

Delay Spread and Coherence Bandwidth

- **Delay spread (T_d)**

- Difference of the propagation delays of the longest and shortest paths (with significant energy)

$$T_d = \max_{i,j} |\tau_i(t) - \tau_j(t)|$$

- Delay coherence \rightarrow frequency coherence: how quickly the channel changes in frequency
- At time t , frequency response is

$$H(f, t) = \sum_i a_i(t) e^{-j2\pi f \tau_i(t)}$$

- For multiple paths, differential phase = $2\pi f (\tau_i(t) - \tau_k(t))$
- The phase changes significantly as f changes by $\frac{1}{2T_d}$

Delay Spread and Coherence Bandwidth

- **Coherence Bandwidth (W_c)**

$$W_c = \frac{1}{2T_d}$$

- Multipath spread \uparrow , coherence bandwidth \downarrow
- **Flat fading v.s. Frequency selective fading**
- Flat fading: $W_c \gg W$
 - Delay spread \ll symbol time $\frac{1}{W}$ \rightarrow **one filter tap** is sufficient to represent the channel
- Flat fading: $W_c \ll W$
 - Channel is represented by multiple taps

Delay Spread and Coherence Bandwidth

- In most literature, T_d is defined by the **r.m.s. delay spread**

$$\sigma_\tau = \sqrt{\overline{\tau^2} - \tau^2}$$
$$\overline{\tau^2} = \frac{\sum_i a_i^2(t) \tau_i^2(t)}{\sum_i a_i^2(t)}$$
$$\tau = \frac{\sum_i a_i^2(t) \tau_i(t)}{\sum_i a_i^2(t)}$$

- In some literature,
 - If W_c is defined as BW over which frequency correlation is above **0.9**, $W_c \approx \frac{1}{50\sigma_\tau}$
 - If W_c is defined as BW over which frequency correlation is above **0.5**, $W_c \approx \frac{1}{5\sigma_\tau}$

Doppler Spread and Coherence Time

- $$h_l[m] = \sum_i a_i^b \left(\frac{m}{W}\right) \text{sinc}(l - W\tau_i(m/W))$$
$$= \sum_i a_i \left(\frac{m}{W}\right) e^{-j2\pi f_c \tau_i(m/W)} \text{sinc}(l - W\tau_i(m/W))$$

- Phase of the i -th path change greatly at intervals of $1/(4D_i)$
- $D_i = f_c \tau_i'(t)$: Doppler shift of the i -th path

- **Doppler Spread (D_s) :**

$$D_s = \max_{i,j} f_c |\tau_i'(t) - \tau_j'(t)|$$

Largest difference between Doppler shifts

When the Doppler shifts are different, the fastest changes in the filter taps are significant over delay changes of $\frac{1}{4D_s}$.

Doppler Spread and Coherence Time

- **Coherence time T_c**

The interval over which $h_l[m]$ changes significantly.

$$T_c = \frac{1}{4D_s}$$

- Doppler spread \uparrow , the coherence time \downarrow
- **Fast fading v.s. slow fading**
 - Fast fading: $T_c \ll$ delay requirement of the application
 - Slow fading: $T_c \gg$ delay requirement of the application
 - For voice transmission, the delay requirement is less than 100 ms. For data, the delay requirement is laxer.
 - If signal BW $\gg D_s$, effects of Doppler spread is negligible.

Doppler Spread and Coherence Time

- In some literature, coherence time has a different definition:
- T_c is a statistical measure of the time duration over which the channel is essentially invariant.
- Usually, $T_c \approx 1/D_s$
- If T_c is defined by the time duration over which the time correlation is above 0.5,

$$T_c \approx \frac{9}{16\pi D_s}$$

- Another popular definition is geometric mean of above two values:

$$T_c \approx 0.423/D_s$$

Outline

2.1 Wireless channels

2.2 Outage probability and error probability over fading channels

- 2.2.1 Average error probabilities over Rayleigh fading channels
- 2.2.2 Outage probability for Rayleigh fading channels

2.3 Diversity Techniques

AWGN Channel

- Recall the AWGN channel

$$y[m] = x[m] + w[m]$$

- $SNR = \frac{E[|x[m]|^2]}{E[|w[m]|^2]} = \frac{E_s}{N_0}$ depends only the transmit energy
- For BPSK signaling, $x[m] \in \{\pm\sqrt{E_s}\}$

- $x[m]$ is detected by

$$\hat{x}[m] = \begin{cases} \sqrt{E_s}, & \Re\{y[m]\} \geq 0 \\ -\sqrt{E_s}, & \Re\{y[m]\} < 0 \end{cases}$$

- The error probability is

$$P_e = Q\left(\frac{\sqrt{E_s}}{\sqrt{N_0/2}}\right) = Q\left(\sqrt{2SNR}\right) < e^{-SNR}$$

$$Q(x) < e^{-x^2/2}$$

- P_e decays exponentially in SNR
- It demands SNR= 7 dB to have $P_e = 10^{-3}$

Coherent detection in fading channel

- In flat fading channel

$$y = hx + w$$

- Rayleigh fading $h \sim CN(0,1)$
- AWGN $w \sim CN(0, N_0)$
- Assume h is known at rx \Rightarrow coherent detection

$$r = \frac{h^*}{|h|}y = |h|x + \frac{h^*}{|h|}w$$

- The noise $h^*w/|h| \sim CN(0, N_0)$
- With BPSK signaling, the coherent detection is

$$\hat{x} = \begin{cases} \sqrt{E_s}, & \Re\{r\} \geq 0 \\ -\sqrt{E_s}, & \Re\{r\} < 0 \end{cases}$$


Average error probabilities over Rayleigh fading channels (I)

- For a given channel h , the error probability is

$$P_{e|h} = Q\left(\frac{\sqrt{E_s}|h|}{\sqrt{N_0/2}}\right) = Q\left(\sqrt{2|h|^2 SNR}\right)$$

