Robust Good-Deal Bounds in Incomplete Markets

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Abstract

In this study, we extend Cochrane and Saá-Requejo's (2000) analysis to derive good-deal bounds on asset prices when investors are concerned about model uncertainty and seek robust pricing decisions in incomplete markets. Under the assumption that asset prices are driven by geometric Brownian motion processes, we propose a framework that is meaningful and very natural for investors' decision-making problems involving uncertainty regarding asset price processes and derive closed-form solutions for the pricing bounds of a European option. Further, we investigate properties of the proposed pricing bounds and apply these bounds to value a European option whose underlying asset is a non-traded stock index. We find that, under certain circumstances of model uncertainty, the proposed pricing bounds can include sufficient amounts of the actual option prices, which is in contrast with the empirical finding of the good-deal bounds proposed by Cochrane and Saá-Requejo (2000).

1 Introduction

One of the most important breakthroughs in modern asset pricing theory is that, under the complete markets assumption, complex financial instruments can be perfectly replicated by sophisticated dynamic trading strategies that involve simpler securities. However, in many realistic situations, perfect replication is impracticable and impossible due to non-traded underlying assets or market frictions, which may further lead to the collapse of complete-market conditions. To circumvent this problem, several seminal papers, including Cochrane and Saá-Requejo (2000), Bernardo and Ledoit (2000) and Carr et al. (2001) propose various novel methods to deal with the valuation of securities in incomplete markets. In particular, Cochrane and Saá-Requejo (2000) argue that no portfolio traded in the market has more than twice the market Sharpe ratio. Thus, they propose an approach to calculate asset pricing bounds conditional on the absence of arbitrage and high Sharpe ratios. These pricing bounds (hereafter, good-deal bounds) are useful in situations for which a relative pricing approach is appropriate but perfect replication is not possible. For example, banks can use these good-deal bounds as bid and ask prices to synthesize non-traded securities.[1]

While these good-deal bounds are convenient and useful in certain applications, our empirical study finds that good-deal bounds are not sufficiently wide to cover the actual prices of options, particularly during the recent financial crisis. Specifically, we value a European call option whose underlying asset is the Taiwan Stock Exchange Capitalization Weighted Stock Index (TAIEX). Because the TAIEX is not a traded asset, we use the exchange traded fund of the Taiwan 50 index (TWETF) as an approximate hedge and calculate good-deal bounds conditional on the assumption that no portfolio has more than twice the Sharpe ratio from January 2, 2006 to December 28, 2012. We find that approximately 54% of the option prices fall outside the good-deal bounds during the entire sample period. Even though we widen these pricing bounds associated with four-times the market Sharpe ratio to allow for "unbelievably good deals," these pricing bounds

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[1]Cochrane and Saá-Requejo (2000) outline various methods for utilizing good-deal bounds, such as traders using the bounds as buy and sell points and brokers using these bounds as economic measures of the accuracy of option pricing formulas.

are not sufficiently large to cover the option prices. Moreover, we find that during the financial crisis in late 2008, only 27% of the option prices remained within the good-deal bounds. These results suggest that good-deal bounds should be used with caution. For example, our results imply that if banks would use these pricing bounds as bid and ask prices, these bounds would cause great losses for banks.

A possible cause for these results is that, under the assumption of Cochrane and Saá-Requejo (2000), investors have perfect knowledge of the true probability law governing the stochastic processes of asset prices, even in the face of the global financial crisis. However, in many situations, investors are uncertain regarding the true probability law; hence, any particular probability law or model used to describe the asset prices would be subject to potential model misspecification.[2] For these reasons, the objective of this study is to develop an asset pricing model in which investors account explicitly for model uncertainty in incomplete markets. Inspired by Cochrane and Saá-Requejo (2000), Maenhout (2004), and Hansen and Sargent (2008), we derive good-deal bounds that are robust to a particular type of model misspecification, stemming from the parameter uncertainty of asset processes. More specifically, we consider a European option whose underlying asset is non-tradable, but it is correlated with a traded asset. To obtain analytic-form solutions of pricing bounds, we assume that the non-tradable and traded assets are driven by geometric Brownian motions with some perturbation parameters. By controlling these perturbation parameters, we design a collection of models that comprise a broad range of alternative processes whose Kullback-Leibler divergence from the benchmark model is bounded by a specified value. Then, we use the collection of models to characterize a particular form of model misspecification for investors' decisions. Finally, we solve the optimization problem encountered by investors who maximize their utilities and consider the worst-case scenario under circumstances of model misspecification.

By eliminating investments with high Sharpe ratios, the derived pricing bounds (hereafter referred to as robust good-deal bounds) have several features. First, when mar-

[2]For example, Pástor and Stambaugh (2001) argue that there exists substantial uncertainty regarding the model that generates equity premium. Based on empirical analysis, Cochrane (1997) also suggests that there is a wide band of uncertainty regarding true market return.

kets are incomplete and investors have perfect knowledge of the data generating processes (DGPs) of asset prices, the proposed pricing bounds are reduced to those discussed in Cochrane and Saá-Requejo (2000). Second, when markets are incomplete but investors are concerned that the stochastic processes of asset prices are misspecified, our robust good-deal bounds are wider than those of Cochrane and Saá-Requejo (2000). The greater the uncertainty regarding the asset price processes, the wider the pricing bounds appear. This result is rather natural. For example, in practice, the bid-ask spread tends to be larger when banks are uncertain of their asset price processes. Finally, to provide a possible explanation for the finding that Cochrane and Saá-Requejo's (2000) pricing bounds cannot cover actual prices, a calibration to the TAIEX example in the Taiwan stock market is re-investigated using our robust good-deal bounds. We find that, under certain circumstances of model misspecification, the robust good-deal bounds can contain sufficient amounts of actual option prices. More importantly, compared to other time periods, it is also found that investors were less confident regarding the stochastic processes of asset prices in the face of the recent financial crisis

The remainder of this paper is organized in the following manner. In section 2, we describe the good-deal bounds given by Cochrane and Saá-Requejo (2000); moreover, we also present an empirical study on the TAIEX. In section 3, we introduce robust good-deal bounds and the study on the TAIEX is re-examined by applying these robust pricing bounds. In section 4, we present some concluding remarks.

