11}^{-1} Z_{12} \Gamma Z_{t+1}$$

$$\begin{aligned} Z_{11}^{-1} X^b_{t+1} - Z_{11}^{-1} Z_{12} \Gamma Z_{t+1} &= S_{11}^{-1} T_{11} \left(Z_{11}^{-1} X^b_t - Z_{11}^{-1} Z_{12} \Gamma Z_t \right) \\ &\quad + S_{11}^{-1} \left(T_{12} \Gamma - S_{12} \Gamma \Phi + R_1 \right) Z_t - Z_{11}^{-1} Z_{12} \Gamma \Psi \epsilon_{t+1} \end{aligned}$$

$$\begin{aligned} X^b_{t+1} - Z_{12} \Gamma Z_{t+1} &= Z_{11} S_{11}^{-1} T_{11} \left(Z_{11}^{-1} X^b_t - Z_{11}^{-1} Z_{12} \Gamma Z_t \right) \\ &\quad + Z_{11} S_{11}^{-1} \left(T_{12} \Gamma - S_{12} \Gamma \Phi + R_1 \right) Z_t - Z_{12} \Gamma \Psi \epsilon_{t+1} \end{aligned}$$

$$Z_{t+1} = \mathbf{\Phi} Z_t + \mathbf{\Psi} \epsilon_{t+1}$$

$$\begin{aligned} X^b_{t+1} &= Z_{11} S_{11}^{-1} T_{11} Z_{11}^{-1} X^b_t \\ &+ \left[Z_{11} S_{11}^{-1} \left(T_{12} \mathbf{\Gamma} - S_{12} \mathbf{\Gamma} \mathbf{\Phi} - T_{11} Z_{11}^{-1} Z_{12} \mathbf{\Gamma} + R_1 \right) + Z_{12} \mathbf{\Gamma} \mathbf{\Phi} \right] Z_t \end{aligned}$$

- The dynamics of the predetermined variables is

$$X^b_{t+1} = M_x X^b_t + M_z Z_t$$

$$M_x = Z_{11} S_{11}^{-1} T_{11} Z_{11}^{-1}$$

$$M_z = Z_{11} S_{11}^{-1} \left(T_{12} \Gamma - S_{12} \Gamma \Phi - T_{11} Z_{11}^{-1} Z_{12} \Gamma + R_1 \right) + Z_{12} \Gamma \Phi$$

5 Computing variables of interest

$$N_y Y_t = N_x X_t + N_z Z_t = \begin{bmatrix} N_{x1} & N_{x2} \end{bmatrix} \begin{bmatrix} X_t^b \\ X_t^f \end{bmatrix} + N_z Z_t = N_{x1} X_t^b + N_{x2} X_t^f + N_z Z_t$$

$$N_y Y_t = N_x X_t + N_z Z_t = N_{x1} X_t^b + N_{x2} \left(F_x X_t^b + F_z Z_t \right) + N_z Z_t$$

$$Y_t = N_y^{-1} \left(N_{x1} + N_{x2} F_x \right) X_t^b + N_y^{-1} \left(N_z + N_{x2} F_z \right) Z_t$$

$$Y_t = P_x X_t^b + P_z Z_t$$

$$P_x \equiv N_y^{-1} \left(N_{x1} + N_{x2} F_x \right)$$

$$P_z \equiv N_y^{-1} (N_z + N_{x2} F_z)$$

- Express the solution of the whole system is state-space form

$$X_{t+1}^b = M_x X_t^b + M_z Z_t$$

$$Z_{t+1} = \Phi Z_t + \Psi \epsilon_{t+1}$$

$$X_t^f = F_x X_t^b + F_z Z_t$$

$$Y_t = P_x X_t^b + P_z Z_t$$

6 When $X_{t+1}^b - E_t X_{t+1}^b = \xi_{t+1}$

- Klein (2000) assumes $X_{t+1}^b - E_t X_{t+1}^b = \xi_{t+1}$

$$X_t^b = Z_{11}\omega_t^b + Z_{12}\omega_t^f$$

$$X_{t+1}^b - E_t X_{t+1}^b = \xi_{t+1}$$

$$Z_{11}\left(\omega_{t+1}^b - E_t\omega_{t+1}^b\right) + Z_{12}\left(\omega_{t+1}^f - E_t\omega_{t+1}^f\right) = \xi_{t+1}$$

$$\omega_{t+1}^b = E_t\omega_{t+1}^b - Z_{11}^{-1}Z_{12}\left(\omega_{t+1}^f - E_t\omega_{t+1}^f\right) + Z_{11}^{-1}\xi_{t+1}$$

$$\omega_{t+1}^{b} = E_t \omega_{t+1}^{b} - Z_{11}^{-1} Z_{12} \Gamma \Psi \epsilon_{t+1} + Z_{11}^{-1} \xi_{t+1}$$

