Detail proofs of some results in the article: Classification and evolution of bifurcation curves for the one-dimensional perturbed Gelfand equation with mixed boundary conditions*

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In this article, we give detail proofs of Lemma 3.1(I)(ii)–(I)(iv), (I)(vi)–(I)(viii), (I)(x)–(I)(xv) and (III)(v)–(III)(ix) in the article “Classification and evolution of bifurcation curves for the one-dimensional perturbed Gelfand equation with mixed boundary conditions” [1], where these proofs are not given. To start with, we recall some definitions of functions and Lemma 3.1 therein. Remark that following labels (3.5) and (3.7)–(3.19) are consistent with those in [1].

Define \( f(s) = \exp \left( \frac{as}{a+s} \right) \), \( F(s) = \int_0^s f(t) \, dt \), \( \theta(s) = 2F(s) - sf(s) \) and

\[
\begin{align*}
P_1(\rho, s) &= \frac{\rho f(\rho) - sf(s)}{F(\rho) - F(s)}, \\
P_2(\rho, s) &= \frac{\rho^2 f(\rho) - s^2 f(s)}{\rho f(\rho) - sf(s)}.
\end{align*}
\]

(3.5)

Lemma 3.1. Define

\[
\begin{align*}
M_1(\rho) &= \frac{\rho f(\rho)}{F(\rho)} \text{ for } \rho > 0, \\
M_2(\rho) &= \frac{\rho f(\rho)}{f(\rho)} = \frac{a^2 \rho}{(a + \rho)^2} \text{ for } \rho \geq 0, \\
Q(\rho, s) &= \theta(\rho) - \theta(s) + M_1(\rho)[F(\rho) - F(s)] \text{ for } 0 \leq s \leq \rho,
\end{align*}
\]

(3.7)

(3.8)

(3.9)

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\[ R(\rho, s) = 2[\theta(\rho) - \theta(s)] - \frac{f(\rho)}{\sqrt{F(\rho)}} \sqrt{\rho} \left( \frac{F(\rho) - F(s)^{3/2}}{\sqrt{s}} \right) \quad \text{for } 0 \leq s < \rho, \quad (3.10) \]

\[ K_1(\rho) = \frac{8\rho^2 + 2\rho + 1}{25\rho^4 + 2\rho + 1} - \frac{1}{50} (16\rho - 3) \quad \text{for } \rho \geq 0, \quad (3.11) \]

\[ K_2(\rho) = \frac{8 - 132a + 165}{50a + 25} \rho^2 + \frac{2}{3} \rho + 1 \quad \text{for } \rho \geq 0, \quad (3.12) \]

\[ K_3(\rho) = M_1(\rho) + 2\rho \left[ f(\rho) - \frac{f'(\rho)}{f(\rho)} \right] \quad \text{for } \rho > 0, \quad (3.13) \]

\[ L_1(\rho, s) = \frac{8}{25} s - M_2(s) - \frac{8}{25} \frac{F'(\rho) - F(s)}{f(s)} + M_1(\rho) - 1 \quad \text{for } 0 \leq s \leq \rho, \quad (3.14) \]

\[ L_2(\rho, s) = P_1(\rho, s) - \frac{8}{25} s - M_1(\rho) \quad \text{for } 0 \leq s < \rho, \quad (3.15) \]

\[ L_3(\rho, s) = P_2(\rho, s) - \frac{3}{2} P_1(\rho, s) + \frac{37}{25} \quad \text{for } 0 \leq s < \rho. \quad (3.16) \]

Then the following assertions (I)–(III) hold:

(I) For \( a > 0 \), the following assertions (i)–(xv) hold:

(i) \( f(\rho, a) (= f(\rho)) \) is a strictly increasing function of \( \rho \) on \([0, \infty)\) and of \( a \) on \((0, \infty)\) when the other variable is fixed. Moreover, \( 1 < f(\rho) < \min\{e^\rho, e^a\} \) for \( \rho, a > 0 \).

(ii) \( M_2(\rho) < M_1(\rho) \) for \( \rho > 0 \).

(iii) \( 1 < \frac{1}{2}[M_2(\rho) + 2] < M_1(\rho) \) for \( \rho > 0 \).

(iv) \( M_1(\rho) < M_2(\rho) + 1 \) for \( 0 < \rho \leq a \).

(v) \( M_1(\rho) \) is strictly increasing on \((0, a]\).

(vi) \( M_2(\rho) \) satisfies

\[
M'_2(\rho) = \frac{a^2(a - \rho)}{(a + \rho)^3} \begin{cases} 
> 0 & \text{when } \rho \in (0, a), \\
= 0 & \text{when } \rho = a, \\
< 0 & \text{when } \rho \in (a, \infty), 
\end{cases} \quad (3.17)
\]

and \( M_2(\rho) \leq M_2(a) (= a/4) \) for \( \rho > 0 \).

(vii) \( P_1(\rho, 0) = M_1(\rho), \lim_{s \to \rho^-} P_1(\rho, s) = M_2(\rho) + 1 \) for \( \rho > 0 \) and \( \lim_{\rho \to s^+} P_1(\rho, s) = M_2(s) + 1 \) for \( s \geq 0 \).

(viii) \( P_1(\rho, s) > 1 \) for \( 0 \leq s < \rho \).

(ix) For \( 0 < \rho \leq a \), \( P_1(\rho, s) \) is a strictly increasing function of \( s \) on \([0, \rho]\). Moreover, \( M_1(\rho) \leq P_1(\rho, s) < M_2(\rho) + 1 \) for \( 0 \leq s < \rho \leq a \), where the equality holds if and only if \( s = 0 \).

(x) For \( 0 \leq s < a \), \( P_1(\rho, s) \) is a strictly increasing function of \( \rho \) on \((s, a]\). Moreover, \( P_1(\rho, s) > M_2(s) + 1 \) for \( 0 \leq s < \rho \leq a \).

(xi) \( P_2(\rho, 0) = M_2(\rho) \) and \( \lim_{s \to \rho^-} P_2(\rho, s) = \frac{2a^2(\rho) + a^2\rho^2}{f(\rho) + \rho f'(\rho)} \) for \( \rho > 0 \).
(xii) $M_2(\rho) < P_2(\rho, s)$ for $0 < s < \rho \leq a$.

(xiii) $Q(\rho, s) > 0$ for $0 \leq s < \rho$.

(xiv) $\left[ \frac{f(\rho)}{\sqrt{F(\rho)}} \right]' + \frac{M_2(\rho) + 1}{2\rho} \frac{f(\rho)}{\sqrt{F(\rho)}} > 0$ for $0 < \rho < a$.

(xv) For $\rho > 0$, $M_1(\rho, a)$ and $M_2(\rho, a)$ are both strictly increasing functions of $a$ on $(0, \infty)$.

(II) For $0 < a \leq a_0 (\approx 0.501)$, $R(\rho, s) > 0$ for $0 \leq s < \rho$. Here

\[ a_0 \text{ is the unique positive zero of } -a/2 - e^a \sqrt{a/4} + 1 + 2. \tag{3.18} \]

(III) For $a \geq 4$, the following assertions (i)–(ix) hold:

(i) $R(\rho, s) < 0$ for $\rho \geq a$ and $0 \leq s < \rho$.

(ii) For $0 < \rho < a$, $R(\rho, s) > 0$ for $0 < s < \rho$ if $\rho$ satisfies $2[1 - M_2(\rho)] - f(\rho)\sqrt{M_1(\rho)} > 0$; while $R(\rho, s) < 0$ for $0 < s < \rho$ if $\rho$ satisfies $2[2 - M_1(\rho)] - f(\rho) < 0$.

(iii) $R(\rho, s) > 0$ for $0 \leq s < \rho \leq \rho_1$. Here $\rho_1 (\approx 0.286) > 7/25 = 0.28$ is the unique positive zero of $2(1 - \rho) - \sqrt{\rho e^{3\rho}}/(e^\rho - 1)$.

(iv) $R(1/2, s) < 0$ for $0 \leq s < 1/2$.

(v) $K_1(\rho) < M_1(\rho) < K_2(\rho)$ for $1/4 \leq \rho \leq 1/2$.

(vi) $K_3(\rho) < M_1(\rho) + \frac{132a+165}{50a+25} \rho^2$ for $1/4 \leq \rho \leq 1/2$.

(vii) For any $\rho \in [1/4, 1/2]$, there exists a unique positive $s^* (= s^*(\rho, a))$ on $(0, \rho)$ such that

\[ L_1(\rho, s) \begin{cases} > 0 & \text{when } s \in (0, s^*), \\ = 0 & \text{when } s = s^*, \\ < 0 & \text{when } s \in (s^*, \rho]. \end{cases} \tag{3.19} \]

(viii) $L_2(\rho, s) \geq 0$ for $1/4 \leq \rho \leq 1/2$ and $0 \leq s < \rho$.

(ix) $L_3(\rho, s) \geq 0$ for $7/25 \leq \rho \leq 1/2$ and $0 \leq s < \rho$.

\textbf{Proof of Lemma 3.1}. In the following, we would give detail proofs of Lemma 3.1 for parts (I)(ii)–(I)(iv), (I)(vi)–(I)(viii), (I)(x)–(I)(xv) and (III)(v)–(III)(ix).