2   Pricing Bounds and an Empirical Study on the TAIEX

2.1   Good-Deal Asset Pricing Bounds

First, we briefly describe the arbitrage-free, good-deal bounds proposed by Cochrane and Saá-Requejo (2000). Consider a traded basis asset that yields a stream of payoffs or dividends, $D_t \mathrm{d}t$. Under the assumption of expected utility maximization, the basic pricing model for the basis asset can be derived in the following manner:

$$S_t = \mathbb{E}_t\bigg(\int_{\tau=0}^{\infty} \frac{\Lambda_{t+\tau}}{\Lambda_t} D_{t+\tau}\mathrm{d}\tau \bigg), \tag{1}$$

where S_t denotes the price of the basis asset and Λ_t is a stochastic discount factor. The process S_t in (1) is assumed to satisfy

$$\frac{\mathrm{d}S_t}{S_t} = \mu_S \mathrm{d}t + \sigma_S \mathrm{d}B_t, \tag{2}$$

where $\mathrm{d}B_t$ is a Brownian motion with $\mathbb{E}(\mathrm{d}B_t \mathrm{d}B_t) = 1$. Now, consider a derivative whose underlying asset is non-tradable. This derivative pays continuous dividends at the rate $x^c(V_t)\mathrm{d}t$ at time t, where V_t is the value of the non-tradable asset. The process V_t is assumed to satisfy

$$\frac{\mathrm{d}V_t}{V_t} = \mu_V \mathrm{d}t + \sigma_B \mathrm{d}B_t + \sigma_W \mathrm{d}W_t, \tag{3}$$

where $\mathrm{d}W_t$ is a Brownian motion with $\mathbb{E}(\mathrm{d}W_t \mathrm{d}W_t) = 1$ and $\mathbb{E}(\mathrm{d}B_t \mathrm{d}W_t) = 0$.[3]

Because the payoff $x^c(V_t)$ depends on the non-tradable asset, the derivative is not able to replicate perfectly, and hence the basic pricing model in (1) is not directly applicable for evaluating this derivative. To deal with this problem, Cochrane and Saá-Requejo (2000) considered arbitrage-free, good-deal bounds and showed that the lower pricing bound, $\underline{c}_t$, can be obtained by solving the following constrained optimization problem:

$$\min_{\Lambda_\tau \geq 0,\, t \leq \tau \leq T} \mathbb{E}_t \left(\int_{\tau=t}^T \frac{\Lambda_\tau}{\Lambda_t} x^c(V_\tau)\mathrm{d}\tau \right) + \mathbb{E}_t \left(\frac{\Lambda_T}{\Lambda_t} x^c(V_T) \right), \tag{4}$$

subject to the following two constraints: that the discount factor prices the basis assets S_t correctly at each moment and that the volatility of the discount factor process, $\mathbb{E}_t(\mathrm{d}\Lambda_t^2/\Lambda_t^2)$, is less than a pre-specified value:

$$\mathbb{E}_t \left(\frac{\mathrm{d}\Lambda_t^2}{\Lambda_t^2} \right) \leq A^2 \mathrm{d}t \tag{5}$$

with $A^2 = (1 + h^2)/(1 + r_f)^2$, where r_f is the instantaneously risk-free rate and h is the pre-specified volatility bound. As shown in Hansen and Jagannathan (1991), the constraint in (5) implies that no portfolio priced by Λ_t can have a Sharpe ratio greater than h. That is, the pricing bounds rule out "good deals" with a high Sharpe ratio if h is sufficiently large. Because Rose (1976) and Cochrane and Saá-Requejo (2000) argue

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[3]Cochrane and Saá-Requejo (2000) consider more general cases in which the driving forces in $\mathrm{d}S_t$ and $\mathrm{d}V_t$ (i.e., $\mathrm{d}B_t$ and $\mathrm{d}W_t$) can be multidimensional vector of independent Brownian motions, and μ_S, σ_S, μ_V, σ_B and σ_W may not be constant.

that no portfolio traded in the market has more than twice the market Sharpe ratio, in practice, the term h can be pre-specified to the value of twice the market Sharpe ratio. The upper bound $\overline{c}_t$ follows from replacing min with max in the above optimization.

In particular, when there is only one extra noise, dW_t, driving dV_t, the good-deal bounds for a European option are given by

$$\underline{c}_t \text{ or } \overline{c}_t = V_0 e^{\eta T} \Phi\left(d + 0.5\sigma_V \sqrt{T}\right) - K e^{-r_f T} \Phi\left(d - 0.5\sigma_V \sqrt{T}\right), \tag{6}$$

where $\Phi(\cdot)$ denotes the standard normal cumulative distribution function, K is the strike price, $\sigma_V^2 = \sigma_B^2 + \sigma_W^2$,

$$d = \frac{\ln(V_0/K) + (\eta + r_f)T}{\sigma_V \sqrt{T}},$$

$$\eta = \left[h_V - h_S\left(\rho - a\sqrt{\frac{A^2}{h_S^2} - 1} \ \sqrt{1 - \rho^2}\right)\right]\sigma_V, \tag{7}$$

$$h_S = \frac{\mu_S - r_f}{\sigma_S}, \quad h_V = \frac{\mu_V - r_f}{\sigma_V}, \quad \rho = \frac{\sigma_B}{\sigma_V}, \quad A^2 = \frac{1 + h^2}{(1 + r_f)^2},$$

$a = +1$ for the upper bound and $a = -1$ for the lower bound; see Cochrane and Saá-Requejo (2000) for more details. Note that the larger the term h is (i.e., more "unbelievably good deals" are assumed to survive in markets), the wider the good-deal bounds appear.