- In Rayleigh fading, $|h|^2$ is exponential with $\mathbf{E}[|h|^2]=1$
- The average error probability is

$$\begin{aligned}
 P_e &= \int_0^\infty Q(\sqrt{2uSNR})e^{-u}du \\
 &= \frac{1}{\sqrt{2\pi}} \int_0^\infty \int_{\sqrt{2uSNR}}^\infty e^{-\frac{t^2}{2}} e^{-u} dt du \\
 &= \frac{1}{\sqrt{2\pi}} \int_0^\infty \int_0^{\frac{t^2}{2SNR}} e^{-u} e^{-\frac{t^2}{2}} du dt \\
 &= \frac{1}{\sqrt{2\pi}} \int_0^\infty \left(1 - e^{-\frac{t^2}{2SNR}}\right) e^{-\frac{t^2}{2}} dt \\
 &= \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{t^2(1+SNR)}{2SNR}} dt \\
 &= \frac{1}{2} \left(1 - \sqrt{\frac{SNR}{1+SNR}}\right)
 \end{aligned}$$


 $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{t^2}{2}} dt$

Average error probabilities over Rayleigh fading channels (2)

- For Rayleigh fading channel, the error probability is

$$P_e = \frac{1}{2} \left(1 - \sqrt{\frac{SNR}{1+SNR}} \right)$$

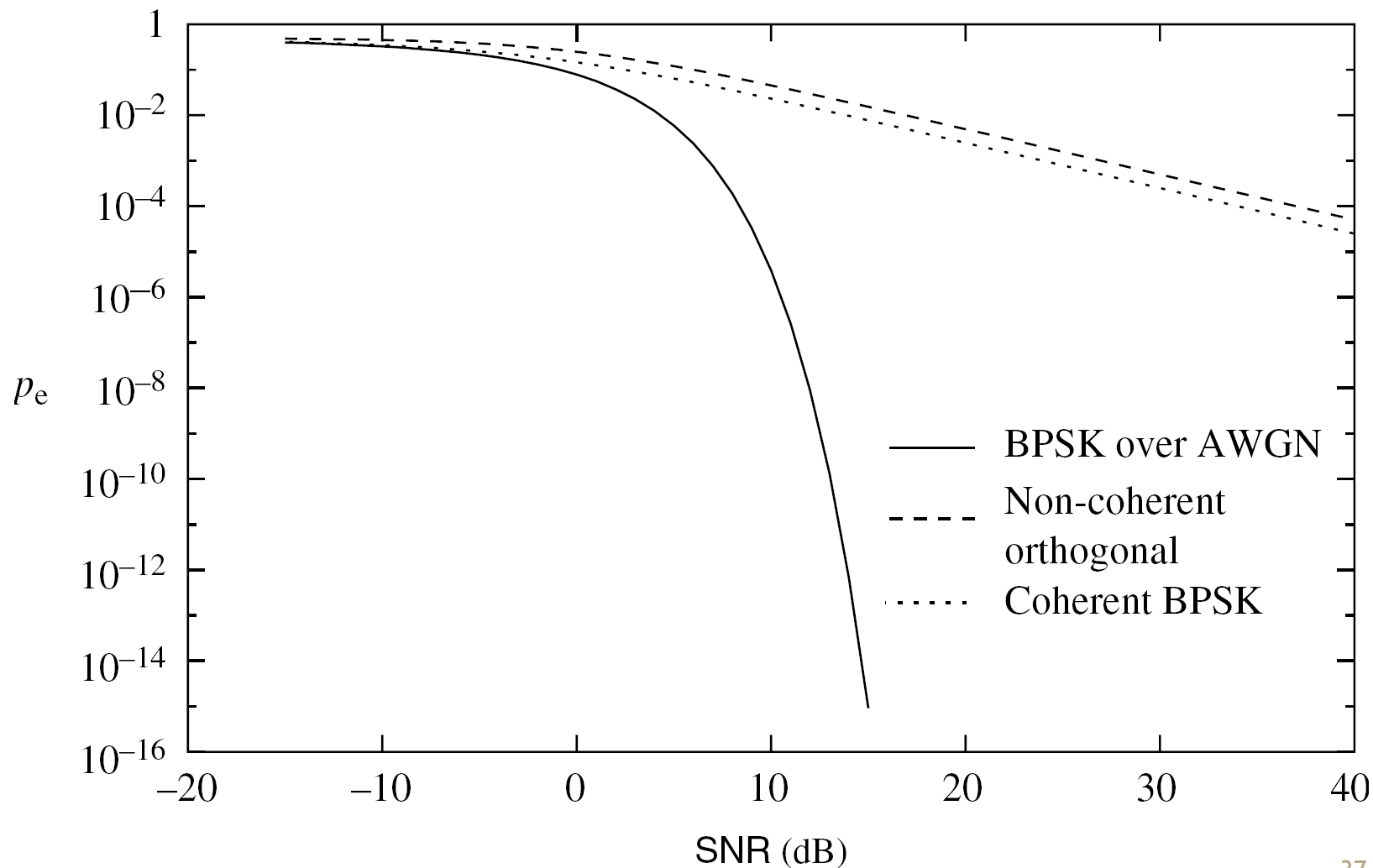
- At high SNR,

$$\sqrt{\frac{SNR}{1+SNR}} = 1 - \frac{1}{2SNR} + \mathcal{O}\left(\frac{1}{SNR^2}\right) \approx 1 - \frac{1}{2SNR}$$

$$\Rightarrow P_e \approx \frac{1}{4SNR}$$

- P_e decays inversely with SNR
- It demands SNR= 24 dB to have $P_e = 10^{-3}$
 \Rightarrow 17 dB loss compared with AWGN

Average error probabilities over Rayleigh fading channels (3)



