Lecture #7

Functions of Random Variables

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Function of Single Variable

Theorem

Suppose that X is a discrete random variable with probability distribution $f_X(x)$. Let $Y = h(X)$ define a one-to-one transformation between the values of X and Y so that the equation $y = h(x)$ can be solved uniquely for x in terms of y. Let the solution be $x = u(y)$. Then the probability distribution of the random variable y is

$$P(Y = y) = f_Y(y) = f_X(u(y)) = P(X = u(y)) \tag{1}$$

Function of Single Variable: Example

Example

Let X be a geometric random variable with probability distribution

$$f_X(x) = p(1-p)^{x-1},\ x = 1, 2, \ldots$$

Find the probability distribution of $Y = X^2$.

- Since $X > 0$, $Y = X^2$ (or $X = \sqrt{Y}$) is one-to-one transformation.
- Therefore, the distribution of the random variable Y is

$$f_Y(y) = f_X(\sqrt{y}) = p(1-p)^{\sqrt{y}-1},\ y = 1, 4, 9, \ldots$$

Function of Two Random Variables

Theorem

Suppose that X_1 and X_2 are two discrete random variables with joint probability distribution $f_{X_1X_2}(x_1,x_2)$ (or $f(x_1,x_2)$). Let $Y_1 = h_1(X_1,X_2)$ and $Y_2 = h_2(X_1,X_2)$ define one-to-one transformations between the points (x_1,x_2) and (y_1,y_2) so that equations $y_1 = h_1(x_1,x_2)$ and $y_2 = h_2(x_1,x_2)$ can be solved uniquely for x_1 and x_2 in terms of y_1 and y_2, i.e., $x_1 = u_1(y_1,y_2)$ and $x_2 = u_2(y_1,y_2)$. Then the joint probability distribution of Y_1 and Y_2 is

$$f_{Y_1,Y_2}(y_1,y_2) = f_{X_1X_2}[u_1(y_1,y_2),u_2(y_1,y_2)] \tag{2}$$

Note that the distribution of Y_1 can be found by summing over the y_2, i.e., the marginal probability distribution of Y_1,

$$f_{Y_1}(y_1) = \sum_{y_2} f_{Y_1,Y_2}(y_1,y_2)$$

MGF of Sum Two Independent Poisson RVs I

Example

Suppose that X_1 and X_2 are two independent Poisson random variables with parameters λ_1 and λ_2. Find the probability distribution of $Y = X_1 + X_2$.

- Let $Y = Y_1 = X_1 + X_2$ and $Y_2 = X_2$. Hence, $X_1 = Y_1 - Y_2$ and $X_2 = Y_2$.
- Since X_1 and X_2 are independent, the joint distribution of X_1 and X_2 is

$$f_{X_1X_2}(x_1, x_2) = \frac{\lambda_1^{x_1} e^{-\lambda_1}}{x_1!} \cdot \frac{\lambda_2^{x_2} e^{-\lambda_2}}{x_2!}$$

$$x_1 = 0, 1, 2, \ldots, \; x_2 = 0, 1, 2, \ldots$$

MGF of Sum Two Independent Poisson RVs II

- The joint distribution of Y_1 and Y_2 is

$$\begin{aligned}
f_{Y_1,Y_2}(y_1,y_2) &= f_{X_1X_2}[u_1(y_1,y_2),u_2(y_1,y_2)] \\
&= f_{X_1X_2}(y_1-y_2,y_2) \\
&= \frac{\lambda_1^{y_1-y_2}e^{-\lambda_1}}{(y_1-y_2)!}\cdot\frac{\lambda_2^{y_2}e^{-\lambda_2}}{y_2!} \\
&= e^{-(\lambda_1+\lambda_2)}\frac{\lambda_1^{y_1-y_2}\lambda_2^{y_2}}{(y_1-y_2)!y_2!} \\
& \quad y_1=0,1,2,\ldots,\; y_2=0,1,2,\ldots,y_1
\end{aligned}$$

Note that $x_1 \geq 0$ so that $y_1 - y_2 \geq 0$.

MGF of Sum Two Independent Poisson RVs III

- The marginal distribution of Y_1 is

$$
\begin{aligned}
f_{Y_1}(y_1) &= \sum_{y_2=0}^{y_1} f_{Y_1,Y_2}(y_1, y_2) \\
&= e^{-(\lambda_1+\lambda_2)} \sum_{y_2=0}^{y_1} \frac{\lambda_1^{y_1-y_2}\lambda_2^{y_2}}{(y_1-y_2)!y_2!} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{y_1!} \sum_{y_2=0}^{y_1} \frac{y_1!}{(y_1-y_2)!y_2!} \lambda_1^{y_1-y_2}\lambda_2^{y_2} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{y_1!} (\lambda_1+\lambda_2)^{y_1}
\end{aligned}
$$

- The random variable $Y = Y_1 = X_1 + X_2$ has Poisson distribution with parameter $\lambda_1 + \lambda_2$.