\textit{Proof of Lemma 3.1(I)(ii)}. Let

\[ n_1(\rho) \equiv (a + \rho)^2 \frac{f(\rho)}{\rho} \left[ M_1(\rho) - M_2(\rho) \right] \]

\[ = (a + \rho)^2 \frac{f(\rho)}{\rho} \left[ \rho f(\rho) \frac{a^2}{F(\rho)} - \frac{a^2}{f(\rho)} \right] \quad \text{(by (3.7) and (3.8))} \]

\[ = (a + \rho)^2 f(\rho) - a^2 F(\rho) \]
for $\rho > 0$. Then we compute that $\lim_{\rho \to 0^+} n_1(\rho) = a^2 > 0$ and $n_1'(\rho) = 2(a + \rho)f(\rho) > 0$ for $\rho > 0$. It implies that, for $\rho > 0$, $n_1(\rho) > 0$ and hence $M_1(\rho) > M_2(\rho)$. Hence the proof of Lemma 3.1(I)(ii) is complete.

**Proof of Lemma 3.1(I)(iii).** First, for $\rho > 0$, $\frac{1}{2}[M_2(\rho) + 2] > 1$ since $M_2(\rho) > 0$ by (3.8). Next, we show that $\frac{1}{2}[M_2(\rho) + 2] < M_1(\rho)$ for $\rho > 0$. Let

$$n_2(\rho) \equiv \frac{\rho f(\rho)}{\frac{1}{2}[M_2(\rho) + 2]} - F(\rho) = \frac{\rho f(\rho)}{\frac{1}{2}[\rho f'(\rho)/f(\rho) + 2]} - F(\rho)$$

for $\rho > 0$. Then we compute that $n_2(0) = 0$ and

$$n_2'(\rho) = \frac{a^2 \rho^2(4\rho + a^2 + 4a)}{(2\rho^2 + a^2 \rho + 4\rho + 2a^2)^2} f(\rho) > 0$$

for $\rho > 0$. It implies that, for $\rho > 0$, $n_2(\rho) > 0$ and hence $\frac{1}{2}[M_2(\rho) + 2] < M_1(\rho) (= \rho f(\rho)/F(\rho))$ by (3.7). Hence the proof of Lemma 3.1(I)(iii) is complete.

**Proof of Lemma 3.1(I)(iv).** Let

$$n_3(\rho) \equiv \frac{\rho f(\rho)}{M_2(\rho) + 1} - F(\rho) = \frac{\rho f(\rho)}{\rho f'(\rho)/f(\rho) + 1} - F(\rho)$$

for $\rho > 0$. Then we compute that $n_3(0) = 0$ and

$$n_3'(\rho) = -\frac{a^2 \rho(a^2 - \rho^2)}{(\rho^2 + a^2 \rho + 2a \rho + a^2)} f(\rho) < 0$$

for $0 < \rho < a$. It implies that, for $0 < \rho \leq a$, $n_3(\rho) < 0$ and hence $M_1(\rho) (= \rho f(\rho)/F(\rho)) < M_2(\rho) + 1$ by (3.7). Hence the proof of Lemma 3.1(I)(iv) is complete.

**Proof of Lemma 3.1(I)(v).** The proof is trivial and hence it is omitted.

**Proof of Lemma 3.1(I)(vii).** The first equality in the statement of Lemma 3.1(I)(vii) is clear while the second and the third ones follow from the L’Hôpital’s rule directly.

**Proof of Lemma 3.1(I)(viii).** By integration by parts, we have that, for $0 \leq s < \rho$, $F(\rho) - F(s) = \int_s^\rho f(t)dt = \rho f(\rho) - sf(s) - \int_s^\rho tf'(t)dt < \rho f(\rho) - sf(s)$ since $f'(t) = \frac{a^2}{(a+t)^2} f(t) > 0$ for $t > 0$. Hence $P_1(\rho, s) > 1$ by (3.5), which completes the proof of Lemma 3.1(I)(viii).

**Proof of Lemma 3.1(I)(ix).** Fix $s$ on $[0, a)$. Then we compute that

$$\frac{\partial}{\partial \rho} P_1(\rho, s) = \frac{f(\rho) \left\{ [M_2(\rho) + 1] - P_1(\rho, s) \right\}}{F(\rho) - F(s)} > 0$$

for $s < \rho \leq a$ by Lemma 3.1(I)(ix). Hence, for $0 \leq s < a$, $P_1(\rho, s)$ is a strictly increasing function of $\rho$ on $(s, a]$. It implies that $P_1(\rho, s) \geq \lim_{\rho \to s^+} P_1(\rho, s) = M_2(s) + 1$ for $0 \leq s < \rho \leq a$ by Lemma 3.1(I)(vii). Hence the proof of Lemma 3.1(I)(ix) is complete.

**Proof of Lemma 3.1(I)(xii).** By (3.5) and (3.8), for $0 < s < \rho \leq a$, we compute that

$$P_2(\rho, s) - M_2(\rho) = \frac{\rho^2 f'(\rho) - s^2 f'(s)}{\rho f(\rho) - sf(s)} - \frac{\rho f'(\rho)}{f(\rho)} = \frac{sf(s) [M_2(\rho) - M_2(s)]}{[\rho f(\rho) - sf(s)]} > 0,$$
since $M_2(\rho)$ is strictly increasing on $(0, a]$ by (3.17). Hence the proof of Lemma 3.1(I)(xii) is complete.

**Proof of Lemma 3.1(I)(xiii).** We prove that $Q(\rho, s) = \theta(\rho) - \theta(s) + M_1(\rho)[F(\rho) - F(s)] > 0$ for $0 \leq s < \rho$ in the following three cases.

**Case 1:** $0 \leq s < \rho \leq a$. We compute that

$$Q(\rho, s) = [F(\rho) - F(s)][M_1(\rho) + 2 - P_1(\rho, s)]$$

$$\geq [F(\rho) - F(s)] \{M_1(\rho) + 2 - [M_1(\rho) + 1]\} \ \text{(by Lemma 3.1(I)(ii) and (I)(ix))}$$

$$= F(\rho) - F(s) > 0.$$  \hfill (3.20)

**Case 2:** $0 \leq s \leq a < \rho$. Note first that, since $Q(a, s) > 0$ for $0 \leq s < a$ by (3.20) and $Q(a, a) = 0$ by (3.9), we have that $Q(a, s) \geq 0$ for $0 \leq s \leq a$. Moreover, we compute that

$$\frac{\partial}{\partial \rho} [F(\rho)Q(\rho, s)] = f(\rho) \{4F(\rho) - 3F(s) + sf(s) - M_2(\rho)F(s)\}$$

$$\geq f(\rho) \{4F(a) - 3F(s) + sf(s) - M_2(a)F(s)\} \ \text{by (3.17)}$$

$$\equiv f(\rho)n_4(s),$$

where $n_4(s) = 4F(a) - 3F(s) + sf(s) - M_2(a)F(s)$. Since $n_4(a) = F(a) [1 + M_1(a) - M_2(a)] > 0$ by (3.7) and Lemma 3.1(I)(ii) and since $n_4'(s) = f(s) [-2 + M_2(s) - M_2(a)] \leq -2f(s) < 0$ for $0 < s < a$ by (3.17), we have that $n_4(s) > 0$ for $0 \leq s \leq a$. It follows that

$$\frac{\partial}{\partial \rho} [F(\rho)Q(\rho, s)] > 0 \ \text{for} \ 0 \leq s \leq a < \rho.$$ Combining the fact with $F(\rho)Q(\rho, s) \geq 0$ for $0 \leq s \leq a$ as claimed above, we have that $Q(\rho, s) > 0$ for $0 \leq s \leq a < \rho$.

**Case 3:** $a < s < \rho$. Fix $\rho > a$. Suppose it is not true that $Q(\rho, s) > 0$, then there exists some $s_0$ on $(a, \rho)$ such that $Q(\rho, s_0) \leq 0$. Note first that, by the result from Case 2 with $s = a$, we have that $Q(\rho, a) > 0$. On the other hand, since $Q(\rho, \rho) = 0$ by (3.9) and $\lim_{s \to \rho} \frac{\partial}{\partial s} Q(\rho, s) = -f(\rho) [1 + M_1(\rho) - M_2(\rho)] < 0$ by Lemma 3.1(I)(ii), there exists some $s_1$ on $(s_0, \rho)$ such that $Q(\rho, s_1) > 0$. Hence $Q(\rho, s_0) \leq 0 < \min \{Q(\rho, a), Q(\rho, s_1)\}$. It implies that $Q(\rho, s)$ on $[a, s_1]$ has its global minimum at some point $s_2$ on $(a, s_1)$. However, we compute that, for $a < s < \rho$,

$$\frac{\partial}{\partial s} Q(\rho, s) = -f(s) [1 + M_1(\rho) - M_2(s)],$$

$$\frac{\partial^2}{\partial s^2} Q(\rho, s) = -f'(s) [1 + M_1(\rho) - M_2(s)] + f(s)M_2'(s)$$

$$= -\frac{a^2}{(s + a)^2} f(s) [1 + M_1(\rho) - M_2(s)] - f(s) \frac{a^2(s - a)}{(a + s)^3} \ \text{by (3.17)}$$

$$= \frac{a^2}{(s + a)^2} \frac{\partial}{\partial s} Q(\rho, s) - f(s) \frac{a^2(s - a)}{(a + s)^3},$$

which implies that $\frac{\partial^2}{\partial s^2} Q(\rho, s_2) = -f(s_2) \frac{a^2(s_2 - a)}{(s_2 + a)^3} < 0$ since $\frac{\partial}{\partial s} Q(\rho, s_2) = 0$. Consequently, $s_2$ can not be a local minimum point, and we get a contradiction. Hence $Q(\rho, s) > 0$ for $a < s < \rho$. 