Outage probability for Rayleigh fading channels

- In flat fading channel

$$y = hx + w$$

- Given the channel h , the receive SNR is random,

$$\rho = \frac{E_s |h|^2}{N_0}$$

- ρ is exponentially distributed with $\bar{\rho} = \frac{E_s}{N_0} = SNR$

- **Outage probability** (P_{out})

- $P_{out} = \Pr\{\text{Channel is in deep fade}\}$

- $P_{out} = \Pr\{\text{SNR is dropped below a threshold } \rho_{th}\}$

$$P_{out} = \Pr\{\rho < \rho_{th}\} = \int_0^{\rho_{th}} \frac{1}{\bar{\rho}} e^{-\rho/\bar{\rho}} d\rho = 1 - e^{-\rho_{th}/\bar{\rho}}$$

- At high SNR, the outage probability is approximated by

$$P_{out} \approx \frac{\rho_{th}}{\bar{\rho}} = \frac{\rho_{th}}{SNR} \Rightarrow \text{inversely proportional to SNR}$$

Outline

2.1 Wireless channels

2.2 Outage probability and error probability over fading channels

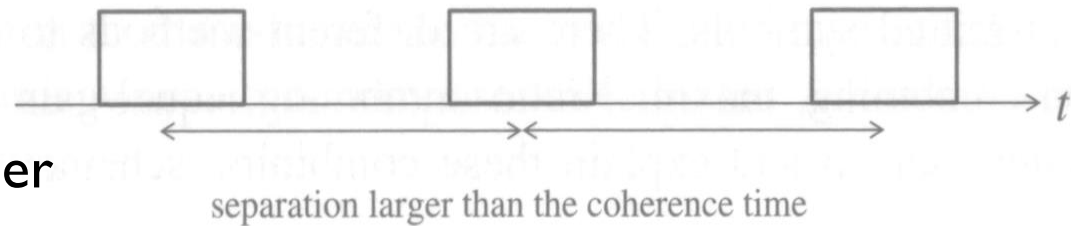
2.3 Diversity Techniques

- 2.3.1 Diversity in time, frequency, and space domains
- 2.3.2 Maximal Ratio Combining
- 2.3.3 Selection Combining

Time/Frequency Diversity

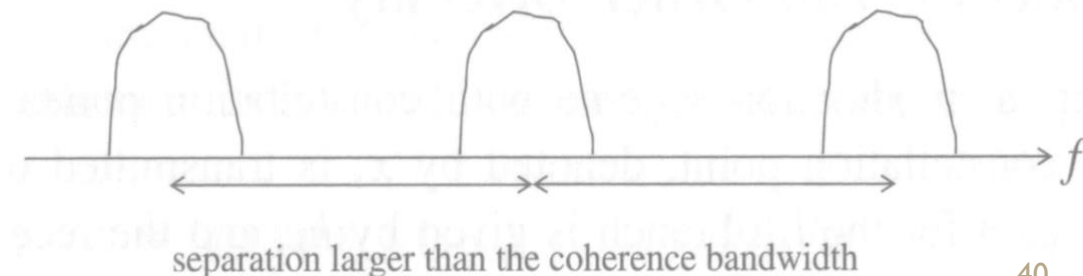
- *Time Diversity*

- Transmit the same signal several times in different time-slots
- In slow fading, it takes a long time to transmit another replica



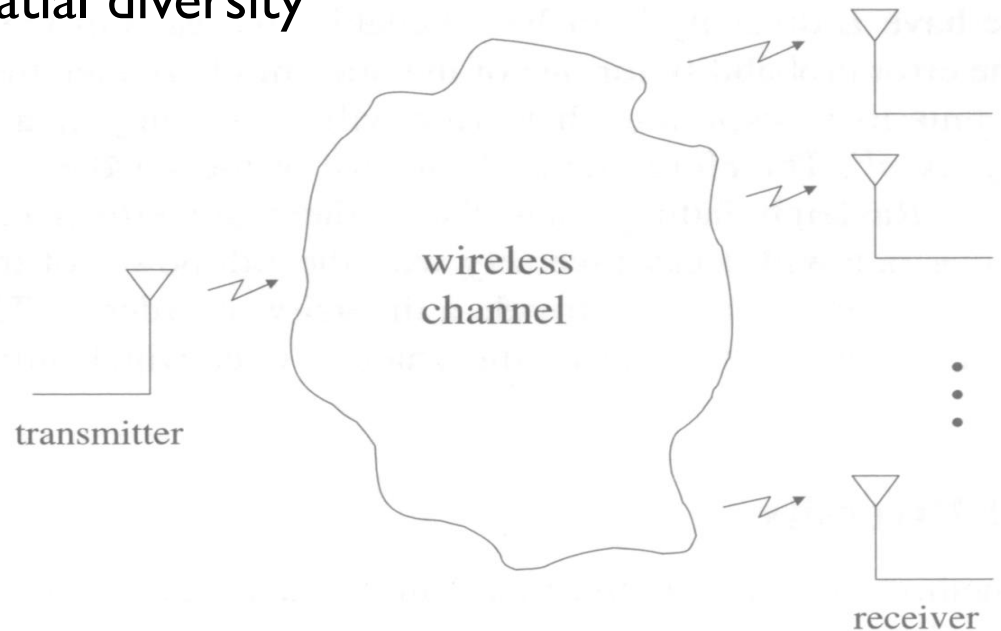
- *Frequency Diversity*

- Transmit a few replicas over different frequency band
- With delay spread = $200\mu s$, minimum frequency separation is 5KHz



Spatial Diversity

- “Space” can be used as a resource to provide diversity
- Assume that the rx has multiple antennas
 - Different replica of the signal are picked up at each signal
 - The separation between antennas is assumed at least $\lambda/2$
 - The received signals undergo different channel fades
⇒ provide spatial diversity