2.2 An Empirical Study on the TAIEX

As an empirical example, we consider a European call option whose underlying asset is the TAIEX, where the TAIEX is a stock market index for over 700 listed companies traded on the Taiwan Stock Exchange (TSE). Because the TAIEX is not a traded asset, we employed the TWETF as an approximate hedge. It must be noted that there is an unavoidable basis risk between the TAIEX call option and the security TWETF because TWETF only tracks the Taiwan 50 index, which comprises the top 50 companies traded on the TSE.[4] We used the one-year certificate of deposit interest rate reported by the Bank of Taiwan as the risk-free rate and collected all the data needed for the period

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[4]Note also that the aggregate market value of these top 50 listed companies accounted for over 70% of the total market value in the TSE market.

July 1, 2003 – December 28, 2012 from the Taiwan Economic Journal database. To ease exposition, the TWETF is represented by S_t and the TAIEX is represented by V_t. By applying the maximum likelihood method, we recursively estimated the parameters in (2) and (3). For example, we used the sample period of July 1, 2003 – January 2, 2006 to estimate the parameters in January 3, 2006 and obtained the estimated parameters for the subsequent period by using the sample period of July 1, 2003 – January 3, 2006. Employing these estimated parameters, we calculated the market Sharpe ratios and estimated good-deal bounds in (6) and (7) based on the near-maturity and at-the-money TAIEX call options. Finally, we calculated the price-to-bounds ratio (hereafter referred to as the PB ratio) that is defined as

$$\text{PB ratio} = \frac{\text{number of actual option prices that stay within the bounds}}{\text{number of observations}}.$$

We used this PB ratio to assess the empirical relevance of the good-deal bounds in (6) and (7). We summarize the PB ratios in Table 1.

It is evident from Table 1 that the PB ratios vary over time. In addition, the PB ratios increase when the term h increases. For example, conditional on $h_2 = 2 \times SR$ (i.e., twice the estimated market Sharpe ratio), the largest PB ratio is approximately 52.59% in 2010, while the PB ratio is 40.16% in 2008. Further, the PB ratio for the entire sample period is only approximately 46.84%. In particular, we found that only 27.03% of the option prices remained within the good-deal bounds during the financial crisis in late 2008. Even when h increased to $h_4 = 4 \times SR$ (i.e., four-times the estimated Sharpe ratio), the PB ratio was still below 53% during the 2008 financial crisis. In other words, even if "unbelievably good deals" are assumed to survive in the markets, the derived good-deal bounds are not sufficiently large to cover the actual option prices. These results suggest that the good-deal bounds are not sufficiently appropriate to evaluate derivatives whose underlying asset is the TAIEX. These results also indicate that the good-deal bounds should be used with caution. For example, when banks use these good-deal bounds as bid and ask prices, very often they will find that the option prices do not lie within the bid-ask quotes. Such bid-ask quotes may cause great losses because when the option prices lie outside the bid-ask quotes, it implies that banks bid higher than actual prices (or offer

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Table 1: The PB Ratios of Good-Deal Bounds.

	2006	2007	2008	2009	2010	2011	2012
$h_2 = 2 \times SR$	50.00%	46.72%	40.16%	45.42%	52.59%	40.49%	52.40%
$h_3 = 3 \times SR$	64.92%	67.62%	51.81%	47.81%	72.11%	51.01%	60.40%
$h_4 = 4 \times SR$	75.81%	80.74%	63.05%	49.80%	83.27%	65.99%	64.80%

	2006 $\sim$ 2012	2008 Financial Crisis
$h_2 = 2 \times SR$	46.84%	27.03%
$h_3 = 3 \times SR$	59.37%	39.64%
$h_4 = 4 \times SR$	69.02%	52.25%

Note: SR denotes the estimated market Sharpe ratio. The PB ratios of the 2008 Financial Crisis are calculated from Sept. 15, 2008 to Feb. 28, 2009.

lower than actual prices). Moreover, these results imply that the tightening of good-deal bounds, such as by using the approach of Pyo (2011), may not perform better unless there is perfect knowledge that the actual prices will remain within the pricing bounds. This is because when actual prices fall outside good-deal bounds, further tightening the good-deal bounds may lead to a worse performance.

The finding that the good-deal bounds do not contain sufficient amounts of option prices can be caused by the methods of estimating parameters, the assumptions of constant parameters, and the number of extra driving forces in dV_t. Another possible cause of this finding is the assumption that investors have perfect knowledge of the true probability law governing the stochastic processes of asset prices. As suggested by Uppal and Wang (2003), Anderson et al. (2003), and Maenhout (2004), among many others, this assumption is too restrictive because in many situations investors are uncertain of the asset price processes. Therefore, it is desirable to take model uncertainty into account when studying asset pricing bounds in incomplete markets.

