Discrete Dynamic Optimization: Six Examples

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Example 1: the neoclassical growth model

- Uhlig (1999), section 4

- Model principles

- Specify the environment explicitly:
  
  1. Preferences
  
  2. Technologies
3. Endowments

4. Information

- State the object of study:

1. The social planner’s problem

2. The competition equilibrium

3. The game
• The environment:

• Preferences: the representative agent experiences utility according to

\[ U = E_t \left[ \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\eta} - 1}{1 - \eta} \right] \]

• \( 0 < \beta < 1 \)

• \( \eta > 0 \)
• Absolute risk aversion: $-\frac{u''(c)}{u'(c)}$

• Relative risk aversion: $-\frac{cu''(c)}{u'(c)}$

• $\eta$ is the coefficient of relative risk aversion

• Technologies: we assume a Cobb-Douglas production function

\[ Y_t = Z_t K_t^\rho N_t^{1-\rho} \]
• Budget constraint:

\[ K_{t+1} = I_t + (1 - \delta) K_t \]

\[ C_t + I_t = Y_t \]

• or equivalently

\[ C_t + K_{t+1} = Z_t K_t^\rho N_t^{1-\rho} + (1 - \delta) K_t \]
• $0 < \rho < 1$

• $0 < \delta < 1$

• $Z_t$, the total factor productivity, is exogenously evolving according to

\[
\log Z_t = (1 - \psi) \log \tilde{Z} + \psi \log Z_{t-1} + \epsilon_t
\]

\[
\epsilon_t \sim i.i.d. \mathcal{N}(0; \sigma^2)
\]
• $0 < \psi < 1$

• Endowment:

• $N_t = 1$

• $K_0$

• Information: $C_t$, $N_t$, and $K_{t+1}$ need to be chosen based on all information $\mathcal{I}_t$ up to time $t$.

• The social planner's problem
• The objective of the social planner is to maximize the utility of the representative agent subject to feasibility,

\[
\max_{(C_t, K_t)_{t=0}^{\infty}} \mathbb{E}_t \left[ \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\eta} - 1}{1 - \eta} \right]
\]

\[s.t. \quad K_0, Z_0\]

\[C_t + K_{t+1} = Z_t K_t^\rho + (1 - \delta) K_t\]
\[ \log Z_t = (1 - \psi) \log \tilde{Z} + \psi \log Z_{t-1} + \epsilon_t \]

\[ \epsilon_t \sim i.i.d. \mathcal{N}(0; \sigma^2) \]

- To solve it, form the Lagrangian:

\[
L = \max_{(C_t, K_{t+1})_{t=0}^\infty} \mathbb{E}_t \sum_{t=0}^\infty \left[ \beta^t \frac{C_t^{1-\eta} - 1}{1 - \eta} + \beta^t \lambda_t \left( Z_t K_t^\rho + (1 - \delta) K_t - C_t - K_{t+1} \right) \right]
\]
The first order conditions are:

\[
\frac{\partial L}{\partial \lambda_t} : 0 = C_t + K_{t+1} - Z_t K^\rho_t - (1 - \delta) K_t
\]

\[
\frac{\partial L}{\partial C_t} : 0 = C_t^{\eta} - \lambda_t
\]

\[
\frac{\partial L}{\partial K_{t+1}} : 0 = -\lambda_t + \beta E_t \left[ \lambda_{t+1} \left( \rho Z_{t+1} K_{t+1}^{\rho - 1} + (1 - \delta) \right) \right]
\]
Notice that $K_{t+1}$ appears in period $t$ and period $t + 1$.

The first order conditions are often also called Euler equations.

One also obtains the transversality condition.

$$0 = \lim_{T \to \infty} E_0 \left[ \beta^T C_T^{-\eta} K_{T+1} \right]$$

The transversality condition is a limiting Kuhn-Tucker condition.
It means that in net present value terms, the agent should not have capital left over at infinity.

Rewrite the necessary conditions:

\[ C_t = Z_t K_t^\rho + (1 - \delta) K_t - K_{t+1} \]  \hspace{1cm} (1)

\[ R_t = \rho Z_t K_t^{\rho-1} + (1 - \delta) \]  \hspace{1cm} (2)
\[ 1 = E_t \left[ \beta \left( \frac{C_t}{C_{t+1}} \right)^\eta R_{t+1} \right] \] (3)

\[ \log Z_t = (1 - \psi) \log \bar{Z} + \psi \log Z_{t-1} + \epsilon_t, \quad \epsilon_t \sim i.i.d. \mathcal{N}(0; \sigma^2) \] (4)

- Equation accounting.