System model of L^{th} order diversity

- Consider a BPSK symbol $x \in \{\pm\sqrt{E_s}\}$ is transmitted over L diversity branches

$$y_1 = h_1x + w_1$$

$$y_2 = h_2x + w_2$$

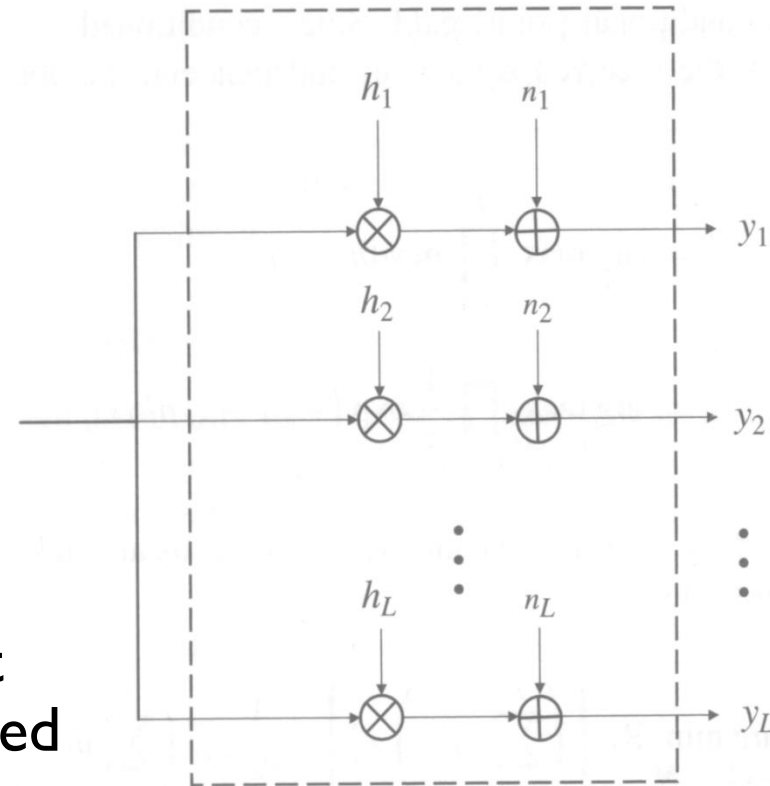
\vdots

$$y_L = h_Lx + w_L$$

- AWGN $w_l \sim CN(0, N_0)$
- Rayleigh fading, $h_l \sim CN(0, 1)$
- SNR of the l^{th} branch

$$\rho_l = E_s |h_l|^2 / N_0$$

- Signals obtained thru different diversity branches are combined to detect the symbol x



Diversity Combining Methods

- Optimum combining:
 - Maximal Ratio Combining (MRC)
- Suboptimal combining:
 - Equal Gain Combining (EGC)
 - Selection Combining (SC)
 - Switch-and-Stay Combining (SSC)
- Diversity order

$$d = - \lim_{SNR \rightarrow \infty} \frac{\log(P_e)}{\log(SNR)} \quad or \quad d = - \lim_{SNR \rightarrow \infty} \frac{\log(P_{out})}{\log(SNR)}$$

Maximal Ratio Combining (I)

- Assume that $\{h_1, h_2, \dots, h_L\}$ are known at rx
- The maximal likelihood (ML) decision is

$$\begin{aligned}\hat{x} &= \arg \max_{x \in \{\pm\sqrt{E_s}\}} p(y_1, y_2, \dots, y_L | x) \\ &= \arg \max_{x \in \{\pm\sqrt{E_s}\}} p(y_1 | x) \times p(y_2 | x) \times \dots \times p(y_L | x) \\ &= \arg \max_{x \in \{\pm\sqrt{E_s}\}} \prod_{\ell=1}^L \frac{1}{\pi N_0} \exp\left(-\frac{|y_\ell - h_\ell x|^2}{N_0}\right) \\ &= \arg \min_{x \in \{\pm\sqrt{E_s}\}} \sum_{\ell=1}^L |y_\ell - h_\ell x|^2 \\ &= \arg \max_{x \in \{\pm\sqrt{E_s}\}} \sum_{\ell=1}^L 2\Re\{y_\ell h_\ell^* x^*\} - \sum_{\ell=1}^L |h_\ell|^2 |x|^2 \\ &= \arg \max_{x \in \{\pm\sqrt{E_s}\}} \Re\left\{\left(\sum_{\ell=1}^L y_\ell h_\ell^*\right) x^*\right\}\end{aligned}$$

Maximal Ratio Combining (2)

- Optimal decision rule:

$$\hat{x} = \arg \max_{x \in \{\pm\sqrt{E_s}\}} \Re \left\{ \left(\sum_{\ell=1}^L h_{\ell}^* y_{\ell} \right) x^* \right\}$$

- Linearly combines the received signals after co-phasing and weighting them with the respective channel gains
- The branches with better channel gains are emphasized more than others since they are more reliable
- The resulting combining is **maximal ratio combining (MRC)**
- Effective channel model of MRC:

$$y = \sum_{\ell=1}^L h_{\ell}^* y_{\ell} = \left(\sum_{\ell=1}^L |h_{\ell}|^2 \right) x + w'$$

- **Noise** $w' = \sum_{\ell} h_{\ell}^* w_{\ell} \sim CN(0, N_0 \sum_{\ell} |h_{\ell}|^2)$

Maximal Ratio Combining (3)

- SNR of L branches of diversity with MRC is

$$\rho_{eff} = \frac{\left(\sum_{\ell=1}^L |h_{\ell}|^2\right)^2 E_s}{\left(\sum_{\ell=1}^L |h_{\ell}|^2\right) N_0} = \sum_{\ell=1}^L \frac{E_s |h_{\ell}|^2}{N_0} = \sum_{\ell=1}^L \rho_{\ell}$$

- ρ_{eff} is Chi-Square distributed with $2L$ degrees of freedom

$$p_{\rho_{eff}}(u) = \frac{u^{L-1} \exp\left(-\frac{u}{E_s/N_0}\right)}{(E_s/N_0)^L (L-1)!}, \quad u > 0$$

Chi-Square Random Variable

Let X_1, X_2, \dots, X_n be i.i.d. Gaussian random variables, $X_i \sim CN(0, \sigma^2)$.
 $X = X_1^2 + X_2^2 + \dots + X_n^2$ is a Chi-Square r.v. with n degrees of freedom.