3 Robust Good-Deal Asset Pricing Bounds

3.1 Robust Good-Deal Bounds

Here, we follow Anderson et al. (2003), Maenhout (2004), and Hansen and Sargent (2008) and take model uncertainty into consideration. To incorporate uncertainty into investors' decisions, we extend the equations (2) and (3) by allowing some perturbation parameters:

$$\frac{\mathrm{d}S_t}{S_t} = (\mu_S + \lambda_1 \sigma_S)\mathrm{d}t + \sigma_S \mathrm{d}B_t,$$

$$\frac{\mathrm{d}V_t}{V_t} = (\mu_V + \lambda_1 \sigma_B + \lambda_2 \sigma_W)\mathrm{d}t + \sigma_B \mathrm{d}B_t + \sigma_W \mathrm{d}W_t, \tag{8}$$

where λ_1 and λ_2 are unknown perturbation parameters. Note that, compared with equations (2) and (3), some extra drift terms are added in model (8). The intuition behind model (8) can be described in the following manner. Let $\mathcal{P}$ denote the probability measure of investors' knowledge of price processes. It is often the case that the knowledge of $\mathcal{P}$ is based on some estimation results or prior beliefs. Because investors are not sure if $\mathcal{P}$ is the right model, it is natural that they would consider some alternative probability measures, $\{\mathcal{Q}_1, \mathcal{Q}_2, \ldots\}$, to allow for the possibility of model misspecification. Consider a possible alternative measure $\mathcal{Q}^\xi$, which is given by

$$\mathrm{d}\mathcal{P} = \xi \, \mathrm{d}\mathcal{Q}^\xi,$$

where ξ is a density function and can be regarded as a Radon-Nikodym derivative. According to Girsanov's theorem, considering this alternative measure is equivalent to shifting the drift terms in (8). Thus, model (8) can be used to describe the notion that investors are uncertain regarding the true price processes and thus consider numerous alternative models.

Of course, not all alternative measures should be considered in our framework; alternative measures that are too far away from the reference (or benchmark) measure, that is, $\mathcal{P}$, could be ignored. To illustrate this, we impose the following restriction:

$$\lambda_1^2 + \lambda_2^2 \leq \kappa^2, \tag{9}$$

to exclude some alternative measures, where κ is a pre-specified value. The larger κ^2 is, the more alternative measures that are far away from $\mathcal{P}$ are included. When $\kappa = 0$,

it implies that investors have perfect knowledge of the true probability law governing the stochastic processes of asset prices. The advantages of this constraint are twofold. First, if the notion of distance between measures is measured by the Kullback-Leibler divergence, equation (9) implies

$$\mathcal{D}_{KL}\Big(S_t \,||\, S_t(\lambda_1)\Big) \leq \kappa_1^2 \qquad \text{and} \qquad \mathcal{D}_{KL}\Big(V_t \,||\, V_t(\lambda_1, \lambda_2)\Big) \leq \kappa_2^2, \tag{10}$$

where $\mathcal{D}_{KL}\big(S_t \,||\, S_t(\lambda_1)\big)$ denotes the Kullback-Leibler divergence of S_t in (8) from S_t in (2), and $\mathcal{D}_{KL}\big(V_t \,||\, V_t(\lambda_1, \lambda_2)\big)$ measures the discrepancy or divergence of V_t in (8) from V_t in (3).[5] That is, in the sense of the Kullback-Leibler information criterion, a broad range of alternative models is considered by imposing the restriction in (9). Second, Ben-Tal and Nemirovski (1999) have shown that the robust solution of a linear programming problem with an ellipsoidal uncertainty set (that is, $\lambda_1^2 + \lambda_2^2 \leq \kappa^2$) is mathematically tractable.

Given the framework in (8) that investors are uncertain regarding the true processes of asset prices, we now discuss the optimization problem encountered by investors. Intuitively, when investors are concerned about model uncertainty and disfavour model misspecification, they entertain more conservative and pessimistic views regarding their decisions. This concern keeps the investors from choosing the most pessimistic or the worst-case scenario. Thus, instead of the constrained optimal decision problem in (4), the investors' optimization problem involving uncertainty regarding the true processes are given by

$$\begin{aligned}
\underline{C}_t &= \min_{\lambda_1, \lambda_2} \; \min_{\Lambda_\tau \geq 0, \, t \leq \tau \leq T} \mathbb{E}_t\bigg(\int_{\tau=t}^{T} \frac{\Lambda_\tau}{\Lambda_t} x^c(V_\tau) \mathrm{d}\tau \bigg) + \mathbb{E}_t\bigg(\frac{\Lambda_T}{\Lambda_t} x^c(V_T) \bigg), \\
&\text{s.t. } \lambda_1^2 + \lambda_2^2 \leq \kappa^2, \quad \mathbb{E}_t\bigg(\frac{\mathrm{d}\Lambda_t^2}{\Lambda_t^2} \bigg) \leq A^2 \mathrm{d}t,
\end{aligned} \tag{11}$$

and subject to the constraint that the selected stochastic discount factor should correctly price basis assets, where $\underline{C}_t$ denotes the lower robust pricing bound. Note that the first minimization (i.e., $\min_{\lambda_1, \lambda_2}$) in (11) describes the worst-case scenario considered by

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[5]It can be shown that $\mathcal{D}_{KL}\Big(S_t \,||\, S_t(\lambda_1)\Big) = \lambda_1^2 \, t/2$ and $\mathcal{D}_{KL}\big(V_t \,||\, V_t(\lambda_1, \lambda_2)\big) = (\lambda_1 w_1 + \lambda_2 w_2)^2 \, t/2$, where $w_1 = \sigma_B/(\sigma_B + \sigma_W)$ and $w_2 = \sigma_V/(\sigma_B + \sigma_W)$. By setting $k^2 = 4 \max\big\{k_1^2/t, \, (k_2 + k_1 w_1)^2/(w_2^2 + t)\big\}$, it is easy to verify that (9) implies equation (10).

investors. Again, the dynamics of upper bound $\overline{C}_t$ follow from replacing min with max in the above optimization.