- To solve for the steady state, dropping the time indices and yield
\[ \tilde{C} = \tilde{Z}K^\rho + (1 - \delta) \tilde{K} - \tilde{K} \]

\[ \tilde{R} = \rho \tilde{Z}K^{\rho - 1} + (1 - \delta) \]

\[ 1 = \beta \tilde{R} \]

- From the welfare theorems, the solution to the competitive equilibrium yields the same allocation as the solution to the social planner's problem.
• The case for competitive equilibrium is provided in Uhlig (1999) section 4.4.
Example 2: Hansen’s real business cycle model

- Uhlig (1999), section 4

- Hansen’s real business cycle model

- The model is an extension of the stochastic neoclassical growth model.

- The main difference is to endogenize the labour supply.

- The social planner solves the problem of the representative agent:
\[
\max E_t \sum_{t=1}^{\infty} \beta^t \left( \frac{C_t^{1-\eta} - 1}{1 - \eta} - AN_t \right)
\]

s.t.

\[
C_t + I_t = Y_t
\]

\[
K_{t+1} = I_t + (1 - \delta) K_t
\]
\[ Y_t = Z_t K_t^\rho N_t^{1-\rho} \]

\[ \log Z_t = (1 - \psi) \log \tilde{Z} + \psi \log Z_{t-1} + \epsilon_t, \quad \epsilon_t \sim i.i.d. N(0; \sigma^2) \]

- Hansen only consider the case \( \eta = 1 \), so that the objective function is:

\[ E_t \sum_{t=1}^{\infty} \beta^t (\log C_t - AN_t) \]
• Proof:

\[
\lim_{\eta \to 1} \frac{C_t^{1-\eta} - 1}{1-\eta} = \lim_{\eta \to 1} \frac{e^{(1-\eta)\ln C_t} - 1}{1-\eta} = \lim_{\eta \to 1} \frac{\partial [e^{(1-\eta)\ln C_t} - 1]}{\partial \eta} / \frac{\partial [1 - \eta]}{\partial \eta} = \lim_{\eta \to 1} \frac{e^{(1-\eta)\ln C_t} \cdot (-\ln C_t)}{-1} = \frac{e^0 \cdot (-\ln C_t)}{-1} = \ln C_t
\]

• To solve it, form the Lagrangian:
\[ L = \max_{(C_t, N_t, K_{t+1})_{t=0}^{\infty}} E_t \sum_{t=0}^{\infty} \left[ \beta^t \left( \frac{C_t^{1-\eta}-1}{1-\eta} - AN_t \right) \right. \\
\left. + \beta^t \lambda_t \left( Z_t K_t^\rho N_t^{1-\rho} + (1 - \delta) K_t - C_t - K_{t+1} \right) \right] \]

- The first order conditions are:

\[ \frac{\partial L}{\partial C_t} = C_t^{-\eta} - \lambda_t = 0 \]

\[ \frac{\partial L}{\partial N_t} = -A + \lambda_t (1 - \rho) Z_t K_t^\rho N_t^{-\rho} = 0 \]
\[
\frac{\partial L}{\partial K_{t+1}} = -\lambda_t + \beta E_t \lambda_{t+1} \left( \rho Z_{t+1} K_{t+1}^{\rho - 1} N_{t+1}^{1-\rho} + (1 - \delta) \right) = 0
\]

- After arrangement

\[
A = C_t^{-\eta} (1 - \rho) \frac{Y_t}{N_t}
\]  \hspace{1cm} (1)

\[
R_t = \rho \frac{Y_t}{K_t} + 1 - \delta
\]  \hspace{1cm} (2)
1 = \beta E_t \left[ \left( \frac{C_t}{C_{t+1}} \right)^{\eta} R_{t+1} \right] \quad (3)

- The steady state for the real business cycle model above is obtained by dropping the time subscripts and stochastic shocks in the equations above.

\[
A = \bar{C}^{\eta} (1 - \rho) \frac{\bar{Y}}{N}
\]