If $n = 2m$, the PDF and CDF of X are

PDF:
$$p(x) = \frac{x^{m-1} \exp\left(-\frac{x}{2\sigma^2}\right)}{(m-1)!(2\sigma^2)^m}, \quad x > 0$$

CDF:
$$F(x) = 1 - \exp\left(-\frac{x}{2\sigma^2}\right) \sum_{k=0}^{m-1} \frac{1}{k!} \left(\frac{x}{2\sigma^2}\right)^k, \quad x > 0$$

Maximal Ratio Combining (4)

- Outage Probability with MRC

$$\begin{aligned} P_{out} &= \Pr\{\rho_{eff} < \rho_{th}\} \\ &= 1 - \exp\left(-\frac{\rho_{th}}{E_s/N_0}\right) \sum_{k=0}^{L-1} \frac{1}{k!} \left(\frac{\rho_{th}}{E_s/N_0}\right)^k \end{aligned}$$

(ex) Let $x = \rho_{th} N_0 / E_s$. At high SNR, $x \rightarrow 0$. Assume that $L=3$, and P_{out} is approximated by

$$\begin{aligned} P_{out} &= 1 - e^{-x} \left(1 + x + \frac{x^2}{2}\right) \\ &= 1 - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \mathcal{O}(x^4)\right) \left(1 + x + \frac{x^2}{2!}\right) \\ &= x^3/3! + \mathcal{O}(x^4) \\ &\approx \left(\frac{\rho_{th}}{E_s/N_0}\right)^3 / 3! \propto (E_s/N_0)^{-3} \end{aligned}$$

- Outage probability decays with $1/(SNR)^L$

Maximal Ratio Combining (5)

- Average error probability with MRC

$$\begin{aligned}
 P_{e,MRC} &= \int_0^\infty Q(\sqrt{2u}) p_{\rho_{eff}}(u) du \\
 &= \frac{1}{\sqrt{2\pi}} \int_0^\infty \int_{\sqrt{2u}}^\infty e^{-\frac{t^2}{2}} p_{\rho_{eff}}(u) dt du \\
 &= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{t^2}{2}} \int_0^{t^2/2} p_{\rho_{eff}}(u) du dt \\
 &= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{t^2}{2}} \left(1 - \exp\left(-\frac{t^2/2}{E_s/N_0}\right) \sum_{k=0}^{L-1} \frac{1}{k!} \left(\frac{t^2/2}{E_s/N_0}\right)^k \right) dt \\
 &= \frac{1}{2} - \sum_{k=0}^{L-1} \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{t^2}{2} \frac{E_s+N_0}{E_s}} \frac{1}{k!} \left(\frac{t^2/2}{E_s/N_0}\right)^k dt \\
 &= \frac{1}{2} - \frac{1}{2} \sum_{k=0}^{L-1} \frac{1}{k!} \left(\frac{N_0}{2E_s}\right)^k \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-\frac{t^2}{2} \frac{E_s+N_0}{E_s}} t^{2k} dt \\
 &= \frac{1}{2} - \frac{1}{2} \left(\frac{E_s}{E_s+N_0}\right) \sum_{k=0}^{L-1} \left(\frac{N_0}{E_s+N_0}\right)^k \frac{(2k)!}{4^k k! k!}
 \end{aligned}$$

For $X \sim N(0,1)$

$$\mathbf{E}[X^{2k}] = \frac{(2k)!}{2^k k!}$$

Hard to obtain
diversity order

Maximal Ratio Combining (6)

- To study the behavior of average error probability, use the upper bound on Q-function

$$Q(x) \leq \frac{1}{2}e^{-x^2/2}$$

- $P_{e,MRC} = \int_0^\infty Q(\sqrt{2u}) p_{\rho_{eff}}(u) du$

$$\leq \frac{1}{2} \int_0^\infty e^{-u} \frac{u^{L-1} \exp(-\frac{u}{E_s/N_0})}{(E_s/N_0)^L (L-1)!} du$$

$$= \frac{1}{2(E_s/N_0)^L (L-1)!} \int_0^\infty u^{L-1} \exp\left(-\frac{u(E_s + N_0)}{E_s}\right) du$$

$$= \frac{1}{2(E_s/N_0)^L (L-1)!} \frac{(L-1)!}{\left(\frac{E_s + N_0}{E_s}\right)^L}$$

$$= \frac{1}{2} (1 + (E_s/N_0))^{-L} \leq \frac{1}{2} (E_s/N_0)^{-L}$$

- Average error probability decays with SNR^{-L}

$$\int_0^\infty u^k e^{-au} du = \frac{k!}{a^{k+1}}$$

Selection Combining (I)

- At any time interval, **one branch with largest SNR** is used in demodulation

$$k = \arg \max_{\ell=1, \dots, L} \frac{E_s |h_\ell|^2}{N_0} = \arg \max_{\ell=1, \dots, L} |h_\ell|^2$$

⇒ *Selection combining (SC)*

- The selected signal is

$$y = h_k x + n_k$$

- The effective SNR of selection combining is

$$\rho_{eff} = \max_{\ell=1, \dots, L} \frac{E_s |h_\ell|^2}{N_0} = \frac{E_s}{N_0} \left(\max_{\ell=1, \dots, L} |h_\ell|^2 \right)$$

Selection Combining (2)