In general, the robust pricing bounds in (11) can be approximated by using numerical methods. However, in certain special cases, such as the European call option discussed in section 2.1, closed-form solutions can be obtained. As shown in the Appendix, the robust good-deal bounds for European call options are given by

$$\underline{C}_t \text{ or } \overline{C}_t = V_0 e^{\eta T} \Phi\left(d + 0.5\sigma_V \sqrt{T}\right) - K e^{-r_f T} \Phi\left(d - 0.5\sigma_V \sqrt{T}\right), \tag{12}$$

where $\sigma_V^2 = \sigma_B^2 + \sigma_W^2$,

$$d = \frac{\ln(V_0/K) + (\eta + r_f)T}{\sigma_V \sqrt{T}},$$

$$\eta = \left[g_V - g_S\left(\rho - a\sqrt{\frac{A^2}{g_S^2} - 1} \sqrt{1 - \rho^2}\right)\right] \sigma_V,$$

$$g_S = \frac{\mu_S + \lambda_1^* \sigma_S - r_f}{\sigma_S}, \quad g_V = \frac{\mu_V + \lambda_1^* \sigma_B + \lambda_2^* \sigma_W - r_f}{\sigma_V}, \quad \rho = \frac{\sigma_B}{\sigma_V}, \tag{13}$$

$$\lambda_1^* = -\frac{\kappa h_S}{\kappa + A}, \quad \lambda_2^* = a\frac{\kappa\sqrt{(\kappa + A)^2 - h_S^2}}{\kappa + A}, \quad h_S = \frac{\mu_S - r_f}{\sigma_S}, \quad A^2 = \frac{1 + h^2}{(1 + r_f)^2},$$

$a = +1$ for the upper bound and $a = -1$ for the lower bound. With reference to the robust good-deal bounds in (12), some standard Greeks are also presented in the Appendix (see Table 3).

Several novel features are evident from (12) and (13). First, when markets are complete and the processes dS_t and dV_t are correctly specified, equations (12) and (13) are reduced to the Black-Scholes formula. To see this, let $V_t = S_t$ and $\kappa = 0$. The former represents the situation that the payoff of V_t can be perfectly replicated by the basis asset S_t, while the latter implies that investors have perfect knowledge of the true DGPs of asset prices. In this case, $\rho = 1$, $\lambda_1^* = \lambda_2^* = 0$ and η in (12) becomes zero. Substituting these results into equation (12), the Black-Scholes formula appears as a special case. Second, when markets are incomplete and dS_t and dV_t are correctly specified, equations (12) and (13) become the pricing bounds proposed by Cochrane and Saá-Requejo (2000). This is simply because $\kappa = 0$, and hence $\lambda_1^* = \lambda_2^* = 0$.

Third and more importantly, when markets are incomplete and investors worry about the worst-case scenario that involves model uncertainty regarding the true price processes, the robust good-deal bounds are wider than those discussed in Cochrane and Saá-Requejo (2000). The greater the uncertainty about the asset processes (i.e., κ increases), the wider the robust pricing bounds appear. Such wider pricing bounds are mainly attributed to the conservative and pessimistic decisions of investors in (11), not relaxing the assumption of no-arbitrage or allowing the existence of "unbelievably good deals." The intuition of the result is clear. For example, if banks are not sure whether their asset processes are correctly specified, they may express reservations regarding the pricing models and widen bid-ask quote prices. The only difference is that equation (12) provides closed-form solutions to optimally widen bid-ask quotes. To illustrate, Figure 1 presents the lower and upper good-deal bounds (the dashed lines) based on equation (6) and the following parameter values:

$$\mu_S = 0.13, \quad \sigma_S = 0.14, \quad \mu_V = 0.11, \quad \sigma_B = 0.13, \quad \sigma_W = 0.06,$$

$$V_0 = 80, \quad r_f = 0.05, \quad K = 100, \quad T = 1/4, \quad \ddot{h}_2 = 0.8382,$$

where $\ddot{h}_2 = 2 \times (0.11 - 0.05)/\sqrt{0.13^2 + 0.06^2} \approx 0.8382$. For comparison, Figure 1 also plots the European call option price derived by the Black-Scholes formula (the dotted line). Given the same parameter values, Figure 1 also shows the lower and upper robust good-deal bounds (the solid lines) according to equation (12), with $\kappa = 1$. As can be seen in this figure, the robust good-deal bounds are wider than the pricing bounds discussed in Cochrane and Saá-Requejo (2000), conditional on the same discount factor volatility constraint, $h = 0.8382$.

3.2 Re-examining the TAIEX with Robust Good-Deal Bounds

In this subsection, we re-examine the empirical study in the TAIEX by applying the robust good-deal bounds. Using the same procedure that was discussed in section 2.2, we calculate the PB ratios for the robust good-deal bounds in (12) and (13); we summarize these PB ratios in Table 2. To rule out "good deals" with high Sharpe ratios, in this table we only present the results of $h = h_2$ and $\kappa = \{1, 1.5\}$. For comparison purposes, we also
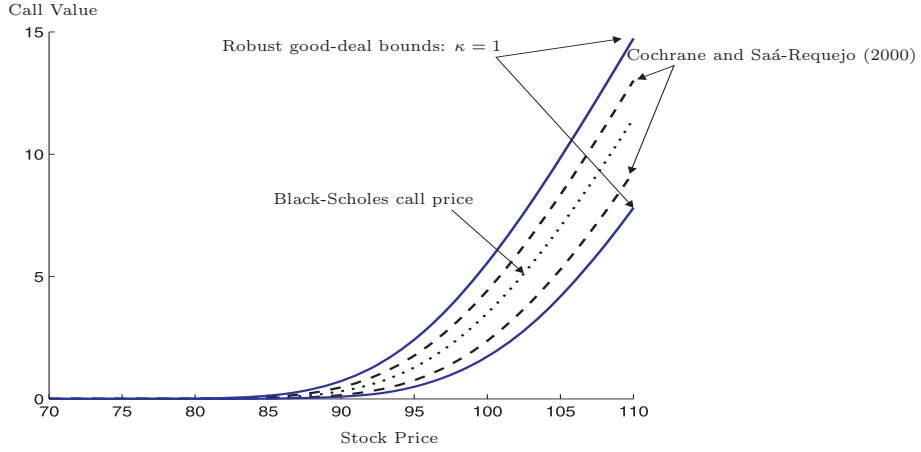
Figure 1: Upper and lower pricing bounds on a European call option.

present the PB ratios of the good-deal bounds of Cochrane and Saá-Requejo (2000) in this table (i.e., h_2 and $\kappa = 0$).