1 = \beta \bar{R}
$\tilde{R} = \rho \frac{\tilde{Y}}{K} + 1 - \delta$

- Exercise: Equation accounting.
Example 3: Brock-Mirman growth model

- Chow (1997), chapter 2

Consider the Brock-Mirman growth model:

$$\max \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \ln c_t$$

$$\{c_t\}$$

s.t.
\[ k_{t+1} = k_t^\alpha z_t - c_t \]

- The Lagrangian is:

\[
L = E_t \sum_{t=0}^{\infty} \left[ \beta^t \ln c_t + \beta^t \lambda_t (k_t^\alpha z_t - c_t - k_{t+1}) \right]
\]

- The first order conditions are:
\[ \frac{\partial L}{\partial c_t} = \left( \beta^t \frac{1}{c_t} - \beta^t \lambda_t \right) = 0 \]

\[ \frac{\partial L}{\partial k_{t+1}} = -\beta^t \lambda_t + \beta^{t+1} E_t \left( \lambda_{t+1} \alpha k_{t+1}^{\alpha-1} z_{t+1} \right) = 0 \]

- The first order conditions can be rearranged as:

\[ \frac{1}{c_t} = \lambda_t \]
\[ \lambda_t = \alpha \beta E_t \left( \lambda_{t+1} k_{t+1}^{\alpha-1} z_{t+1} \right) \]

- Solve the problem explicitly.

- \[ k_{T+1} = 0 \]

- \[ c_T = k_T^\alpha z_T \]

- \[ c_t = d \cdot k_t^\alpha z_t \text{ (guess a solution)} \]
\[ E_t \left( \lambda_{t+1} k_{t+1}^{\alpha-1} z_{t+1} \right) = E_t \left( \frac{1}{c_{t+1}} k_{t+1}^{\alpha-1} z_{t+1} \right) = E_t \left( \frac{1}{d \cdot k_{t+1}^{\alpha}} k_{t+1}^{\alpha-1} z_{t+1} \right) = E_t \left( \frac{1}{d k_{t+1}^{\alpha}} \right) = \frac{1}{d k_{t}^{\alpha} z - c_t} \]

\[ \lambda_t = \alpha \beta E_t \left( \lambda_{t+1} k_{t+1}^{\alpha-1} z_{t+1} \right) \iff \frac{1}{c_t} = \alpha \beta \frac{1}{d k_{t}^{\alpha} z - c_t} \]

\[ c_t = \frac{d}{\alpha \beta} \left( k_{t}^{\alpha} z - c_t \right) \]
\[ d \cdot k_t^\alpha z_t = \frac{d}{\alpha \beta} (k_t^\alpha z_t - d \cdot k_t^\alpha z_t) \]

\[ 1 = \frac{1}{\alpha \beta} (1 - d) \]

\[ d = 1 - \alpha \beta \]

\[ \therefore c_t = d \cdot k_t^\alpha z_t = (1 - \alpha \beta) k_t^\alpha z_t \]
Example 4: Brock-Mirman growth model II

- Chow (1997), chapter 2

- We shall use dynamic programming to solve the Brock-Mirman growth model.

- Consider the Brock-Mirman growth model:

\[
\max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t \ln c_t
\]
The Bellman equation is:

\[ k_{t+1} = k_t^\alpha z_t - c_t \]

- Control variable: \( c_t \)
- State variables: \( k_t, z_t \)
- The Bellman equation is:
\[ V(k_t, z_t) = \max_{c_t} [\ln c_t + \beta E_t V(k_{t+1}, z_{t+1})] \]

- We conjecture value function takes the form:

\[ V(k_t, z_t) = a + b \ln k_t + c \ln z_t \]

- \(a, b, c\) are coefficients to be determined.

- \(E_t V(k_{t+1}, z_{t+1})\)
\[ V(k_{t+1}, z_{t+1}) = a + b \ln k_{t+1} + c \ln z_{t+1} \]
\[ = a + b \ln (k_t^{\alpha} z_t - c_t) + c \ln z_{t+1} \]

\[ \therefore E_t \ln z_{t+1} = 0 \]

\[ \therefore E_t V(k_{t+1}, z_{t+1}) = a + b \ln (k_t^{\alpha} z_t - c_t) \]

- It follows
\[ V (k_t, z_t) = \max_{c_t} [\ln c_t + \beta E_t V (k_{t+1}, z_{t+1})] \]
\[ = \max_{c_t} \{ \ln c_t + \beta \left[ a + b \ln \left( k_t^\alpha z_t - c_t \right) \right] \} \]
\[ \equiv \max_{c_t} \{ \} \]