- CDF of ρ_{eff}

$$\begin{aligned} F_{\rho_{eff}}(u) &= Pr\left\{\frac{E_s}{N_0} \max_{\ell} |h_{\ell}|^2 \leq u\right\} \\ &= Pr\left\{\max(|h_1|^2, |h_2|^2, \dots, |h_L|^2) \leq \frac{uN_0}{E_s}\right\} \\ &= Pr\left\{|h_1|^2 \leq \frac{uN_0}{E_s}, |h_2|^2 \leq \frac{uN_0}{E_s}, \dots, |h_L|^2 \leq \frac{uN_0}{E_s}\right\} \\ &= \prod_{i=1}^L Pr\left\{|h_i|^2 \leq \frac{uN_0}{E_s}\right\} \\ &= \left(1 - \exp\left(-\frac{uN_0}{E_s}\right)\right)^L \end{aligned}$$

- PDF of ρ_{eff}

$$\begin{aligned} p_{\rho_{eff}}(u) &= \frac{\partial}{\partial u} F_{\rho_{eff}}(u) = \frac{\partial}{\partial u} \left(1 - \exp\left(-\frac{uN_0}{E_s}\right)\right)^L \\ &= \frac{L}{E_s/N_0} \exp\left(-\frac{u}{E_s/N_0}\right) \left(1 - \exp\left(-\frac{u}{E_s/N_0}\right)\right)^{L-1} \end{aligned}$$

Selection Combining (3)

- Outage probability of selection combining

$$\begin{aligned} P_{out,SC} &= Pr\{\rho_{eff} < \rho_{th}\} = F_{\rho_{eff}}(\rho_{th}) \\ &= \left(1 - \exp\left(-\frac{\rho_{th}}{E_s/N_0}\right)\right)^L \end{aligned}$$

- At high SNR,

$$P_{out,SC} \approx \left(\frac{\rho_{th}}{E_s/N_0}\right)^L$$

- Outage probability decays with $(E_s/N_0)^{-L}$

Selection Combining (4)

- Average error probability of selection combining

$$\begin{aligned}
 P_{e,SC} &= \int_0^\infty Q(\sqrt{2u}) p_{\rho_{eff}}(u) du \\
 &\leq \int_0^\infty \frac{1}{2} e^{-u} p_{\rho_{eff}}(u) du \\
 &= \int_0^\infty \frac{1}{2} e^{-u} \frac{L}{E_s/N_0} \exp\left(-\frac{u}{E_s/N_0}\right) \left(1 - \exp\left(-\frac{u}{E_s/N_0}\right)\right)^{L-1} du
 \end{aligned}$$

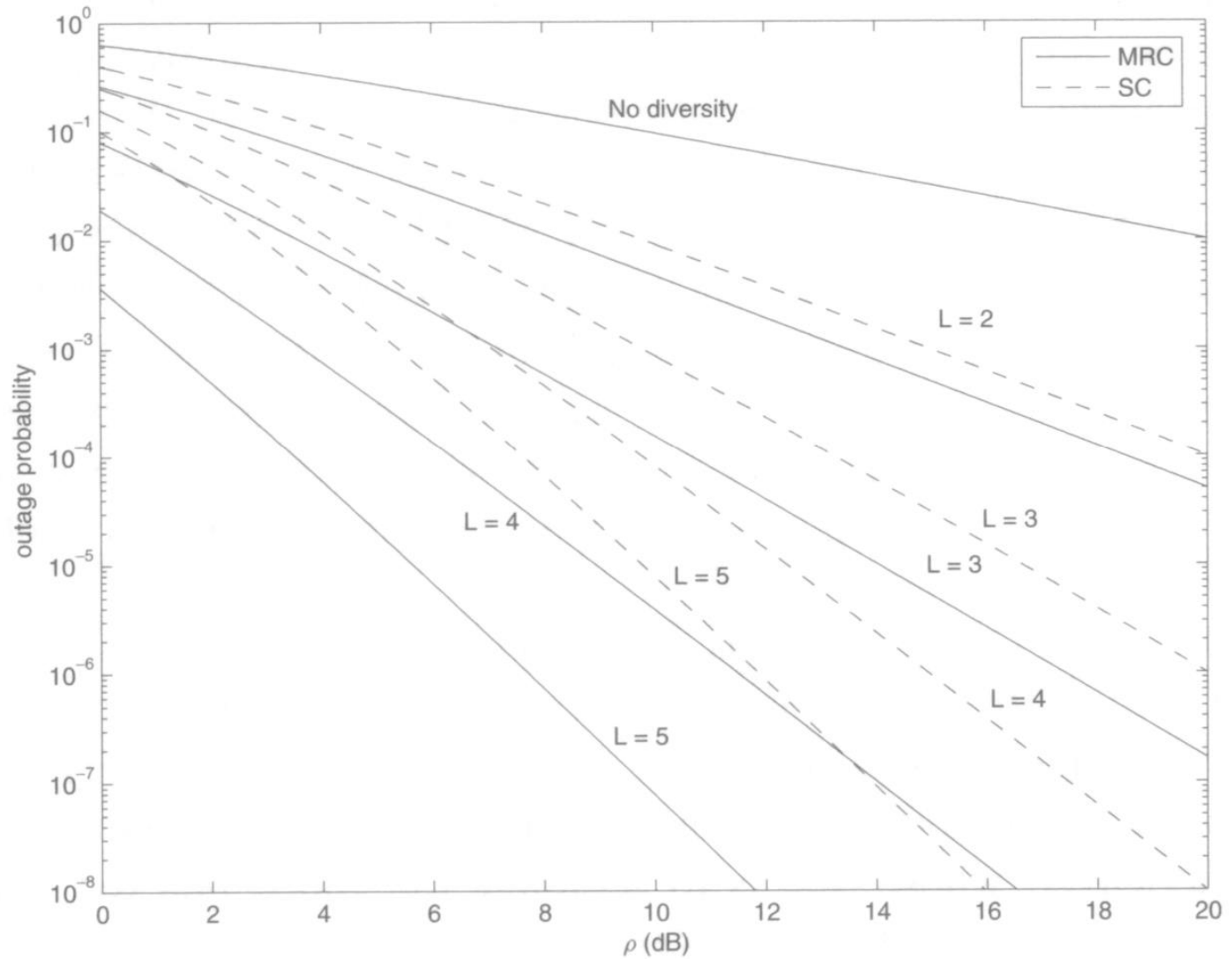
- At high SNR, ($1 - e^{-x} \approx x$)