By Cases 1–3, we complete the proof of Lemma 3.1(I)(xiii).

Proof of Lemma 3.1(I)(xiv). For \( 0 < \rho \leq a \), we compute that, by (3.7) and (3.8),

\[
\begin{align*}
\left[ \frac{f(\rho)}{\sqrt{F(\rho)}} \right]' + \frac{M_2(\rho) + 1}{2\rho} \frac{f(\rho)}{\sqrt{F(\rho)}} &= \frac{f(\rho)}{2\rho \sqrt{F(\rho)}} \left[ 3M_2(\rho) + 1 - M_1(\rho) \right] \\
&> \frac{f(\rho)}{2\rho \sqrt{F(\rho)}} \left[ 3M_2(\rho) + 1 - [M_2(\rho) + 1] \right] \\
&= \frac{f(\rho)}{\rho \sqrt{F(\rho)}} M_2(\rho) > 0.
\end{align*}
\]

Hence the proof of Lemma 3.1(I)(xiv) is complete.

Proof of Lemma 3.1(I)(xv). For \( \rho > 0 \), we compute that \( \frac{\partial}{\partial a} f(\rho, a) = \frac{\rho^2}{(\rho + a)^2} f(\rho, a) \) and

\[
\begin{align*}
\frac{\partial}{\partial a} M_1(\rho, a) &= \frac{\partial}{\partial a} \frac{\rho f(\rho, a)}{F(\rho, a)} = \rho \frac{\frac{\partial^2}{\partial^2} f(\rho, a) F(\rho, a) - f(\rho, a) \frac{\partial^2}{\partial a} F(\rho, a)}{[F(\rho, a)]^2} \\
&= \rho \frac{\frac{\partial^2}{\partial a} f(\rho, a) \int_0^\rho f(s, a) ds - f(\rho) \left[ \int_0^\rho s^2 f(s, a) ds \right]}{[F(\rho, a)]^2} \\
&= \rho \frac{\rho f(\rho, a)}{[F(\rho, a)]^2} \int_0^\rho \left[ \frac{\rho^2}{(\rho + a)^2} - \frac{s^2}{(s + a)^2} \right] f(s, a) ds \\
&= \frac{\rho f(\rho, a)}{[F(\rho, a)]^2} \int_0^\rho \frac{(\rho - s)(\rho a + 2s^2)}{(\rho + a)^2(s + a)^2} f(s, a) ds > 0,
\end{align*}
\]

Hence, for \( \rho > 0 \), \( M_1(\rho, a) \) and \( M_2(\rho, a) \) are both strictly increasing functions of \( a \) on \((0, \infty)\), which completes the proof of Lemma 3.1(I)(xv).

Proof of Lemma 3.1(III)(v). (I). We prove that \( K_1(\rho) < M_1(\rho) \) for \( 1/4 \leq \rho \leq 1/2 \) and \( a \geq 4 \). Note first that, for \( 1/4 \leq \rho \leq 1/2 \),

\[
K_1(\rho) = \frac{-3200\rho^3 + 19800\rho^2 + 2702\rho + 7959}{50(200\rho^2 + 50\rho + 153)} > \frac{\rho^2 (-3200/2 + 19800)}{50(200\rho^2 + 50\rho + 153)} > 0.
\]

Hence, for \( 1/4 \leq \rho \leq 1/2 \) and \( a \geq 4 \), to prove that \( K_1(\rho) < M_1(\rho, a) = \rho f(\rho, a)/F(\rho, a) \) is equivalent to prove that

\[
n_5(\rho, a) \equiv \frac{\rho f(\rho, a)}{K_1(\rho)} - F(\rho, a) > 0. \tag{3.21}
\]

Indeed, we compute that,

\[
\frac{\partial}{\partial \rho} n_5(\rho, a) = \frac{n_6(\rho, a)}{(\rho + a)^2 (3200\rho^3 - 19800\rho^2 - 2702\rho - 7959)^2} f(\rho, a), \tag{3.22}
\]

where

\[
n_6(\rho, a) \equiv n_{6,2}(\rho)a^2 + n_{6,1}(\rho)a + n_{6,0}(\rho)
\]
and
\[
\begin{align*}
n_{6,2}(\rho) &= -42240000\rho^6 + 316720000\rho^5 - 116707200\rho^4 \\
&\quad + 284753400\rho^3 - 187854404\rho^2 + 57670914\rho - 2459331, \\
n_{6,1}(\rho) &= -20480000\rho^7 + 253440000\rho^6 - 337494400\rho^5 \\
&\quad + 93876800\rho^4 - 45684408\rho^3 - 6430872\rho^2 - 4918662\rho, \\
n_{6,0}(\rho) &= -10240000\rho^8 + 126720000\rho^7 - 168747200\rho^6 \\
&\quad + 46938400\rho^5 - 22842204\rho^4 - 3215436\rho^3 - 2459331\rho^2.
\end{align*}
\]

Then it can be verified easily (but tediously and hence omitted) that, for \(1/4 \leq \rho \leq 1/2\) and \(a \geq 4\),
\[
\begin{align*}
n_6(\rho, 4) &= -10240000\rho^8 + 44800000\rho^7 + 169172800\rho^6 + 37644800\rho^5 \\
&\quad - 1720230204\rho^4 + 2725461332\rho^3 - 3033853283\rho^2 + 903059976\rho - 39349296 \\
&\equiv n_7(\rho) > 0, \quad \text{(See Fig. 1(a).)} \quad (3.23) \\
\frac{\partial}{\partial a} n_6(\rho, 4) &= -20480000\rho^7 - 84480000\rho^6 + 2196265600\rho^5 - 839780800\rho^4 \\
&\quad + 1821182792\rho^3 - 1509266104\rho^2 + 456448650\rho - 19674648 \\
&\equiv n_8(\rho) > 0, \quad \text{(See Fig. 1(b).)} \quad (3.24) \\
\frac{\partial^2}{\partial a^2} n_6(\rho, a) &= -84480000\rho^6 + 633440000\rho^5 - 233414400\rho^4 \\
&\quad + 569506800\rho^3 - 375708808\rho^2 + 115341828\rho - 4918662 \\
&\equiv n_9(\rho) > 0. \quad \text{(See Fig. 1(c).)} \quad (3.25)
\end{align*}
\]

It implies that, for \(1/4 \leq \rho \leq 1/2\) and \(a \geq 4\), \(n_6(\rho, a) > 0\) and hence, by (3.22),
\[
\frac{\partial}{\partial \rho} n_5(\rho, a) > 0. \quad (3.26)
\]

Fig. 1. Function \(n_7(\rho)\) (resp., \(n_8(\rho)\) and \(n_9(\rho)\)) defined in (3.23) (resp., (3.24) and (3.25)) is positive on the interval \([1/4, 1/2]\).
Next, we show that \( n_5(1/4, a) > 0 \) for \( a \geq 4 \). It is easy to see that
\[
\int_0^{1/4} \left[ (6b_1 - 48b_0 + 192)(s - 1/4)^2 + b_1(s - 1/4) + b_0 \right] ds = 1
\]
for any \( b_0, b_1 > 0 \). Thus, if we let \( b_0 = \frac{9822}{2225} \) and \( b_1 = \frac{9822}{2225} \frac{\partial f(1/4,a)}{f(1/4,a)} \), then
\[
n_5(1/4, a) = \frac{1}{K_1(1/4)} f(1/4, a) - F(\rho, a) = \frac{2225}{9822} f(1/4, a) - F(1/4, a)
\]
\[
= \int_0^{1/4} \left\{ \frac{2225}{9822} f(1/4, a) \left[ (6b_1 - 48b_0 + 192)(s - 1/4)^2 + b_1(s - 1/4) + b_0 \right] - f(s, a) \right\} ds
\]
\[
\equiv \int_0^{1/4} n_{10}(s, a) ds.
\] (3.27)