- Recall that

$$E_t \omega_{t+1}^{b} = S_{11}^{-1} T_{11} \omega_t^{b} + S_{11}^{-1} T_{12} \omega_t^{f} - S_{11}^{-1} S_{12} E_t \omega_{t+1}^{f} + S_{11}^{-1} R_1 Z_t$$

$$\omega_{t+1}^{b} = S_{11}^{-1} T_{11} \omega_t^{b} + S_{11}^{-1} \left(T_{12} \Gamma - S_{12} \Gamma \Phi + R_1 \right) Z_t - Z_{11}^{-1} Z_{12} \Gamma \Psi \epsilon_{t+1} + Z_{11}^{-1} \xi_{t+1}$$

$$\omega_t^b = Z_{11}^{-1} X_t^b - Z_{11}^{-1} Z_{12} \Gamma Z_t$$

$$\omega_{t+1}^b = Z_{11}^{-1} X_{t+1}^b - Z_{11}^{-1} Z_{12} \Gamma Z_{t+1}$$

$$\begin{aligned} Z_{11}^{-1} X_{t+1}^b - Z_{11}^{-1} Z_{12} \Gamma Z_{t+1} \quad = \quad & S_{11}^{-1} T_{11} \left(Z_{11}^{-1} X_t^b - Z_{11}^{-1} Z_{12} \Gamma Z_t \right) \\ & + S_{11}^{-1} \left(T_{12} \Gamma - S_{12} \Gamma \Phi + R_1 \right) Z_t \\ & - Z_{11}^{-1} Z_{12} \Gamma \Psi \epsilon_{t+1} + Z_{11}^{-1} \xi_{t+1} \end{aligned}$$

$$\begin{aligned} X^b_{t+1} - Z_{12}\Gamma Z_{t+1} &= Z_{11}S_{11}^{-1}T_{11}\left(Z_{11}^{-1}X^b_t - Z_{11}^{-1}Z_{12}\Gamma Z_t\right) \\ &+ Z_{11}S_{11}^{-1}\left(T_{12}\Gamma - S_{12}\Gamma\Phi + R_1\right)Z_t - Z_{12}\Gamma\Psi\epsilon_{t+1} + \xi_{t+1} \end{aligned}$$

$$Z_{t+1} = \Phi Z_t + \Psi\epsilon_{t+1}$$

$$\begin{aligned} X^b_{t+1} &= Z_{11}S_{11}^{-1}T_{11}Z_{11}^{-1}X^b_t \\ &+ \left[Z_{11}S_{11}^{-1}\left(T_{12}\Gamma - S_{12}\Gamma\Phi - T_{11}Z_{11}^{-1}Z_{12}\Gamma + R_1\right) + Z_{12}\Gamma\Phi\right]Z_t + \xi_{t+1} \end{aligned}$$

- The dynamics of the predetermined variables is

$$X_{t+1}^b = M_x X_t^b + M_z Z_t + \xi_{t+1}$$

$$M_x = Z_{11} S_{11}^{-1} T_{11} Z_{11}^{-1}$$

$$M_z = Z_{11} S_{11}^{-1} \left(T_{12} \Gamma - S_{12} \Gamma \Phi - T_{11} Z_{11}^{-1} Z_{12} \Gamma + R_1 \right) + Z_{12} \Gamma \Phi$$

7 In practice

- In practice, we can treat exogenous shocks as part of the pre-determined variables.

- In other words, we redefine the dynamic system as

$$AE_tX_{t+1} = BX_t + \begin{bmatrix} \emptyset \\ \Psi\epsilon_{t+1} \\ \emptyset \end{bmatrix}$$

$$X_t = \begin{bmatrix} X_t^b \\ Z_t \\ X_t^f \end{bmatrix} \equiv \begin{bmatrix} \mathcal{X}_t^b \\ X_t^f \end{bmatrix}$$

$$\mathcal{X}_t^b \equiv \begin{bmatrix} X_t^b \\ Z_t \end{bmatrix}$$

- The solution to the system becomes

$$\mathcal{X}_{t+1}^b = M_x \mathcal{X}_t^b + \xi_{t+1} \tag{6}$$

$$M_x = Z_{11} S_{11}^{-1} T_{11} Z_{11}^{-1}$$

$$\xi_{t+1} = \begin{bmatrix} \emptyset \\ \mathbf{\Psi} \epsilon_{t+1} \end{bmatrix}$$

$$X_t^f = F_x X_t^b \tag{7}$$

$$F_x = Z_{21} Z_{11}^{-1}$$

- This is the solution form employed in Klein's MATLAB code.

- To implement Paul Klein's method, you need 3 MATLAB m files: *solab.m*; *qzswitch.m*; and *qzdiv.m*.

- These MATLAB m files are available from Paul Klein's website.

- The 2 MATLAB m files, *qzswitch.m* and *qzdiv.m*, are originally written by C. Sims.

References

[1] Anderson, Gary S. (2008), "Solving Linear Rational Expectations Models: A Horse Race," *Computational Economics*, 31, pp. 95-113.

[2] Klein, Paul (2000), "Using the generalized Schur form to solve a multivariate linear rational expectations model," *Journal of Economic Dynamics and Control*, 24, pp. 1405-1423.