- First order condition.

\[ \frac{\partial \{ \} }{\partial c_t} = \frac{1}{c_t} - \frac{\beta b}{k_t^\alpha z_t - c_t} = 0 \]
\[ c_t = \frac{k_t^\alpha z_t}{1 + \beta b} \]

- Substitute this optimal \( c_t = \frac{k_t^\alpha z_t}{1 + \beta b} \) into \( \max_c \{ \} \) and equate the expression to the value function, we can solve the coefficients \( a, b, c \), which after some algebraic manipulation are:

\[
a = (1 - \beta)^{-1} \left[ \ln (1 - \alpha \beta) + \beta \alpha (1 - \alpha \beta)^{-1} \ln (\alpha \beta) \right]
\]

\[
b = \alpha (1 - \alpha \beta)^{-1}
\]
\[ c = (1 - \alpha \beta)^{-1} \]
Example 5: the money-in-the-utility function (MIU) model

- Walsh (2003), chapter 2

- Utility function, labour supply, and production function.

\[ u(c_t, m_t, l_t) = \frac{C_t^{1-\Phi}}{1 - \Phi} + \Psi \frac{l_t^{1-\eta}}{1 - \eta} \]
\[ C_t \equiv \left[ ac_t^{1-b} + (1 - a) m_t^{1-b} \right]^{\frac{1}{1-b}} \]

\[ n_t = 1 - l_t \]

\[ y_t = e^{zt} k_t^\alpha n_t^{1-\alpha} = f(k_t, n_t, z_t) \]

\[ z_t = \rho z_{t-1} + e_t \]
• Two state variables.

\[ a_t \equiv \tau_t + \frac{(1 + i_{t-1}) b_{t-1}}{1 + \pi_t} + \frac{m_{t-1}}{1 + \pi_t} \]

\[ k_t \]

• The objective function is:
The Bellman equation is:

\[ E_t \sum_{t=0}^{\infty} \beta^t u \left( c_t, m_t, 1 - n_t \right) \]

\[ s.t. \]

\[ f \left( k_t, n_t, z_t \right) + (1 - \delta) k_t + a_t = c_t + k_{t+1} + m_t + b_t \]

- The Bellman equation is:
\[ V(a_t, k_t) = \max \left[ u \left( c_t, m_t, 1 - n_t \right) + \beta E_t V(a_{t+1}, k_{t+1}) \right] \]

- Use the definition of \( a_t \) to substitute for \( a_{t+1} \).

\[ a_{t+1} \equiv \tau_{t+1} + \left( \frac{1 + i_t}{1 + \pi_{t+1}} \right) b_t + \frac{m_t}{1 + \pi_{t+1}} \]

- Use the budget constraint to eliminate \( k_{t+1} \).
\[ k_{t+1} = f(k_t, n_t, z_t) + (1 - \delta) k_t + a_t - c_t - m_t - b_t \]

- Rewrite the Bellman equation as:

\[
V(a_t, k_t) = \max \left[ u(c_t, m_t, 1 - n_t) + \beta E_t V(a_{t+1}, k_{t+1}) \right]
\]

\[
= \max \left[ + \beta E_t V \left( \tau_{t+1} + \frac{(1+i_t)b_t}{1+\pi_{t+1}} + \frac{m_t}{1+\pi_{t+1}}, f(k_t, n_t, z_t) \right) \right]
\]

\[
= \max_{c_t, b_t, m_t, n_t} \{ \}
\]
• The first order conditions are:

\[
\frac{\partial \{\} }{\partial c_t} : u_c (c_t, m_t, 1 - n_t) - \beta E_t V_k (a_{t+1}, k_{t+1}) = 0
\]

\[
\frac{\partial \{\} }{\partial b_t} : \beta E_t \left( \frac{1 + i_t}{1 + \pi_{t+1}} \right) V_a (a_{t+1}, k_{t+1}) - \beta E_t V_k (a_{t+1}, k_{t+1}) = 0
\]

\[
\frac{\partial \{\} }{\partial m_t} : u_m (c_t, m_t, 1 - n_t) + \beta E_t \left( \frac{1}{1 + \pi_{t+1}} \right) V_a (a_{t+1}, k_{t+1}) - \beta E_t V_k (a_{t+1}, k_{t+1}) = 0
\]