Note that \( n_{10}(1/4, a) = 0 \), \( \frac{\partial}{\partial s} n_{10}(1/4, a) = 0 \) and we compute that, for \( 0 < s < 1/4 \) and \( a \geq 4 \),
\[
\frac{\partial^2}{\partial s^2} n_{10}(s, a)
\]
\[
= \frac{2225}{9822} (6b_1 - 48b_0 + 192) f(1/4, a) - \frac{\partial^2}{\partial s^2} f(s, a)
\]
\[
= \frac{32}{1637} \frac{2446a^2 - 3688a - 461}{(4a + 1)^2} f(1/4, a) - \frac{\partial^2}{\partial s^2} f(s, a)
\]
\[
> \frac{32}{1637} \frac{2446a^2 - 3688a - 461}{(4a + 1)^2} - \left[ \frac{a^2(a^2 - 2a - 2s)}{(a + s)^4} \right]_{s=0} e^{1/4}
\]
\[
(\text{since } \frac{\partial^2}{\partial s^2} f(s, a) = \frac{a^2(a^2 - 2a - 2s)}{(a + s)^4} f(s, a) < \frac{a^2(a^2 - 2a - 2s)}{(a + s)^4} e^{1/4} \text{ by Lemma 3.1(I)(i)}
\]
\[
\text{and } \frac{\partial}{\partial s} a^2(a^2 - 2a - 2s) = -\frac{2a^2(2a^2 - 3a - 3s)}{(a + s)^5} \leq -\frac{2a^2(2a^2 - 3a - 3/4)}{(a + s)^5} < 0
\]
\[
= \frac{(78272 - 26192e^{1/4})a^3 - (118016 - 39288e^{1/4})a^2 + (24555e^{1/4} - 14752)a + 3274e^{1/4}}{1637a(4a + 1)^2}
\]
\[
> \frac{a^2 \left[ (78272 - 26192e^{1/4})a - (118016 - 39288e^{1/4}) \right] + 3274e^{1/4}}{1637a(4a + 1)^2}
\] (since \( 24555e^{1/4} - 14752 \approx 16777 > 0 \))
\[
\geq \frac{a^2 \left[ 4(78272 - 26192e^{1/4}) - (118016 - 39288e^{1/4}) \right] + 3274e^{1/4}}{1637a(4a + 1)^2}
\] (since \( 78272 - 26192e^{1/4} \approx 4464 > 0 \))
\[
> \frac{3274e^{1/4}}{1637a(4a + 1)^2}
\] (since \( 4(78272 - 26192e^{1/4}) - (118016 - 39288e^{1/4}) = 195072 - 65480e^{1/4} \approx 110994 > 0 \))
\[
> 0.
\]
So \( n_{10}(s, a) > 0 \) for \( 0 < s < 1/4 \) and \( a \geq 4 \) and hence \( n_5(1/4, a) > 0 \) for \( a \geq 4 \) by (3.27).
Combining this fact with (3.26), we conclude that, for $1/4 \leq \rho \leq 1/2$ and $a \geq 4$, $n_5(\rho, a) > 0$ and hence $K_1(\rho) < M_1(\rho, a)$ by (3.21).

(II). We prove that $M_1(\rho) < K_2(\rho)$ for $0 < \rho \leq 1/2$ and $a \geq 4$. Note first that, for $0 \leq \rho \leq 1/2$,

$$
K_2(\rho, a) \geq \left[ \frac{8}{3} - \frac{132a + 165}{50a + 25} \right]_{a=4} \rho^2 + \frac{2}{3} \rho + 1 = -\frac{31}{75} \rho^2 + \frac{2}{3} \rho + 1 > 0,
$$

since $\left[ -\frac{31}{75} \rho^2 + \frac{2}{3} \rho + 1 \right]_{\rho=1/2} = 123/100 > 0$. Hence, for $0 < \rho \leq 1/2$ and $a \geq 4$, to prove that $M_1(\rho, a) \equiv \varphi(\rho, a)/F(\rho, a)) < K_2(\rho, a)$ is equivalent to prove that

$$
n_{11}(\rho, a) \equiv F(\rho, a) - \frac{\rho f(\rho, a)}{K_2(\rho, a)} > 0. \tag{3.28}
$$

By direct computations, we have that $n_{11}(0, a) = 0$, $\frac{\partial}{\partial \rho} n_{11}(0, a) = 0$ and $\frac{\partial^2}{\partial \rho^2} n_{11}(0, a) = 1/3 > 0$. Hence, if we can prove that $\frac{\partial}{\partial \rho} n_{11}(\rho, a) < 0$ for $0 < \rho < 1/2$ and $a \geq 4$ and that $n_{11}(1/2, a) > 0$ for $a \geq 4$, then $n_{11}(\rho, a), a \geq 4$, is either strictly increasing or first strictly increasing and then strictly decreasing on $[0, 1/2]$. Consequently, $n_{11}(\rho, a) > \min \{n_{11}(0, a), n_{11}(1/2, a)\} = 0$ for $0 < \rho < 1/2$ and $a \geq 4$, which completes the proof.

Indeed, we compute that, for $0 < \rho < 1/2$ and $a \geq 4$,

$$
\frac{\partial^2}{\partial \rho^2} n_{11}(\rho, a) = -\frac{n_{12}(\rho, a)}{(a + \rho)^6[(4\rho^2 + 100\rho + 150)a - (295\rho^2 - 50\rho - 75)]^4} f(\rho, a), \tag{3.29}
$$

where

$$
n_{12}(\rho, a) \equiv n_{12,10}(\rho)a^{10} + n_{12,9}(\rho)a^9 + n_{12,8}(\rho)a^8 + n_{12,7}(\rho)a^7 + n_{12,6}(\rho)a^6 + n_{12,5}(\rho)a^5 + n_{12,4}(\rho)a^4 + n_{12,3}(\rho)a^3 + n_{12,2}(\rho)a^2 + n_{12,1}(\rho)a + n_{12,0}(\rho)
$$

and

$$
n_{12,10}(\rho) = -256\rho^8 - 16000\rho^7 - 307200\rho^6 - 1182400\rho^5 + 12142400\rho^4 + 3180000\rho^3
\quad + 120960000\rho^2 - 195750000\rho + 256500000
\geq \rho^4 \left( -256\rho^4 - 16000\rho^3 - 307200\rho^2 - 1182400\rho + 12142400 \right)
\quad + (-195750000\rho + 256500000)
\geq \rho^4 \left[ -256\rho^4 - 16000\rho^3 - 307200\rho^2 - 1182400\rho + 12142400 \right]_{\rho=1/2}
\quad + (-195750000/2 + 256500000)
\quad = 12100524\rho^4 + 158625000 \geq 0,
$$

$$
n_{12,9}(\rho) = -512\rho^9 + 24832\rho^8 + 1471200\rho^7 + 7319200\rho^6 - 137460800\rho^5 + 706615200\rho^4
\quad + 619515000\rho^3 + 6933735000\rho^2 - 1235250000\rho + 4502250000
\geq \rho^8 \left( -512\rho + 24832 \right) + \rho^4 \left( -137460800\rho + 706615200 \right)
\quad + (-1235250000\rho + 4502250000)
\geq \rho^8 \left( -512/2 + 24832 \right) + \rho^4 \left( -137460800/2 + 706615200 \right)
\quad + (-1235250000/2 + 4502250000)
\quad = 24576\rho^8 + 637884800\rho^4 + 3884625000 \geq 3884625000,
$$

with $\varphi(\rho, a) \equiv f(\rho, a) = \rho(\rho^4 - 3\rho^3 - 2\rho^2 + 4\rho - 1)$.

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$$n_{12.8}(\rho) = -256\rho^{10} + 126976\rho^9 + 1865760\rho^8 + 34983000\rho^7 + 1316010400\rho^6$$
$$+ 845606700\rho^5 + 16752839500\rho^4 + 27908092500\rho^3$$
$$+ 58279702500\rho^2 + 300645000000\rho + 5356125000$$
$$\geq \rho^9 (-256\rho + 126976) \geq \rho^9 (-256/2 + 126976) = 126848\rho^9 \geq 0,$$

$$n_{12.7}(\rho) = 77056\rho^{10} - 11357120\rho^9 - 261518000\rho^8 - 4619532550\rho^7 - 19422836750\rho^6$$
$$- 21716243000\rho^5 + 18152965000\rho^4 + 348903296250\rho^3$$
$$+ 135101756250\rho^2 + 45186187500\rho + 1601437500$$
$$\geq \rho^4 \left[-11357120\rho^5 - 261518000\rho^4 - 4619532550\rho^3 - 19422836750\rho^2ight]_{\rho=1/2} = 7379971835\rho^9/4 \geq 0,$$

$$n_{12.6}(\rho) = 512\rho^{11} - 8756320\rho^{10} + 425016800\rho^9 + 6148320775\rho^8 + 35661957875\rho^7$$
$$- 89732426875\rho^6 + 65490918125\rho^5 + 814563403125\rho^4$$
$$+ 395949740625\rho^3 + 141649171875\rho^2 + 21321984375\rho + 126562500$$
$$\geq \rho^9 \left[-8756320\rho^8 + 425016800\rho^7 + 6148320775\rho^6 + 35661957875\rho^5 + 814563403125\rho^4ight]_{\rho=1/2} = 7379971835\rho^9/4 \geq 0.$$

$$n_{12.5}(\rho) = -151040\rho^{11} + 449582000\rho^{10} - 5398556050\rho^9 + 47297392000\rho^8 - 143387352500\rho^7$$
$$+ 366205973750\rho^6 + 1173195525000\rho^5 + 784271750000\rho^4$$
$$+ 21553312500\rho^3 + 67318593750\rho^2 + 2813906250\rho - 126562500$$
$$\geq \rho^8 \left[-151040\rho + 449582000\rho^2 + 5398556050\rho^6 + 47297392000\rho^7 + 143387352500\rho^7ight]_{\rho=1/2} = 7379971835\rho^9/4 \geq 0,$$

$$n_{12.4}(\rho) = 16708800\rho^{11} - 9208180225\rho^{10} + 4495767950\rho^9 - 211526101875\rho^8$$
$$+ 50291908250\rho^7 + 1158044534375\rho^6 + 107548537500\rho^5$$
$$+ 224914921875\rho^4 + 8467031250\rho^3 + 1170703150\rho^2 - 189843750\rho$$
$$\geq \rho^8 \left[-9208180225\rho^8 - 211526101875\rho^7 - 189843750\rho^8\right]_{\rho=1/2} = -1049712587725/1024 > -1025110000,$$