$$\begin{aligned}
 &\approx \int_0^\infty \frac{1}{2} e^{-u} \frac{L}{E_s/N_0} \exp\left(-\frac{u}{E_s/N_0}\right) \left(\frac{u}{E_s/N_0}\right)^{L-1} du \\
 &= \frac{L}{2(E_s/N_0)^L} \int_0^\infty \exp\left(-\frac{u(E_s+N_0)}{E_s}\right) u^{L-1} du \\
 &= \frac{L}{2(E_s/N_0)^L} (L-1)! \left(\frac{E_s/N_0}{1+E_s/N_0}\right)^L \\
 &= \frac{L!}{2} (1 + E_s/N_0)^{-L} \approx \frac{L!}{2} (E_s/N_0)^{-L}
 \end{aligned}$$

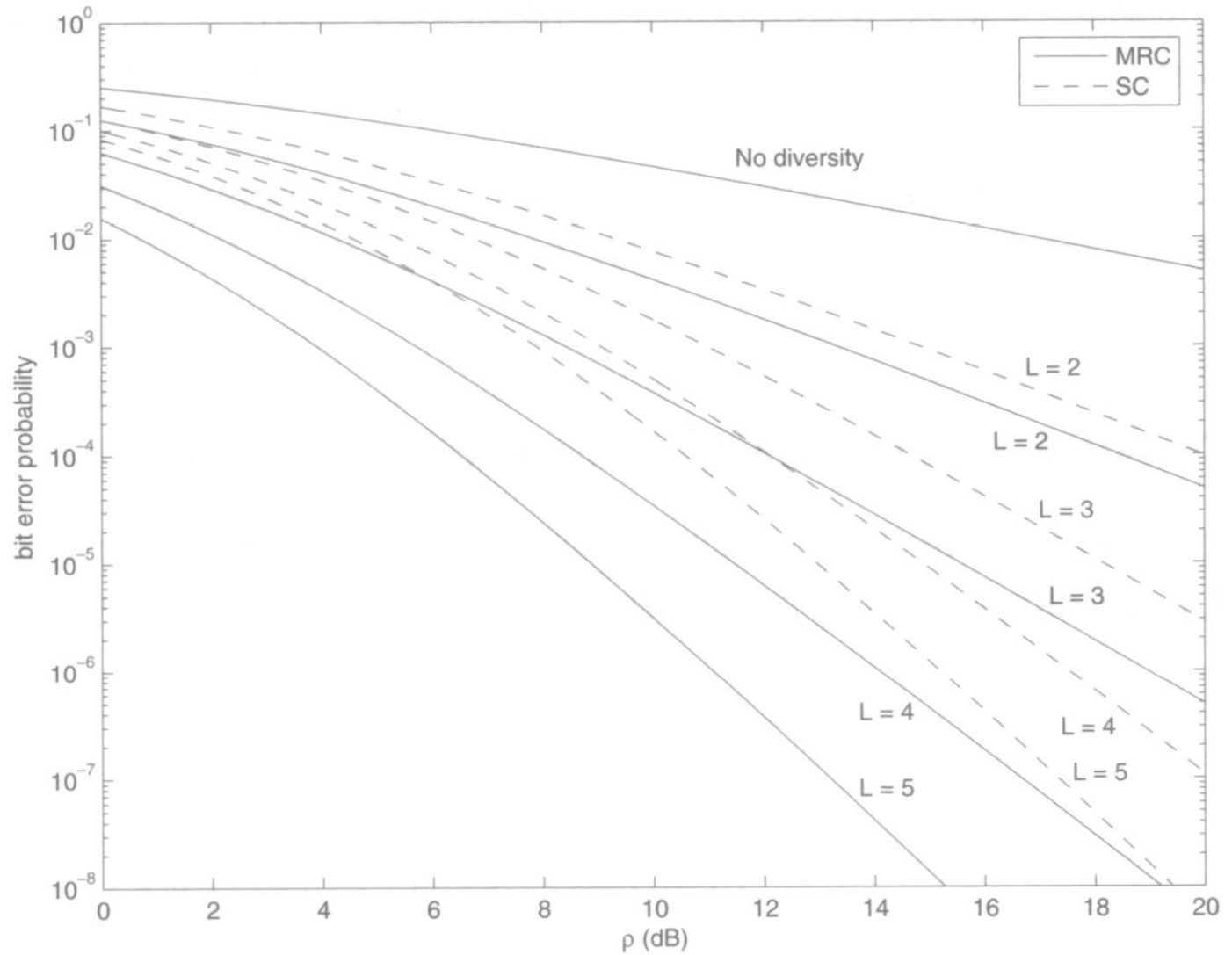
$$\int_0^\infty u^k e^{-au} du = \frac{k!}{a^{k+1}}$$

- Average error probability decays with $(E_s/N_0)^{-L}$

MRC v.s. SC (I)



MRC v.s. SC (2)