Several features emerge in Table 2. First, the PB ratios of the robust good-deal bounds vary over different time periods. For example, given that $\kappa = 1$, the largest PB ratio was 85.66% in 2010, while the PB ratio was 65.06% in 2008; the PB ratio was approximately 71.79% for the entire sample period. In addition, the PB ratios increase when the level of model uncertainty (i.e., κ) increases.[6] Second, compared with the results of the good-deal bounds (i.e., $\kappa = 0$), the PB ratios of the robust good-deal bounds rise substantially, particularly during the 2008 financial crisis. For example, during the times of financial crisis in late 2008, the PB ratio increases from 27.03% ($\kappa = 0$) to 65.77% ($\kappa = 1.5$). This result suggests that to obtain a reasonable PB ratio, the possibility of model misspecification should be taken into account. Third, compared with other time periods, the PB ratios during the 2008 financial crisis are the smallest values, regardless of the values of κ. That is, to sustain the same PB ratio, the value of κ during the 2008 financial crisis should be larger than these values in other periods. For example, to sustain the PB ratio at 68.55%, the value of κ was equal to 1 in 2006, while the value of κ should be larger than 1.5 during the financial crisis. This result implies that the investors entertain more conservative and pessimistic views regarding

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[6]As long as κ increases, the robust good-deal bounds can include all the actual prices.

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Table 2: The PB Ratios of Robust Good-Deal Bounds.

$h = h_2$	2006	2007	2008	2009	2010	2011	2012
$\kappa = 0$	50.00%	46.72%	40.16%	45.42%	52.59%	40.49%	52.40%
$\kappa = 1$	68.55%	75.00%	65.06%	66.93%	85.66%	72.47%	68.80%
$\kappa = 1.5$	74.19%	81.15%	73.09%	75.30%	89.64%	80.16%	73.60%

$h = h_2$	$2006 \sim 2012$	2008 Financial Crisis
$\kappa = 0$	46.84%	27.03%
$\kappa = 1$	71.79%	56.76%
$\kappa = 1.5$	78.16%	65.77%

Note: The PB ratios of the 2008 Financial Crisis are calculated from Sept. 15, 2008 to Feb. 28, 2009. For comparison, the PB ratios of Cochrane and Saá-Requejo's (2000) good-deal bounds are also presented here (i.e., $\kappa = 0$).

their decisions during the 2008 financial crisis.

4 Conclusion

In this study, we developed a framework that formalizes the problem of investors who are concerned about model uncertainty and seek robust pricing decisions in incomplete markets. Intuitively, because investors disfavour model misspecification and understand that the distributions of asset prices are not estimated with perfect precision, they are more conservative regarding their decisions. This concern may keep investors from choosing the worst-case scenario and leads to wider pricing bounds as compared to Cochrane and Saá-Requejo's (2000) good-deal bounds. In addition, when the degree of model uncertainty is high, investors are less confident regarding their asset price processes, which may widen the pricing bounds.

For the application, we first assumed that asset prices are driven by geometric Brownian motion processes and then we derived closed-form solutions for the robust pricing bounds of the European option. By applying the proposed pricing bounds (and the good-deal bounds) to evaluate a European option whose underlying asset is a non-traded

TAIEX, we found several interesting results. First, we found that the good-deal bounds of Cochrane and Saá-Requejo (2000) are not sufficiently wide to cover the actual prices of the options. This result suggests that the good-deal bounds may be not applicable for evaluating these derivatives. More importantly, it implies that tightening the pricing bounds may not be of significance, unless the actual prices are guaranteed to stay within the pricing bounds. Second, we find that, under certain circumstances of model uncertainty, the proposed pricing bounds can include the actual prices of the options. Third, compared to other time periods, we found that investors were less confident regarding the price processes when faced with the recent financial crisis. This result provides a possible explanation for the contention that the PB ratios of good-deal bounds tend to be low during the times of financial crisis in late 2008.

Appendix

Recall that our objective is to price a European option under the conditions that markets are incomplete and price processes may be misspecified. Along the lines of Cochrane and Saá-Requejo (2000) and Hansen and Sargent (2001), our objective is to solve the following constrained optimization problem for the lower pricing bounds on date $t = 0$:

$$\min_{\lambda_1, \lambda_2} \min_{\Lambda_\tau \geq 0, t \leq \tau \leq T} \mathbb{E}_t \left(\int_{\tau=t}^{T} \frac{\Lambda_\tau}{\Lambda_t} x^c(V_\tau) \mathrm{d}\tau \right) + \mathbb{E}_t \left(\frac{\Lambda_T}{\Lambda_t} x^c(V_T) \right),$$

$$\text{s.t.} \quad \lambda_1^2 + \lambda_2^2 \leq \kappa^2, \quad \mathbb{E}_0 \left(\frac{\mathrm{d}\Lambda_t^2}{\Lambda_t^2} \right) \leq A^2 \mathrm{d}t,$$