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\( n_{12,3}(\rho) = -821516000\rho^{11} + 25332638950\rho^{10} - 124121515500\rho^9 + 271303753750\rho^8 \\
+ 711758365000\rho^7 + 913136531250\rho^6 + 2360829375000\rho^5 \\
+ 42858281250\rho^4 + 171281250000\rho^3 \\
\geq \rho^{10}(-821516000\rho + 25332638950) + \rho^8(-124121515500\rho + 271303753750) \\
\geq 249218809500\rho^{10} + 209242996000\rho^8 \geq 0, \\
\)
\( n_{12,2}(\rho) = 15146701250\rho^{11} - 11202448000\rho^{10} + 88360833750\rho^9 + 261156046250\rho^8 \\
+ 405446975000\rho^7 + 200884687500\rho^6 + 10065937500\rho^5 \\
+ 236082937500\rho^4 + 913136531250\rho^3 \\
\geq \rho^{10}(-11202448000\rho + 88360833750) \geq \rho^9(-11202448000/2 + 88360833750) \\
= 82759609750\rho^9 \geq 0, \\
\)
\( n_{12,1}(\rho) = 22635202500\rho^{10} + 69315412500\rho^9 + 70012350000\rho^8 + 93814875000\rho^7 \\
- 6802312500\rho^6 + 49865625000\rho^5 \\
\geq \rho^5(-6802312500\rho + 4986562500) \geq \rho^5(-6802312500/2 + 4986562500) \\
= 1585406250\rho^6 \geq 0, \\
\)
\( n_{12,0}(\rho) = 11552568750\rho^{10} + 1762256250\rho^9 - 199125000\rho^8 + 831093750\rho^7 \\
\geq \rho^8(1762256250\rho^2 - 199125000\rho + 831093750) \\
\geq \rho^8[1762256250\rho^2 - 199125000\rho + 831093750]_{\rho=199125000/(2\cdot1762256250)} \\
= 774843750\rho^6 \geq 0. \\
\)

Above inequalities imply that, for \( a \geq 4 \) and \( 0 \leq \rho \leq 1/2 \),
\( n_{12}(\rho,a) > (3884625000a^9 - 244687500a^6 - 126562500a^5 - 102511000a^4) \\
= 2488265000a^9 + 244687500(a^9 - a^6) + 126562500(a^9 - a^5) + 102511000(a^9 - a^4) > 0 \\
\)
and hence, by (3.29),
\[
\frac{\partial^3}{\partial \rho^3} n_{11}(\rho,a) < 0. (3.30)
\]

Next, we show that \( n_{11}(1/2,a) > 0 \) for \( a \geq 4 \). It is easy to see that
\[
\int_0^{1/2} [(3b_1 - 12b_0 + 24)(s - 1/2)^2 + b_1(s - 1/2) + b_0] \, ds = 1
\]
for any \( b_0, b_1 > 0 \). If we let \( b_0 \equiv \frac{268a+35}{100a+50} \) and \( b_1 \equiv \frac{268a+35}{100a+50} \), then

\[ n_{11}(1/2, a) = F(1/2, a) - \frac{1}{2K_2(1/2, a)} f(1/2, a) = F(1/2, a) - \frac{100a + 50}{268a + 35} f(1/2, a) \]

\[ = \int_0^{1/2} \left\{ f(s, a) - \frac{100a + 50}{268a + 35} f(1/2, a) \left[ (3b_1 - 12b_0 + 24)(s - 1/2)^2 + b_1 (s - 1/2) + b_0 \right] \right\} ds \]

\[ = \int_0^{1/2} n_{13}(s, a) ds. \quad (3.31) \]

Note that \( n_{13}(1/2, a) = 0 \), \( \frac{\partial}{\partial s} n_{13}(1/2, a) = 0 \) and we compute that, for \( 0 < s < 1/2 \) and \( a \geq 4 \),

\[
\frac{\partial^2}{\partial s^2} n_{13}(s, a) = \frac{\partial^2}{\partial s^2} f(s, a) - \frac{200a + 100}{268a + 35} (3b_1 - 12b_0 + 24)f(1/2, a) \\
= \frac{\partial^2}{\partial s^2} f(s, a) - \frac{-24(4a^3 - 23a^2 - 192a - 65)}{(268a + 35)(2a + 1)^2} f(1/2, a) \\
> \left[ \frac{a^2(a^2 - 2a - 2s)}{(a + s)^4} \right]_{a=4} [f(s, a)]_{a=4} + \left[ \frac{24(4a^3 - 23a^2 - 192a - 65)}{(268a + 35)(2a + 1)^2} \right]_{a=4} e^{1/2} \\
(by \ Lemma \ 3.1(I)(i) \ and \ since \ \frac{\partial^2}{\partial s^2} f(s, a) = \frac{a^2(a^2 - 2a - 2s)}{(a + s)^4} f(s, a), \\
\frac{\partial}{\partial a} \frac{a^2(a^2 - 2a - 2s)}{(a + s)^4} = \frac{2a [(1 + 2s)a^2 - sa - 2s^2]}{(a + s)^5} \geq \frac{2a (a^2 - \frac{1}{2}a - \frac{1}{2})}{(a + s)^5} > 0, \\
\frac{d}{d a} \left[ \frac{24(4a^3 - 23a^2 - 192a - 65)}{(268a + 35)(2a + 1)^2} \right] = \frac{48736a^3 + 100040a^2 + 58175a + 9900}{(268a + 35)^2(2a + 1)^3} > 0) \\
eq \frac{16(-2s + 8)}{(s + 4)^4} e^{4s} - \frac{280}{1107} e^{1/2} \\
\geq \left[ \frac{16(-2s + 8)}{(s + 4)^4} e^{4s} \right]_{s=1/2} - \frac{280}{1107} e^{1/2} \\
(since \ \frac{d}{ds} \left[ \frac{16(-2s + 8)}{(s + 4)^4} e^{4s} \right] = \frac{32(3s^2 - 24s - 16)}{(s + 4)^6} e^{4s} < \frac{32(3/4 - 16)}{(s + 4)^6} e^{4s} < 0) \\
= \frac{1792}{6561} e^{4/9} - \frac{280}{1107} e^{1/2} (\approx 0.009) > 0. \\
\]

So \( n_{13}(s, a) > 0 \) for \( 0 < s < 1/2 \) and \( a \geq 4 \) and hence \( n_{13}(1/2, a) > 0 \) by (3.31). Combining this fact with (3.30), we conclude that, for \( 1/4 \leq \rho \leq 1/2 \) and \( a \geq 4 \), \( n_{11}(\rho, a) > 0 \) and hence \( M_1(\rho, a) < K_2(\rho, a) \) by (3.28).

Hence the proof of Lemma 3.1(III)(v) is complete.

**Proof of Lemma 3.1(III)(vi).** Let

\[ n_{14}(\rho, a) \equiv \frac{1}{\rho} \left\{ M_1(\rho, a) + \frac{132a + 165}{50a + 25} \rho^2 \right\} - K_3(\rho, a) = \frac{132a + 165}{50a + 25} \rho + 2 \frac{a^2}{(a + \rho)^2} - 2 e^{\frac{a^2}{a + \rho}} \]
for $0 < \rho \leq 1/2$ and $a \geq 4$. Then

$$\lim_{\rho \to 0^+} n_{14}(\rho, a) = 0,$$  \hspace{1cm} (3.32)

and

$$n_{14}(1/2, a) = \frac{(664a^2 + 462a + 165) - 100(4a^2 + 4a + 1)e^{24a^2 + 1}}{50(2a + 1)^2} = 8a^2 - \frac{99}{25}a + \frac{33}{10} - 2e^a$$

$$\equiv n_{15}(\alpha),$$

where $\alpha \equiv \frac{a}{2a+1}$ satisfies $4/9 \leq \alpha \leq 1/2$ for $a \geq 4$. We have that $n_{15}(\alpha) > 0$ for $4/9 \leq \alpha \leq 1/2$ since $n_{15}(4/9) = 12637/4050 - 2e^{4/9} (\approx 0.001) > 0$, $n_{15}^\prime(4/9) = 709/225 - 2e^{4/9} (\approx 0.032) > 0$ and $n_{15}^\prime\prime(\alpha) = 16 - 2e^\alpha > 16 - 2e^{1/2} (\approx 12.703) > 0$ for $4/9 \leq \alpha \leq 1/2$. Hence

$$n_{14}(1/2, a) > 0$$  \hspace{1cm} (3.33)

for $a \geq 4$. Moreover, we compute that, for $0 < \rho \leq 1/2$ and $a \geq 4$,

$$\frac{\partial^2}{\partial \rho^2} n_{14}(\rho, a) = -\frac{2a^2}{(a + \rho)^4} \left[ (a^2 - 2a - 2\rho)\frac{\partial}{\partial \rho} e^{2\rho} + 6 \right]$$

$$\leq -\frac{2a^2}{(a + \rho)^4} \left\{ [a^2 - 2a - 2/2]_{a=4} - 6 \right\} = -\frac{2a^2}{a + \rho})^4 < 0.$$

Combing this fact with (3.32) and (3.33), we conclude that, for $0 < \rho \leq 1/2$ and $a \geq 4$, $n_{14}(\rho, a) > 0$ and hence $K_3(\rho) < M_1(\rho) + \frac{192a + 165}{50a + 25} \rho^2$. Hence the proof of Lemma 3.1(III)(vi) is complete.

