Newton’s Method and the Finding the Yields for Bonds

Newton’s method is one of the best basic and important search routines available to the numerical analyst. It is a general method for finding the root of a polynomial, although it should be pointed out that it does not work under all circumstances.

While the excerpt from *Numerical Recipes in C* provides an excellent overview of the general technique, we wish to understand how it applies to the particular problem that we face in the bond analyzer: finding the yield on a bond given an observed (i.e. known) price. To do this we need to frame and define the problem that we are going to solve. The first step in this process is fully defining the bond and bond pricing mechanism.

**Treasury Bonds**

For now we will limit ourselves to a discussion of the mechanics of treasury bonds and of the jargon that is frequently used in this field. You should be aware that from the perspective of mechanics and pricing there is no substantive difference between Treasury Bonds and Treasury Notes, and we will not make a distinction here.

A Treasury bond has a *face* or principal amount of $1,000. This is the nominal amount that the US Treasury is borrowing from the lender. This face amount is frequently referred to as the bond’s *par* value.

The bond makes semi-annual interest payments, on the anniversary date of the bond’s maturity and exactly 6 months later. By convention the payments are made on the 15th of the month in which they are due. These payments are made on the bond’s coupon rate (denoted $C$) and face amount. The amount of the payment is given by:

\[
\text{Semi-annual Coupon Payment} = \$1,000 \times \frac{C}{2}.
\]  

(1)

Note that the coupon payment is constant over the life of the bond. On the maturity date of the bond the bondholder receives both a coupon payment and the face amount of the bond.

**Example 1:** Consider a US Treasury bond which matures on February 15, 2027 and which has a coupon of 6%. If today in August 15, 1997, what are the cash flows this bond will generate in the future?

The bond will make coupon payment of $1,000 * .06/2 = $30 on every February 15th and August 15th until February 15, 2027. On February 15, 2027 the bond will make a final coupon payment of $30 along with a principal payment of $1,000.
The coupon rate of the bond is set by the bond covenant. Typically it is set to the market rate of interest on the date that the bond is issued. Of course, over time the market interest rate will change, while the coupon rate will not. The price of the bond is determined by discounting the remaining cash flows of the bond at the current market rate, denoted \( r \). The current market rate is also referred to as the *yield* of the bond.

**Example 2:** If the current market rate is 8%, what is the price of the bond in example 1?

To answer this, simply discount the cash flows in example 1 at 8%. Generally, it is known that the present value of a series of cash flows is simply the sum of the present values of the individual cash flows, i.e.:

\[
\text{Price} = \sum_{i=1}^{n} \frac{CF_i}{(1 + \frac{r}{2})^i}
\]

In the particular case here there are a total of 59 cash flows remaining (two per year for years 1998-2026 inclusive, plus 1 on February 15, 2027). Note that the first 58 cash flows are for $30, and the last cash flow is for $1030. At an 8% discount rate this gives a formula of:

\[
\text{Price} = \sum_{i=1}^{58} \frac{30}{(1 + \frac{.08}{2})^i} + \frac{1030}{(1 + \frac{.08}{2})^{59}} = $774.72
\]

Note that it is sometimes more convenient to use the present value of a stream of equal payments formula instead of having to do the summation of the 58 separate cash flows:

\[
\text{Price} = 30 \times \left( 1 - \frac{1}{\left( 1 + \frac{.08}{2} \right)^{58}} \right) + \frac{1030}{\left( 1 + \frac{.08}{2} \right)^{59}} = $774.72
\]

So, if you are given the yield of the bond, calculating the price is a rather trivial project. Frequently, however, you do not know the yield, but rather observe a price at which the bond is trading and wish to know what yield (i.e. discount rate) is implicit in that price.

**Example 3:** Again considering the above bond, let’s say that you did not know the market interest yield for the bond, but did observe the bond trading at a
price of $800 in the market. (By the way, prices are usually quoted as a percentage of par, so a price of $800 would be quoted as 80.) What discount rate (i.e. yield) is the market using to price this bond?

$$800.00 = \sum_{i=1}^{58} \frac{30}{\left(1 + \frac{r}{2}\right)^i} + \frac{1030}{\left(1 + \frac{r}{2}\right)^{59}}$$

(5)

Notice that there is not a closed form solution for finding this equation. This implies that we must search for this correct value of r. One method is to simply try by trial and error. That is, simply guess a value for r. Plug it into equation (3) or (4), if the price you calculate using that rate is equal to $800, then you have found the right yield. If the price you calculate is less than (greater than) $800, your guess was too high (low) and you should guess a lower (higher) value for r. You continue doing this until you find the right value of r. In this case that value is: 7.73%

The search algorithm outlined in example 3 is not particularly efficient. That is, it tends to take many guesses before you typically find the value of r that is correct. Fortunately, we can take advantage of Newton’s method (sometimes called the Newton-Raphson method) to quickly find the solution.

To begin our discussion of the implementation of Newton’s method, let’s first define some terms. Let P* be the price that we observe in the market for the bond. Let P(r) be the price formula (3) or (4) calculates when the interest rate r is used. Define Err(r) as the difference between P* and P(r), i.e.

$$Err = P^* - P(r)$$

(6)

Our goal is to find the value of r that sets Err(r) = 0.

The general method for Newton’s method is the following:
1. Start with an initial guess for r.
2. Calculate P(r).
3. Calculate Err(r).
4. If Err(r)=0, stop and report r.
5. If Err(r)<0, calculate the derivative of Err(r) with respect to r, i.e.

$$\frac{dErr(r)}{dr}$$

(7)

6. Adjust r by an amount equal to:

$$dr = \frac{Err(r)}{\left[\frac{dErr(r)}{dr}\right]}$$

(8)

7. Go to step 2 and repeat.
The only real difficulty in this algorithm is the calculation of the derivative in step (7). This, however, is not nearly as difficult as it might at first seem. Writing out the equation (6) in long form:

\[ \text{Err}(r) = P^* - \sum_{i=1}^{n} \left( \frac{CF_i}{1 + \frac{r}{2}} \right) \tag{9} \]

When taking the derivative with respect to \( r \), notice since \( P^* \) is a constant, its derivative with respect to \( r \) is 0, thus leaving:

\[ \frac{d\text{Err}(r)}{dr} = -\sum_{i=1}^{n} \frac{-i*CF_i}{\left( 1 + \frac{r}{2} \right)^{i+1}} \tag{10} \]

**Example 4**: Returning to the example problem presented above, let’s assume that your initial guess for \( r \) were .07. The price equation (5) would generate at that \( r \), i.e. \( P(r) \), would be $875.91. The error function, therefore will have value:

\[ \text{Err}(.07) = 800 - 875.91 = -75.91 \tag{11} \]

The change of the price with respect to \( r \) is given by:

\[ \frac{d\text{Err}(.07)}{dr} = \sum_{i=1}^{58} \frac{-i*30}{(1.035)^{i+1}} + \frac{59*1030}{(1.035)^{60}} \tag{12} \]

and would have value of –22342. From equation (8), therefore your next guess of \( r \) should be equal to the current guess plus the calculated \( dr \), i.e.:

\[ \text{new } r = .07 + \frac{-75.91}{-22342} = .07 + .003397 = .073397 \tag{13} \]

If you now plug .073397 into equation (5) you get a price of 839.23, and a derivative of –20857.2. From equation 13, then your next adjustment should be to a new \( r \) of 0.075279. If you continue this process, you will get the correct value for \( r \) of 7.73% in approximately 7 iterations.