\[
\frac{\partial \{\} }{\partial n_t} : -u_n (c_t, m_t, 1 - n_t) + \beta E_t V_k (a_{t+1}, k_{t+1}) f_n (k_t, n_t, z_t) = 0
\]
\[
\frac{\partial V(a_t, k_t)}{\partial a_t} = \frac{\partial \{\} : V_a(a_t, k_t) = \beta E_t V_k(a_{t+1}, k_{t+1})}{\partial a_t}
\]

\[
\frac{\partial V(a_t, k_t)}{\partial k_t} = \frac{\partial \{\} : V_k(a_t, k_t) = \beta E_t V_k(a_{t+1}, k_{t+1}) [f_k(k_t, n_t, z_t) + 1 - \delta]}{\partial k_t}
\]

- Resources constrains.

\[
y_t + (1 - \delta) k_t = c_t + k_{t+1}
\]
We can eliminate the partial differentiations of value function in the F.O.C. and then simplify the equilibrium conditions into 9 equations with 9 endogenous variables to be determined.

Equilibrium conditions include F.O.C., definitions, and resources constraints.

\[ y_t, c_t, k_{t+1}, m_t, n_t, R_t, \pi_t, z_t, u_t \]

\[ u_c(c_t, m_t, 1 - n_t) = u_m(c_t, m_t, 1 - n_t) + \beta E_t \left[ \frac{u_c(c_{t+1}, m_{t+1}, 1 - n_{t+1})}{1 + \pi_{t+1}} \right] \]

(1)
\[ u_l(c_t, m_t, 1 - n_t) = u_c(c_t, m_t, 1 - n_t) f_n(k_t, n_t, z_t) \]  \hspace{1cm} (2)

\[ u_c(c_t, m_t, 1 - n_t) = \beta E_t R_t u_c(c_{t+1}, m_{t+1}, 1 - n_{t+1}) \]  \hspace{1cm} (3)

\[ R_t = 1 - \delta + f_k(k_t, n_t, z_t) \]  \hspace{1cm} (4)

\[ y_t + (1 - \delta) k_t = c_t + k_{t+1} \]  \hspace{1cm} (5)
\[ y_t = f(k_t, n_t, z_t) \]  \hspace{1cm} (6)

\[ m_t = \left( \frac{1 + \theta_t}{1 + \pi_t} \right) m_{t-1} \]  \hspace{1cm} (7)

\[ z_t = \rho z_{t-1} + e_t \]  \hspace{1cm} (8)

\[ u_t = \gamma u_{t-1} + \phi z_{t-1} + \varphi_t \]  \hspace{1cm} (9)
Exercises: use Lagrangian to solve the MIU model.

\[ L = \sum_{t=0}^{\infty} \beta^t u \left( c_t, m_t, 1 - n_t \right) \]
\[ + \sum_{t=0}^{\infty} \beta^{t+1} E_t \lambda_{t+1} \left[ f(k_t, n_t, z_t) + (1 - \delta) k_t + \tau_t + \frac{(1+i_{t-1})b_{t-1}}{1+\pi_t} + \frac{m_{t-1}}{1+\pi_t} \right] \]

\[ \frac{\partial L}{\partial c_t}, \frac{\partial L}{\partial m_t}, \frac{\partial L}{\partial n_t}, \frac{\partial L}{\partial k_{t+1}}, \frac{\partial L}{\partial b_t} \]

Show that combing the F.O.C. of the Lagrangian method gives us the same equations as by using the Bellman equation.
Example 6: the cash-in-advance (CIA) model

- Walsh (2003), chapter 3, a certainty case

- Utility function:

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

- CIA constraint:
\( P_t c_t \leq M_{t-1} + T_t \)

- CIA constraint in real terms:

\[
ct \leq \frac{m_{t-1}}{\Pi_t} + \tau_t
\]

\[
\Pi_t \equiv \frac{P_t}{P_{t-1}}, \quad \tau_t \equiv \frac{T_t}{P_t}, \quad I_t = 1 + i_t
\]
• The budget constraint in real terms:

\[ f(k_t) + (1 - \delta) k_t + \tau_t + \frac{m_{t-1} + I_{t-1}b_{t-1}}{\Pi_t} \geq c_t + k_{t+1} + m_t + b_t \]

\[ w_t \equiv f(k_t) + (1 - \delta) k_t + \tau_t + \frac{m_{t-1} + I_{t-1}b_{t-1}}{\Pi_t} \]