**Proof of Lemma 3.1(III)(vii).** We first show that (3.19) holds for function $L_1(\rho, s)$ at the two endpoints $s = 0$ and $\rho$ when $\rho$ is fixed on $[1/4, 1/2]$ and $a \geq 4$. That is, $L_1(\rho, 0) > 0$ and $L_1(\rho, \rho) < 0$ for $1/4 \leq \rho \leq 1/2$ and $a \geq 4$. Indeed, we compute that, for $1/4 \leq \rho \leq 1/2$ and $a \geq 4$,

$$L_1(\rho, 0) = -\frac{8}{25} F(\rho, a) + M_1(\rho, a) - 1$$

$$> -\frac{8}{25} \int_0^\rho e^s ds + K_1(\rho) - 1 \text{ (by Lemma 3.1(I)(i) and (III)(v))}$$

$$= -\frac{8}{25} (e^\rho - 1) + \frac{8}{3} \rho^2 + \frac{2}{3} \rho + \frac{1}{50} - \frac{1}{50} (16\rho - 3) - 1 \text{ (by (3.11))}$$

$$\equiv n_{16}(\rho) > 0. \text{ (see Fig. 2.)}$$  \hspace{1cm} (3.34)

Remark that the proof of (3.34) is easy but tedious and hence we omit it here.

On the other hand, for $1/4 \leq \rho \leq 1/2$ and $a \geq 4$, since $L_1(\rho, \rho) = \frac{8}{25} \rho - M_2(\rho, a) + M_1(\rho, a) - 1 = \frac{\rho f(\rho, a)}{F(\rho, a)} - [M_2(\rho, a) + 1 - \frac{8}{25} \rho]$ by (3.7) and $M_2(\rho, a) + 1 - \frac{8}{25} \rho \geq M_2(\rho, a) + 1 - \frac{4}{25} = M_2(\rho, a) + \frac{24}{25} > 0$, we have that $L_1(\rho, \rho) < 0$ if and only if

$$F(\rho, a) - \frac{\rho f(\rho, a)}{M_2(\rho, a) + 1 - \frac{8}{25} \rho} > 0.$$  \hspace{1cm} (3.35)
Fig. 2. Function $n_{16}(\rho)$ defined in (3.34) is positive on the interval $[1/4,1/2]$.

By direct computation, we have that

$$
\frac{\partial}{\partial a} \left[ F(\rho,a) - \frac{\rho f(\rho,a)}{M_2(\rho,a) + 1 - \frac{8}{25} \rho} \right] = \int_0^\rho \frac{\partial}{\partial a} f(\rho,a) ds + 25 \rho^3 f(\rho,a) \frac{(-17\rho + 25)a^2 + 16\rho^2 a + (8\rho^3 - 25\rho^2)}{((17\rho + 25)a^2 - (16\rho - 50)a - (8\rho - 25)\rho^2)^2} > 0,
$$

since $\frac{\partial}{\partial a} f(\rho,a) \geq 0$ by Lemma 3.1(I)(i) and since $(-17\rho + 25)a^2 + 16\rho^2 a + (8\rho^3 - 25\rho^2) \geq \frac{33}{2}a^2 + a - \frac{21}{4} > 0$ for $1/4 \leq \rho \leq 1/2$ and $a \geq 4$. Hence $L_1(\rho,\rho) < 0$ for $1/4 \leq \rho \leq 1/2$ and $a \geq 4$ if (3.35) holds for $1/4 \leq \rho \leq 1/2$ and $a = 4$ or equivalently, $[L_1(\rho,\rho)]_{a=4} < 0$ for $1/4 \leq \rho \leq 1/2$. Indeed, by Lemma 3.1(III)(v), we have that

$$
[L_1(\rho,\rho)]_{a=4} = \frac{8}{25} \rho - M_2(\rho) + M_1(\rho) - 1 \leq \frac{8}{25} \rho - M_2(\rho) + K_2(\rho) - 1
$$

$$
\leq -\frac{\rho}{75(4+\rho)^2} \left[ 31\rho^3 + 174\rho^2 - 96\rho + 16 \right] \leq -\frac{\rho}{75(4+\rho)^2} \left( 174\rho^2 - 96\rho + 16 \right)
$$

$$
\leq -\frac{80\rho}{2175(4+\rho)^2} \left( 174\rho^2 - 96\rho + 16 \right)_{\rho=96/(2\cdot174)} = -\frac{80\rho}{2175(4+\rho)^2} < 0
$$

for $1/4 \leq \rho \leq 1/2$. Hence we have that

$$
L_1(\rho,\rho) < 0
$$

(3.36)

for $1/4 \leq \rho \leq 1/2$ and $a \geq 4$.

Now we show that (3.19) holds in the following two cases, which completes the proof.
Case 1: $4 \leq a \leq 5$. Fix $1/4 \leq \rho \leq 1/2$. Then we compute that, for $0 < s < \rho$,

$$\frac{\partial^2}{\partial s^2} L_1(\rho, s)$$

$$= \frac{8}{25} \frac{a^2(a^2 + 2a + 2s)}{(a + s)^4 f(s, a)} \left\{ -4a^2 + (-8s + 50)a - (4s^2 + 25s) \right\} f(s, a) - [F(\rho, a) - F(s, a)] \right\}$$

$$\geq \frac{8}{25} \frac{a^2(a^2 + 2a + 2s)}{(a + s)^4 f(s, a)} \left\{ \left[ -4a^2 + (-8s + 50)a - (4s^2 + 25s) \right] f(s, a) - f(s_0, a)(\rho - s) \right\}$$

(since $\frac{\partial}{\partial a} \left[ -4a^2 + (-8s + 50)a - (4s^2 + 25s) \right] = -27a^2 - \frac{19}{2}a - \frac{73}{2} < 0$ and by the Mean Value Theorem for some $s_0 \in [s, \rho] \subset [0, 1/2]$)

$$\geq \frac{8}{25} \frac{a^2(a^2 + 2a + 2s)}{(a + s)^4 f(s, a)} \left[ 8e^{1/2} - 4s^2 + 136e^{1/2} - 65s + (150 - 70e^{1/2}) \right] > 0,$$

since $8e^{1/2} - 4 \approx 9 > 0$, $136e^{1/2} - 65 \approx 159 > 0$ and $150 - 70e^{1/2} \approx 35 > 0$. Combining this fact with (3.34) and (3.36), we conclude that (3.19) holds for $4 \leq a \leq 5$.

Case 2: $a > 5$. Fix $1/4 \leq \rho \leq 1/2$. Then we compute that, for $0 < s < \rho$,

$$\frac{\partial}{\partial s} L_1(\rho, s)$$

$$= \frac{8}{25} \frac{a^2(a^2 - s)}{(a + s)^3 f(s, a)} - \frac{8}{25} \left\{ -1 - \frac{9}{8}a^2 f(s, a) \right\} [F(\rho, a) - F(s, a)]$$

$$\leq \frac{8}{25} \frac{a^2}{(a + s)^2 f(s, a)} \left\{ \left[ -9a^3 - 73sa^2 - 48s^2a - 16s^3 \right] f(s, a) + [F(\rho, a) - F(s, a)] \right\}$$

(since $\frac{\partial}{\partial a} \left[ -9a^3 - 73sa^2 - 48s^2a - 16s^3 \right] = -s(41a^3 + 48sa^2 + 48s^2a + 16s^3) < 0$

and by the Mean Value Theorem for some $s_0 \in [s, \rho] \subset [0, 1/2]$)

$$\leq \frac{8}{25} \frac{a^2}{(a + s)^2 f(s, a)} \left[ -16s^3 - 240s^2 - 1825s + 1125 + e^{1/2}(1/2 - s) \right]$$

(since $-16s^3 - 240s^2 - 1825s + 1125 \geq [-16s^3 - 240s^2 - 1825s + 1125]_{s=1/2} = 301/2 > 0$ and $1 \leq f(s, a) < e^{1/2}$ by Lemma 3.1(I)(i))

$$= a^2 \frac{[16s^3 - (200e^{1/2} - 240)e^{1/2} + (1825 - 900e^{1/2})e^{1/2}] - (1125 - 500e^{1/2})}{625(5 + s)(a + s)^2 f(s, a)} < 0$$
3.1(I)(vii) and 3.1(III)(vii) is now complete.

Then there exists some $s$ defined in $(0, \rho)$ for $0 \leq s < \rho$ such that there exists some $s' > 0$ with $0 < s < s'$. Let $L_2(\rho, s) = M_1(\rho) - M_1(\rho) = 0$ and $\lim_{s \to 0^+} \frac{\partial}{\partial s} L_2(\rho, s) = M_2(\rho) + 1 - \frac{8}{25} \rho - M_1(\rho) = L_1(\rho, \rho) > 0$ by Lemma 3.1(I)(vii), we have that there exists some $t_0$ on $(0, s_0]$ such that $L_2(\rho, t_0) = 0$ and $\frac{\partial}{\partial s} L_2(\rho, t_0) \leq 0$. Note that, by the definition of $L_2(\rho, s)$ in (3.15), we have that $P_1(\rho, t_0) = \frac{8}{25} t_0 + M_1(\rho)$. Moreover, we compute that

$$\frac{\partial}{\partial s} L_2(\rho, t_0) = \frac{f(t_0)}{F(\rho) - F(t_0)} \left[ P_1(\rho, t_0) - 1 - M_2(t_0) \right] \frac{8}{25} \frac{F(\rho) - F(t_0)}{f(t_0)} \left[ 8 t_0 + M_1(\rho) - 1 - M_2(t_0) \right] \frac{8}{25} \frac{F(\rho) - F(t_0)}{f(t_0)} > 0,$$

by (3.14) and (3.19), which makes a contradiction with $\frac{\partial}{\partial s} L_2(\rho, t_0) \leq 0$. Hence $L_2(\rho, s) > 0$ for $0 < s < s^*$. Hence $L_2(\rho, s) > 0$ for $0 < s < s^*$. Hence $L_2(\rho, s) > 0$ for $0 < s < s^*$. Hence $L_2(\rho, s) > 0$ for $0 < s < s^*$. Hence $L_2(\rho, s) > 0$ for $0 < s < s^*$.