where κ^2 and A^2 are pre-specified values and $\mathrm{d}S_t$ and $\mathrm{d}V_t$ are given in (8). We adopt a two-step approach to solve this constrained optimization. In the first step, we solve the second minimization (i.e., $\min_{\Lambda_T \geq 0}$), provided that λ_1 and λ_2 are fixed. According to the proposition 5 of Cochrane and Saá-Requejo (2000), we know that the solution is given by:

$$\frac{\mathrm{d}\Lambda_t^*}{\Lambda_t^*} = -r_f \mathrm{d}t - \ddot{g}_S \mathrm{d}B_t - \sqrt{A^2 - \ddot{g}_S^2} \mathrm{d}W_t,$$

where $\ddot{g}_S = (\mu_S + \lambda_1 \sigma_S - r_f)/\sigma_S$. Thus, together with the specifications of $\mathrm{d}S_t$ and $\mathrm{d}V_t$ in (8), we know that S_T, V_T and Λ_T are jointly lognormally distributed:

$$S_T = S_0 \exp \left[(\mu_S + \lambda_1 \sigma_S - 0.5\sigma_S^2)T + \sigma_S \sqrt{T} \varepsilon_B \right],$$

$$V_T = V_0 \exp \left[(\mu_V + \lambda_1 \sigma_B + \lambda_2 \sigma_W - 0.5\sigma_V^2)T + \sigma_B \sqrt{T} \varepsilon_B + \sigma_W \sqrt{T} \varepsilon_W \right], \tag{14}$$

$$\Lambda_T^* = \Lambda_0^* \exp \left[(-r_f - 0.5A^2)T - \ddot{g}_S \sqrt{T} \varepsilon_B - \sqrt{A^2 - \ddot{g}_S^2} \sqrt{T} \varepsilon_W \right],$$

where ε_B and ε_W are independent $N(0,1)$ random variables and $\sigma_V^2 = \sigma_B^2 + \sigma_W^2$. Now we want to evaluate a European call option with payoff $x^c(V_T) = \max\{0, V_T - K\}$ and $x^c(V_t) = 0$, $\forall t < T$. Substituting V_T in (14) into $(V_T > K)$, we find that the option is in the money only when

$$\delta_1 > \frac{\ln \left(K/V_0 \right) - \left(\mu_V + \lambda_1 \sigma_B + \lambda_2 \sigma_W - 0.5\sigma_V^2 \right)T}{\sigma_V \sqrt{T}} \equiv b_0,$$

where $\delta_1 = (\sigma_B \varepsilon_B + \sigma_W \varepsilon_W)/\sigma_V$. For ease of exposition, we let

$$\delta_2 = \frac{(\sigma_B \varepsilon_W - \sigma_W \varepsilon_B)}{\sigma_V}, \quad b_1 = \frac{-\ddot{g}_S \sigma_B - \sqrt{A^2 - \ddot{g}_S^2} \sigma_W}{\sigma_V}, \quad b_2 = \frac{\ddot{g}_S \sigma_W - \sqrt{A^2 - \ddot{g}_S^2} \sigma_B}{\sigma_V}.$$

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Expressing the expected payoff of the call option in terms of these new variables, we obtain

$$\mathbb{E}_0\left[\frac{\Lambda_T^*}{\Lambda_0^*}\max\left(V_T - K, 0\right)\right] = \int_{\delta_2}\int_{\delta_1 > b_0}\left(V_0 e^{(\mu_V + \lambda_1\sigma_B + \lambda_2\sigma_W - 0.5\sigma_V^2)T + \sigma_V\sqrt{T}\delta_1} - K\right) \times$$

$$e^{(-r_f - 0.5\sigma_V^2)T} \times e^{b_1\sqrt{T}\delta_1 + b_2\sqrt{T}\delta_2} \times \frac{1}{\sqrt{2\pi}}e^{-0.5\delta_1^2} \times \frac{1}{\sqrt{2\pi}}e^{-0.5\delta_2^2}\mathrm{d}\delta_1\mathrm{d}\delta_2.$$

Rearranging these terms, we obtain

$$V_0 e^{\eta T}\Phi\left(d + 0.5\sigma_V\sqrt{T}\right) - Ke^{-r_f T}\Phi\left(d - 0.5\sigma_V\sqrt{T}\right), \tag{15}$$

where

$$d = \frac{\ln(V_0/K) + (\eta + r_f)T}{\sigma_V\sqrt{T}}, \quad \eta = \left[\ddot{g}_V - \ddot{g}_S\left(\rho + 1\sqrt{\frac{A^2}{\ddot{g}_S^2} - 1}\sqrt{1 - \rho^2}\right)\right]\sigma_V, \tag{16}$$

$$\ddot{g}_V = \frac{\mu_V + \lambda_1\sigma_B + \lambda_2\sigma_W - r_f}{\sigma_V},$$

and $\rho = \sigma_B/\sigma_V$.

Given the previous result in (15), in the second step the optimization problem becomes

$$\min_{\lambda_1,\lambda_2} V_0 e^{\eta T}\Phi\left(d + 0.5\sigma_V\sqrt{T}\right) - Ke^{-r_f T}\Phi\left(d - 0.5\sigma_V\sqrt{T}\right),$$

$$\text{s.t. } \lambda_1^2 + \lambda_2^2 \le \kappa^2.$$

Let

$$\mathcal{L} = V_0 e^{\eta T}\Phi\left(d + 0.5\sigma_V\sqrt{T}\right) - Ke^{-r_f T}\Phi\left(d - 0.5\sigma_V\sqrt{T}\right) - 0.5\ell(\lambda_1^2 + \lambda_2^2 - \kappa^2)$$

be the objective function of this optimization problem, where ℓ is the Lagrange multiplier. The first order conditions of this minimization are