• The budget constraint can be rewritten as:

\[ w_t \geq c_t + k_{t+1} + m_t + b_t \]
• Solve the problem

• Two state variables

\[ w_t \]

\[ m_{t-1} \]

• The Bellman equation is:
\[ V(w_t, m_{t-1}) = \max [u(c_t) + \beta V(w_{t+1}, m_t)] \]

\[ s.t. \]

\[ w_t \geq c_t + k_{t+1} + m_t + b_t \]

\[ c_t \leq \frac{m_{t-1}}{\Pi_t} + \tau_t \]
\[ w_t = f(k_t) + (1 - \delta) k_t + \tau_t + \frac{m_t-1 + I_{t-1} b_{t-1}}{\Pi_t} \]

- To solve the problem, first use the definition of \( w_t \) to eliminate \( w_{t+1} \) at the RHS of Bellman equation:

\[
V(w_t, m_{t-1}) = \max \left[ u(c_t) + \beta V \left( f(k_{t+1}) + (1 - \delta) k_{t+1} + \tau_{t+1} + \frac{m_t + I_t b_t}{\Pi_{t+1}}, m_t \right) \right]
\]

- Secondly, from the Lagrangian:
\[ L = \max \left[ u(c_t) + \beta V \left( f(k_{t+1}) + (1 - \delta) k_{t+1} + \tau_{t+1} + \frac{m_t + I_t b_t}{\Pi_{t+1}}, m_t \right) \right] \\
+ \lambda_t (w_t - c_t - k_{t+1} - m_t - b_t) \\
+ \mu_t \left( \frac{m_{t-1}}{\Pi_t} + \tau_t - c_t \right) \\

- The first order conditions are:

\[ \frac{\partial L}{\partial c_t} : u_c(c_t) - \lambda_t - \mu_t = 0 \]
\frac{\partial L}{\partial k_{t+1}} : \beta V_w (w_{t+1}, m_t) [f_k (k_{t+1}) + 1 - \delta] - \lambda_t = 0

\frac{\partial L}{\partial b_t} : \beta V_w (w_{t+1}, m_t) R_t - \lambda_t = 0

R_t \equiv \frac{I_t}{\Pi_{t+1}}

\frac{\partial L}{\partial m_t} : \beta V_w (w_{t+1}, m_t) \frac{1}{\Pi_{t+1}} + \beta V_m (w_{t+1}, m_t) - \lambda_t = 0
\[
\frac{1}{\Pi_{t+1}} = R_t - \frac{i_t}{\Pi_{t+1}}
\]

- Use envelope theorem to eliminate the partial differentiations of value function in the first order conditions.

- Envelope theorem:

\[
\frac{dV}{d\theta} = \frac{\partial L}{\partial \theta} \equiv L_\theta
\]

\[ V_w (w_t, m_{t-1}) = L_w = \lambda_t \]

\[ V_m (w_t, m_{t-1}) = L_m = \frac{\mu_t}{\Pi_t} \]

• Rearrange the first order conditions as:
\[ u_c(c_t) - \lambda_t - \mu_t = 0 \] (1)

\[ \beta \lambda_{t+1} [f_k(k_{t+1}) + 1 - \delta] - \lambda_t = 0 \] (2)

\[ \beta \lambda_{t+1} R_t - \lambda_t = 0 \] (3)

\[ R_t \equiv \frac{I_t}{\Pi_{t+1}} \] (4)
\[ \beta \lambda_{t+1} \frac{1}{\Pi_{t+1}} + \beta \mu_{t+1} \frac{1}{\Pi_{t+1}} - \lambda_t = 0 \]  \hspace{1cm} (5)

\[ \frac{1}{\Pi_{t+1}} = R_t - \frac{i_t}{\Pi_{t+1}} \]  \hspace{1cm} (6)

- Exercises: use Lagrangian method to solve the CIA model.
\[ L = \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \beta^t \lambda_t \left( f(k_t) + (1 - \delta) k_t + \tau_t + \frac{m_{t-1} + I_{t-1} b_{t-1}}{\Pi_t} \right) + \sum_{t=0}^{\infty} \beta^t \mu_t \left( \frac{m_{t-1}}{\Pi_t} + \tau_t - c_t \right) \]
References