By similar arguments, we can prove that $L_2(\rho, s) > 0$ for $s^* < s < \rho$. Hence $L_2(\rho, s) \geq 0$ for $0 \leq s < \rho$, which completes the proof of Lemma 3.1(III)(vii).

Proof of Lemma 3.1(III)(ix). We would show that $L_3(\rho, s) > 0$ for $7/25 \leq \rho \leq 1/2$, $0 < s < \rho$ and $a \geq 4$ by the following strategy. For fixed $\rho$ on $(7/25, 1/2)$ and $a \geq 4$, suppose that there exists some $s_0$ on $(0, \rho)$ such that $L_3(\rho, s_0) = 0$. Then we claim that it must hold that $\frac{\partial}{\partial s} L_3(\rho, s_0) < 0$. However, it makes a contradiction with $\lim_{s \to \rho} L_3(\rho, s) > 0$ as claimed below.

Claim 1: $\lim_{s \to \rho} L_3(\rho, s) > 0$ for $7/25 \leq \rho \leq 1/2$ and $a \geq 4$.

Proof of Claim 1. By Lemma 3.1(I)(vii) and (I)(xi), for $7/25 \leq \rho \leq 1/2$ and $a \geq 4$, we
have that
\[
\lim_{s \to \rho^+} L_3(\rho, s) = \frac{(-25\rho^2 + 24\rho - 1)a^4 - (52\rho^2 + 4\rho)a^3 - (76\rho^3 + 6\rho^2)a^2 - (4\rho^3)a - \rho^4}{50(a + \rho)^2[(\rho + 1)a^2 + 2\rho a + \rho^2]}
\]
\[
> \frac{(-25\rho^2 + 24\rho - 1)a^4 - (52\rho^2 + 4\rho)a^3 - 11a^2 - a - 1}{50(a + \rho)^2[(\rho + 1)a^2 + 2\rho a + \rho^2]}
\]
\[
\equiv \frac{n_{17}(\rho, a)}{50(a + \rho)^2[(\rho + 1)a^2 + 2\rho a + \rho^2]} > 0,
\]
since \(n_{17}(\rho, a) > 0\) which follows from the facts that, for \(7/25 \leq \rho \leq 1/2\),
\[
n_{17}(\rho, 4) = -9728\rho^2 + 5888\rho - 449 \geq \min_{\rho \in \{7/25, 1/2\}} (-9728\rho^2 + 5888\rho - 449) = 63 > 0,
\]
\[
\frac{\partial}{\partial a} n_{17}(\rho, 4) = -8896\rho^2 + 5952\rho - 348 \geq \min_{\rho \in \{7/25, 1/2\}} (-8896\rho^2 + 5952\rho - 348) = 404 > 0,
\]
\[
\frac{\partial^2}{\partial a^2} n_{17}(\rho, 4) = -6048\rho^2 + 4512\rho - 214 \geq \min_{\rho \in \{7/25, 1/2\}} (-6048\rho^2 + 4512\rho - 214) = 530 > 0,
\]
\[
\frac{\partial^3}{\partial a^3} n_{17}(\rho, 4) = -2712\rho^2 + 2280\rho - 96 \geq \min_{\rho \in \{7/25, 1/2\}} (-2712\rho^2 + 2280\rho - 96) = 206112/625 > 0,
\]
\[
\frac{\partial^4}{\partial a^4} n_{17}(\rho, a) = -600\rho^2 + 576\rho - 24 \geq \min_{\rho \in \{7/25, 1/2\}} (-600\rho^2 + 576\rho - 24) = 2256/25 > 0.
\]
So Claim 1 holds.

**Claim 2:** For \(7/25 \leq \rho \leq 1/2\) and \(a \geq 4\), if there exists some \(s_0\) on \((0, \rho)\) such that \(L_3(\rho, s_0) = 0\), then \(\frac{\partial}{\partial a} L_3(\rho, s_0) > 0\).

**Proof of Claim 2.** Fix \(\rho\) on \((7/25, 1/2)\) and \(a \geq 4\). Suppose \(s_0\) be an arbitrary point on \((0, \rho)\) such that \(L_3(\rho, s_0) = 0\). Then, by (3.5) and (3.16), we have that
\[
\rho^2 f'(\rho) - s_0^2 f'(s_0) = \frac{3}{2} \frac{[\rho f(\rho) - s_0 f(s_0)]^2}{F(\rho) - F(s_0)} - \frac{37}{25} [\rho f(\rho) - s_0 f(s_0)].
\]
Moreover,
\[
\frac{\partial}{\partial s} L_3(\rho, s_0) = -\frac{3}{2} \frac{f(s_0) [\rho f(\rho) - s_0 f(s_0)]}{[F(\rho) - F(s_0)]^2} + \frac{3}{2} \frac{f(s_0) + s_0 f'(s_0)}{F(\rho) - F(s_0)} - \frac{2s_0 f'(s_0) + s_0 f''(s_0)}{\rho f(\rho) - s_0 f(s_0)}
\]
\[
+ \frac{f(s_0) + s_0 f'(s_0)}{\rho f(\rho) - s_0 f(s_0)} \left\{ \frac{3}{2} \frac{[\rho f(\rho) - s_0 f(s_0)]^2}{F(\rho) - F(s_0)} - \frac{37}{25} \frac{[\rho f(\rho) - s_0 f(s_0)]}{f(s)} \right\}
\]
\[
= -\frac{3}{2} \frac{f(s_0) [\rho f(\rho) - s_0 f(s_0)]}{[F(\rho) - F(s_0)]^2} + \frac{3}{2} \frac{f(s_0) + s_0 f'(s_0)}{F(\rho) - F(s_0)} - \frac{2s_0 f'(s_0) + s_0 f''(s_0) + \frac{37}{25} f(s) + \frac{37}{25} f'(s_0)}{\rho f(\rho) - s_0 f(s_0)}
\]
\[
= -\frac{f(s_0)}{\rho f(\rho) - s_0 f(s_0)} \left\{ \frac{3}{2} \frac{P_1(\rho, s_0)}{[P_1(\rho, s_0)]^2} - 3 [M_2(s_0) + 1] P_1(\rho, s_0) + \frac{s_0 f''(s_0) + \frac{37}{25} s_0 f'(s_0) + \frac{37}{25} f(s)}{f(s)} \right\} \quad \text{(by (3.5) and (3.8))}
\]
\[
\equiv -\frac{f(s_0)}{\rho f(\rho) - s_0 f(s_0)} n_{18}(\rho, s_0), \quad (3.37)
\]

where
\[
n_{18}(\rho, s) \equiv \frac{3}{2} \frac{[P_1(\rho, s)]^2 - 3 [M_2(s) + 1] P_1(\rho, s) + s_0 f''(s) + \frac{37}{25} s_0 f'(s) + \frac{37}{25} f(s)}{f(s)}. \quad (3.38)
\]

Hence to prove that \( \frac{\partial}{\partial s} L_3(\rho, s_0) < 0 \), it suffices to prove \( n_{18}(\rho, s) > 0 \) for \( 0 < s < \rho, \quad 7/25 \leq \rho \leq 1/2, \) and \( a \geq 4 \). We show it by the following Claims 2(a)–(c), which completes the proof of Claim 2.

**Claim 2(a):** For \( 0 < s < 1/2 \) and \( a \geq 4 \), \( n_{18}(\rho, s) \) is a strictly increasing function of \( \rho \) on \( (s, 1/2) \).

**Proof of Claim 2(a).** Fix \( s \) on \((0, 1/2)\) and \( a \geq 4 \). Then we compute that, for \( s < \rho < 1/2 \) \((< a)\),
\[
\frac{\partial}{\partial \rho} n_{18}(\rho, s) = 3 \left\{ [P_1(\rho, s)] - [M_2(s) + 1] \right\} \frac{\partial}{\partial \rho} P_1(\rho, s)
\]
\[
= 3 \left\{ [P_1(\rho, s)] - [M_2(s) + 1] \right\} \frac{f'(\rho) \left\{ [M_2(\rho) + 1] - P_1(\rho, s) \right\}}{F(\rho) - F(s)} > 0,
\]
by Lemma 3.1(I)(ix) and (I)(x). So Claim 2(a) holds.

By Claim 2(a), for \( a \geq 4 \), to prove \( n_{18}(\rho, s) > 0 \) for \( 0 < s < \rho \) and \( 7/25 \leq \rho \leq 1/2 \), it suffices to prove that \( \lim_{\rho \to s^+} n_{18}(\rho, s) > 0 \) for \( 7/25 \leq s < 1/2 \), and \( n_{18}(7/25, s) > 0 \) for \( 0 < s < 7/25 \). We show them by the following Claims 2(b) and 2(c), respectively.