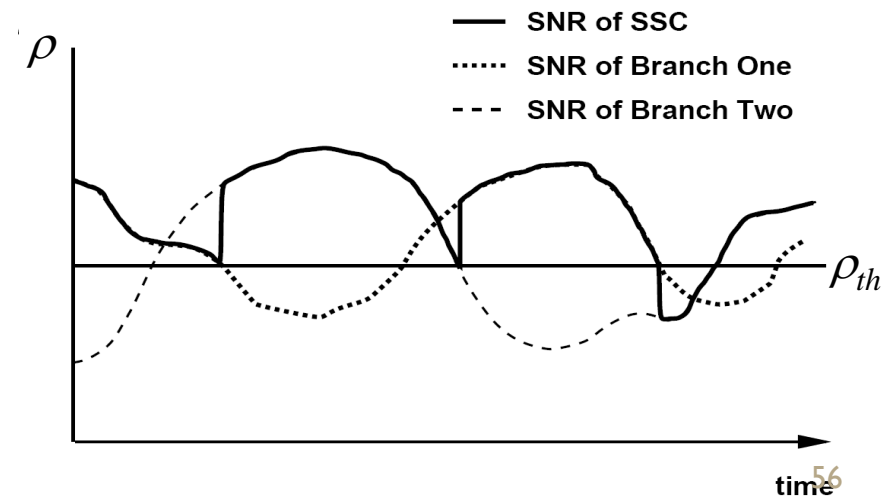


Switch and Stay Combining

- Selection combining can achieve full diversity by switching to the branch with largest SNR
 - May require switch operations frequently
- *Switch and Stay Combining (SSC)*
 1. Rx switches to the branch with largest SNR and stays
 2. When the SNR of the selected branch drops below ρ_{th} , switch to the other branch with highest SNR

- Outage probability

$$\begin{aligned} P_{out,SSC} &= \prod_{\ell=1}^L Pr(\rho_{\ell} < \rho_{th}) \\ &= \left(1 - \exp\left(-\frac{\rho_{th}}{E_s/N_0}\right) \right)^L \\ &= P_{out,SC} \end{aligned}$$



Equal Gain Combining

- **Equal Gain Combining (EGC)**: simply co-phase and combine all branches of signals with equal gain

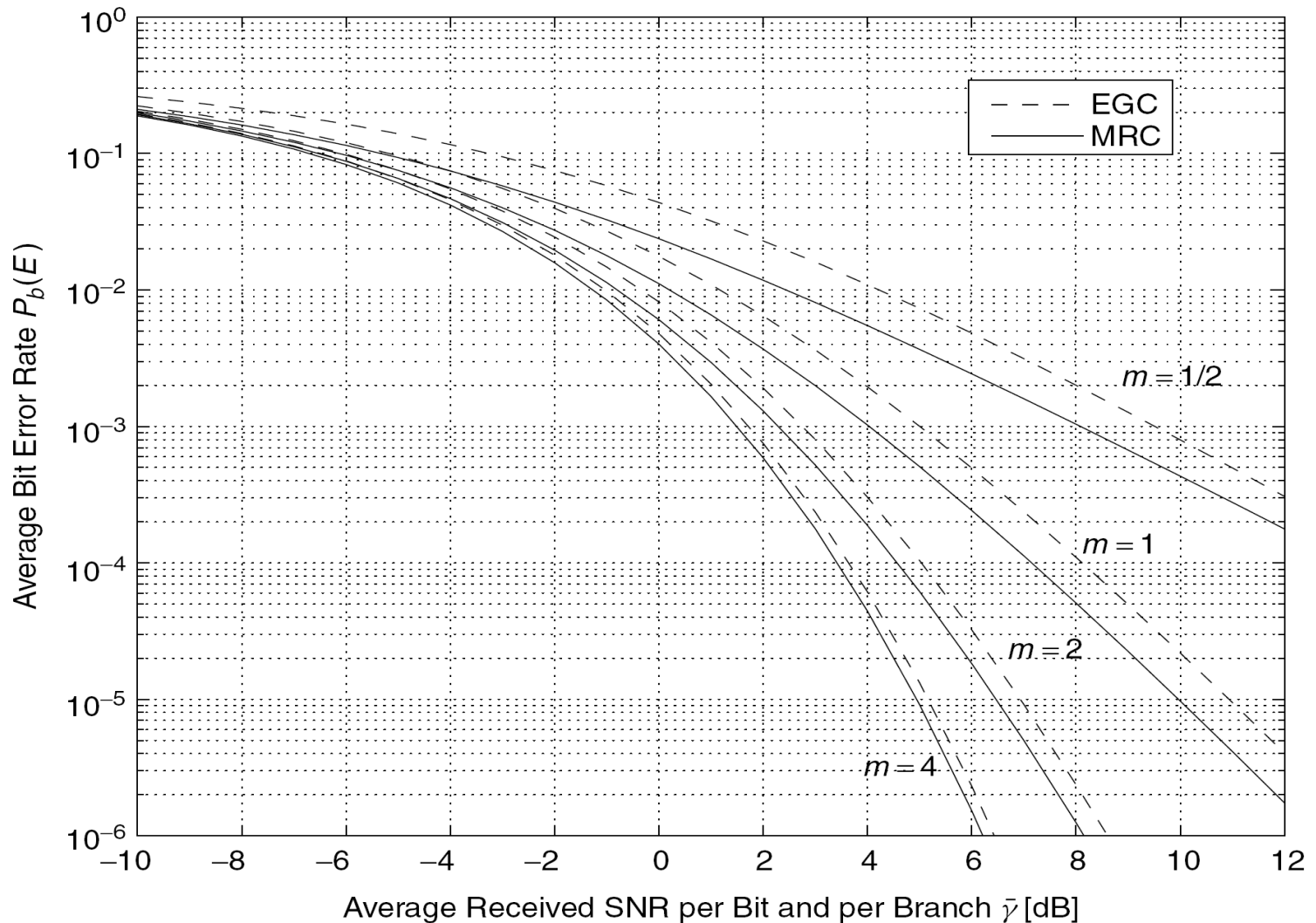
$$y = \sum_{\ell=1}^L \frac{h_{\ell}^*}{|h_{\ell}|} y_{\ell} = \left(\sum_{\ell=1}^L |h_{\ell}| \right) x + w'$$

- AWGN $w' \sim CN(0, LN_0)$
- **Effective SNR of EGC**

$$\rho_{eff} = \frac{E_s}{LN_0} \left(\sum_{\ell=1}^L |h_{\ell}| \right)^2$$

- BER Performance is comparable to MRC within 1dB

Average BER over Nakagami- m fading channels with MRC and EGC ($L = 4$).





References