$$\frac{\partial\mathcal{L}}{\partial\lambda_1} = V_0\frac{\partial\eta}{\partial\lambda_1}e^{\eta T}T\Phi\left(d + 0.5\sigma_V\sqrt{T}\right) + V_0 e^{\eta T}\frac{\partial}{\partial\lambda_1}\Phi\left(d + 0.5\sigma_V\sqrt{T}\right)$$

$$- Ke^{-r_f T}\frac{\partial}{\partial\lambda_1}\Phi\left(d - 0.5\sigma_V\sqrt{T}\right) - \ell\lambda_1 = 0,$$

$$\frac{\partial\mathcal{L}}{\partial\lambda_2} = V_0\frac{\partial\eta}{\partial\lambda_2}e^{\eta T}T\Phi\left(d + 0.5\sigma_V\sqrt{T}\right) + V_0 e^{\eta T}\frac{\partial}{\partial\lambda_2}\Phi\left(d + 0.5\sigma_V\sqrt{T}\right) \tag{17}$$

$$- Ke^{-r_f T}\frac{\partial}{\partial\lambda_2}\Phi\left(d - 0.5\sigma_V\sqrt{T}\right) - \ell\lambda_2 = 0,$$

$$\frac{\partial\mathcal{L}}{\partial\ell} = \lambda_1^2 + \lambda_2^2 - \kappa^2 = 0,$$

where

$$\frac{\partial \eta}{\partial \lambda_1} = \frac{-\sigma_V \sqrt{1-\rho^2}}{\sqrt{A^2/\ddot{g}_S^{\,2} - 1}}, \quad \frac{\partial \eta}{\partial \lambda_2} = \sqrt{1-\rho^2}\sigma_V,$$

$$\frac{\partial}{\partial \lambda_1}\Phi\Big(d \pm 0.5\sigma_V \sqrt{T}\Big) = \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{(d \pm 0.5\sigma_V \sqrt{T})^2}{2}\right\}\frac{-\sqrt{T}\sqrt{1-\rho^2}}{\sqrt{A^2/\ddot{g}_S^{\,2}-1}},$$

$$\frac{\partial}{\partial \lambda_2}\Phi\Big(d \pm 0.5\sigma_V \sqrt{T}\Big) = \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{(d \pm 0.5\sigma_V \sqrt{T})^2}{2}\right\}\Big(\sqrt{T}\sqrt{1-\rho^2}\Big).$$

Rearranging the first two equations in (17), we obtain

$$\left[\frac{1}{\lambda_2} - \frac{-1}{\lambda_1 \sqrt{A^2/\ddot{g}_S^{\,2}-1}}\right]\left\{V_0 \sigma_V T \Phi\Big(d+0.5\sigma_V \sqrt{T}\Big) + \frac{V_0 \sqrt{T}}{2\pi}\exp\Big(-\frac{(d+0.5\sigma_V \sqrt{T})^2}{2}\Big)\right.$$

$$\left. - \frac{K\sqrt{T}}{\sqrt{2\pi}}\exp\Big(-(r_f + \eta)T - \frac{(d-0.5\sigma_V \sqrt{T})^2}{2}\Big)\right\} = 0.$$

Note that the equality holds only when the value within the square brackets is zero:

$$\frac{1}{\lambda_2} - \frac{-1}{\lambda_1 \sqrt{A^2/\ddot{g}_S^{\,2}-1}} = 0. \tag{18}$$

Thus, equation (18), together with the third equation in (17), implies that

$$\lambda_1^* = -\frac{\kappa h_S}{\kappa + A}, \quad \lambda_2^* = -\frac{\kappa \sqrt{(\kappa + A)^2 - h_S^2}}{\kappa + A},$$

where $h_S = (\mu_S - r_f)/\sigma_S$. Thus, substituting λ_1^* and λ_2^* into (15) and (16), we obtain the analytic-form solution of the lower robust good-deal pricing bound. By replacing min with max in the above optimization, we can obtain the upper robust good-deal pricing bound. For ease in application, we also present some standard Greeks of robust good-deal bounds in Table 3.

Table 3: Greeks of Robust Good-Deal Bounds.

Greeks	
delta	$e^{\eta T}\Phi\!\left(d + 0.5\sigma_V\sqrt{T}\right)$
gamma	$\dfrac{e^{\eta T}}{V_0\sigma_V\sqrt{T}}\phi\!\left(d + 0.5\sigma_V\sqrt{T}\right)$
vega	$V_0\sqrt{T}e^{\eta T}\left[a\dfrac{\sqrt{A^2-g_S^2}}{\sqrt{1-\rho^2}}\sqrt{T}\Phi\!\left(d + 0.5\sigma_V\sqrt{T}\right) + \phi\!\left(d + 0.5\sigma_V\sqrt{T}\right)\right]$
theta	$-V_0\eta e^{\eta T}\Phi\!\left(d + 0.5\sigma_V\sqrt{T}\right) - r_f K e^{-r_f T}\Phi\!\left(d - 0.5\sigma_V\sqrt{T}\right) - \dfrac{K\sigma_V}{2\sqrt{T}}e^{-r_f T}\phi\!\left(d - 0.5\sigma_V\sqrt{T}\right)$
rho	$-V_0\sigma_V T e^{\eta T}\left[1 - \rho + a\dfrac{\sqrt{1-\rho^2}}{\sqrt{A^2-g_S^2}}\left(\dfrac{\sigma_V A^2}{1+r_f} - 1\right)\right]\Phi(d + 0.5\sigma_V\sqrt{T}) + TK e^{-r_f T}\Phi(d - 0.5\sigma_V\sqrt{T})$

Note: The term $\phi(\cdot)$ denotes the standard normal density function.

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