**Claim 2(b):** \( \lim_{\rho \to s^+} n_{18}(\rho, s) > 0 \) for \( 7/25 \leq s < 1/2 \) and \( a \geq 4 \).
Proof of Claim 2(b). By Lemma 3.1(I)(vii), we compute that
\[
\lim_{\rho \to s^+} n_{18}(\rho, s) = \frac{3}{2} [M_2(s) + 1]^2 - 3 [M_2(s) + 1]^2 + \frac{s^2 f''(s) + \frac{87}{25} s f'(s) + \frac{37}{25} f(s)}{f(s)}
\]
\[
= (-25s^2 + 24s - 1)a^4 - (52s^2 + 4s)a^3 - (76s^3 + 6s^2)a^2 - 4s^3a - s^4 > 0
\]
for \( a \geq 4 \). Above inequality holds by the exactly same arguments given in Claim 1. So Claim 2(b) holds.

Claim 2(c): \( n_{18}(7/25, s) > 0 \) for \( 0 < s < 7/25 \) and \( a \geq 4 \).
Proof of Claim 2(c). Let \( k = 3/25 \). Then, by (3.38), we have that
\[
n_{18}(7/25, s) = \left\{ \begin{array}{ll}
3/2 \{ P_1(7/25, s) \}^2 - 3(k + 1) P_1(7/25, s) \\
+ 3[k - M_2(s)] P_1(7/25, s) + \frac{s^2 f''(s) + \frac{87}{25} s f'(s) + \frac{37}{25} f(s)}{f(s)}
\end{array} \right.
\]
\[
\geq -\frac{3}{2} (1 + k)^2 + \left\{ 3[k - M_2(s)] P_1(7/25, s) + \frac{s^2 f''(s) + \frac{87}{25} s f'(s) + \frac{37}{25} f(s)}{f(s)} \right\}
\]
\[
(\text{since } \frac{3}{2} x^2 - 3(k + 1)x \geq \left[ \frac{3}{2} x^2 - 3(k + 1)x \right]_{x=k+1})
\]
\[
\geq -\frac{3}{2} (1 + k)^2 \text{ for any } k, x \in \mathbb{R}
\]
\[
= 3[k - M_2(s)] P_1(7/25, s) - \left\{ \frac{3}{2} (1 + k)^2 - \frac{s^2 f''(s) + \frac{87}{25} s f'(s) + \frac{37}{25} f(s)}{f(s)} \right\}
\]
\[
\equiv n_{19}(s) P_1(7/25, s) - n_{20}(s) \equiv n_{21}(s).
\] (3.39)

Note first that \( n_{19}(0) = 3k = 9/25 > 0, n_{19}(7/25) = -\frac{2500a^2-1050a-147}{25(25a+7)^2} < 0 \) for \( a \geq 4 \), and \( n_{19}(s) \) is strictly decreasing on \( (0, 7/25) \) by Lemma 3.1(I)(vi). Thus there exists some \( s_1 \) on \( (0, 7/25) \) such that
\[
n_{19}(s) \begin{cases} > 0 & \text{when } s \in (0, s_1), \\ = 0 & \text{when } s = s_1, \\ < 0 & \text{when } s \in (s_1, 7/25].
\end{cases}
\] (3.40)

Moreover, since \( n_{19}(s_1) = 3[k - M_2(s_1)] = 0 \), i.e., \( M_2(s_1) = k \), we have that, for \( a \geq 4 \),
\[
n_{20}(s_1) = \frac{s^2 f''(s_1)}{f(s_1)} - \frac{87}{25} \frac{s_1 f'(s_1)}{f(s_1)} - \frac{37}{25} + \frac{3}{2} (1 + k)^2
\]
\[
= \frac{f''(s_1) f(s_1)}{f'(s_1)^2} \left[ \frac{s_1 f'(s_1)}{f(s_1)} \right]^2 - \frac{87}{25} \frac{s_1 f'(s_1)}{f(s_1)} - \frac{37}{25} + \frac{3}{2} (1 + k)^2
\]
\[
= -\frac{a^2 - 2a - 2s_1}{a^2} \left[ M_2(s_1) \right]^2 - \frac{87}{25} \frac{M_2(s_1)}{25} - \frac{37}{25} + \frac{3}{2} (1 + k)^2 \quad \text{(by (3.8))}
\]
\[
\leq -\frac{a^2 - 2a - 2s_1}{a^2} \left( k^2 - \frac{37}{25} \right) + \frac{3}{2} (1 + k)^2 \quad \text{(since } 0 < s_1 < 7/25)\]
\[
= \frac{475a^2 - 450a - 126}{15625a^2} \leq \frac{[475a^2 - 450a - 126]_{a=4}}{15625a^2} = \frac{5674}{15625a^2} < 0.
\]
It follows, by (3.39), that
\[ n_{18}(7/25, s_1) = -n_{20}(s_1) > 0. \]

Next, we show that \( n_{18}(7/25, s) > 0 \) for \( s \in (0, s_1) \cup (s_1, 7/25) \) and \( a \geq 4 \). By (3.39), it suffices to show that
\[ n_{21}(s) = n_{19}(s) \left[ P_1(7/25, s) - \frac{n_{20}(s)}{n_{19}(s)} \right] > 0 \]
for \( s \in (0, s_1) \cup (s_1, 7/25) \) and \( a \geq 4 \). We first compute that
\[
\frac{d}{ds} \left[ \frac{n_{20}(s)}{n_{19}(s)} \right] = \frac{5a^2 n_{22}(s, a)}{3(a + s)^3[(25s - 3)a^2 - 6sa - 3s^2]^2},
\]
where
\[
n_{22}(s, a) = (-125s^2 + 30s + 2)a^5 + (125s^3 + 280s^2 - 54s)a^4 \\
+ (220s^3 - 146s^2)a^3 - (30s^4 + 94s^3)a^2 + 24s^4a + 28s^5 \\
\geq a^2 \left[ (-125s^2 + 30s + 2)a^2 + (280s^2 - 54s)a^2 + (-146s^2)a - 175616/78125 \right] \\
(\text{since } [280s^2 - 54s]_{s=1/2} = 175616/78125) \\
\equiv a^2 n_{23}(s, a) > 0,
\]
since, for \( 0 < s < 7/25 \) and \( a \geq 4 \),
\[
n_{23}(s, 4) = -4104s^2 + 1056s + 9824384/78125 \\
\geq \left[ -4104s^2 + 1056s + 9824384/78125 \right]_{s=7/25} = 7787384/78125 > 0,
\]
\[
\frac{\partial}{\partial a} n_{23}(s, 4) = -3906s^2 + 1008s + 96 \\
\geq \left[ -3906s^2 + 1008s + 96 \right]_{s=7/25} = 45006/625 > 0,
\]
\[
\frac{\partial^2}{\partial a^2} n_{23}(s, 4) = -2440s^2 + 612s + 48 \\
\geq \left[ -2440s^2 + 612s + 48 \right]_{s=7/25} = 3508/125 > 0,
\]
\[
\frac{\partial^3}{\partial a^3} n_{23}(s, a) = -750s^2 + 180s + 12 \\
\geq \left[ -750s^2 + 180s + 12 \right]_{s=7/25} = 18/5 > 0.
\]
Hence \( 4 \frac{d}{ds} \left[ \frac{n_{20}(s)}{n_{19}(s)} \right] < 0 \) for \( s \in (0, s_1) \cup (s_1, 7/25) \) and \( a \geq 4 \). Moreover, since \( P_1(7/25, s) \) is strictly increasing on \([0, 7/25]\) by Lemma 3.1(I)(x), \( P_1(7/25, s) - \frac{n_{20}(s)}{n_{19}(s)} \) is strictly increasing on \((0, s_1)\) and \((s_1, 7/25)\). Hence by (3.40) and (3.39), \( n_{18}(7/25, s) > 0 \) for \( s \in (0, s_1) \) if \( n_{21}(0) > 0 \). Similarly, \( n_{18}(7/25, s) > 0 \) for \( s \in (s_1, 7/25) \) if \( \lim_{s \to (7/25)^-} n_{21}(s) > 0 \). Indeed, we
compute that

\[ n_{21}(0) = 3kM_1(7/25) - \left[ \frac{37}{25} + \frac{3}{2}(1 + k)^2 \right] \]
\[ > 3kK_1(7/25) - \left[ \frac{37}{25} + \frac{3}{2}(1 + k)^2 \right] \quad \text{(by Lemma 3.1(III)(v))} \]
\[ = 3k \frac{6373521}{5708750} + \frac{37}{25} - \frac{3}{2}(1 + k)^2 = 3 \frac{7}{25} \frac{6373521}{5708750} + \frac{37}{25} - \frac{3}{2} \left( 1 + \frac{7}{25} \right)^2 \]
\[ = \frac{45839}{142718750} > 0, \]

and, by Lemma 3.1(I)(vii),

\[
\lim_{s \to (7/25)^-} n_{21}(s) = \frac{8984375a^4 - 17500000a^3 - 10994375a^2 - 891800a - 62426}{625(25a + 7)^4} > 0 \\
> \frac{8984375a^4 - (17500000 + 10994375 + 891800 + 62426)a^3}{625(25a + 7)^4} \\
\geq \frac{(8984375a - 29448601)a^3}{625(25a + 7)^4} > 0.
\]

So Claim 2(c) holds.

By Claims 1 and 2, we have that \( L_3(\rho, s) > 0 \) for \( 0 < s < \rho, \, 7/25 \leq \rho \leq 1/2 \) and \( a \geq 4 \), which completes the proof of Lemma 3.1(III)(ix). \( \square \)

**References**