Function of Single Variable I

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Theorem

Suppose that X is a continuous random variable with probability distribution $f_X(x)$. Let $Y = h(X)$ define a one-to-one transformation between the values of X and Y so that the equation $y = h(x)$ can be solved uniquely for x in terms of y. Let the solution be $x = u(y)$. Then the probability distribution of the random variable y is

$$f_Y(y) = f_X(u(y))|J| \tag{3}$$

where $J = \frac{d}{dy}u(y) = u'(y) = \frac{dx}{dy}$ is called the Jacobian of the transformation and the absolute value $|J|$ of J is used.

- Suppose that the function $y = h(x)$ is an increasing function of x.

Function of Single Variable II

- now, we have

$$P(Y \leq a) = P[X \leq u(a)] = \int_{-\infty}^{u(a)} f_X(x)dx$$

- Since $x = u(y)$, we obtain $dx = u'(y)dy$ and

$$P(Y \leq a) = \int_{-\infty}^{a} f_X(u(y))u'(y)dy$$

- The density function of Y is

$$f_Y(y) = f_X(u(y))u'(y) = f_X(u(y))J.$$

- If $y = h(x)$ is a decreasing function of x, a similar argument holds.

Function of Single Variable: Example 1 I

Example

Suppose that X has pdf

$$f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{elsewhere} \end{cases} \tag{4}$$

Let $h(X) = 3X + 1$. How to find the pdf of $Y = h(X)$.

- Since $y = h(x) = 3x + 1$ is an increasing function of x, we obtain $x = u(y) = (y - 1)/3$, $J = u'(y) = 1/3$,

$$f_Y(y) = f_X(u(y))u'(y) = 2(y - 1)/3 \cdot 1/3 = \frac{2}{9}(y - 1).$$

Function of Single Variable: Example 1 II

- Intuitive approach

$$
\begin{aligned}
F_Y(y) &= P(Y \le y) \\
&= P[(3X+1) \le y] = P[X \le (y-1)/3] \\
&= \int_0^{(y-1)/3} 2x dx = [(y-1)/3]^2. \\
f_Y(y) &= \frac{d}{dy} F_Y(y) \\
&= \begin{cases} \frac{2}{9}(y-1), & 1 < y < 4 \\ 0, & \text{elsewhere} \end{cases}
\end{aligned}
$$

Function of Single Variable: Example 2 I

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Discrete RVs
Continuous RVs
Moment Generating Functions

Example

Suppose that X has pdf

$$f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{elsewhere} \end{cases} \tag{5}$$

Let $h(X) = e^{-X}$. How to find the pdf of $Y = h(X)$.

- Since $y = h(x) = e^{-x}$ is a decreasing function of x, we obtain $x = u(y) = -\ln y$, $J = u'(y) = -1/y$,

$$\begin{aligned} f_Y(y) &= f_X(u(y))|J| \\ &= -2\ln y \cdot 1/y = \frac{-2}{y}\ln y, \\ & \quad 1/e < y < 1. \end{aligned}$$

Function of Single Variable: Example 2 II

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- Intuitive approach

$$
\begin{aligned}
F_Y(y) &= P(Y \leq y) \\
&= P(e^{-X} \leq y) = P(X \geq -\ln y) \\
&= \int_{-\ln y}^{1} 2x dx = 1 - (-\ln y)^2. \\
f_Y(y) &= \frac{d}{dy} F_Y(y) = \frac{-2}{y} \ln y, \\
& \quad 1/e < y < 1.
\end{aligned}
$$

Function of Single Variable: Non-monotonic Transformation I

Example

Suppose that X has pdf $f(x)$. Let $h(X) = X^2$. How to find the pdf of $Y = h(X)$.

- Since $y = h(x) = x^2$ is neither an increasing nor a decreasing function of x, we can not directly use the Eq. (3).
- We can obtain the pdf of $Y = X^2$ as follows:

$$\begin{aligned} F_Y(y) &= P(Y \le y) = P(X^2 \le y) \\ &= P(-\sqrt{y} \le X \le \sqrt{y}) \\ &= F_X(\sqrt{y}) - F_X(-\sqrt{y}) \end{aligned}$$

Function of Single Variable: Non-monotonic Transformation II

$$
\begin{aligned}
f_Y(y) &= \frac{d}{dy}F_Y(y) = \frac{d}{dy}[F_X(\sqrt{y}) - F_X(-\sqrt{y})] \\
&= \frac{dx}{dy}\frac{d}{dx}[F_X(\sqrt{y}) - F_X(-\sqrt{y})] \\
&= \frac{f_X(\sqrt{y})}{2\sqrt{y}} - \frac{f_X(-\sqrt{y})}{-2\sqrt{y}} \\
&= \frac{1}{2\sqrt{y}}[f_X(\sqrt{y}) + f_X(-\sqrt{y})]
\end{aligned}
$$

Non-monotonic Transformation: Normal vs Chi-Square I

Example

Suppose that X has pdf $N(0,1)$ Let $h(X) = X^2$. How to find the pdf of $Y = h(X)$.

- X is standard normal distribution with pdf

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

- We can obtain the pdf of $Y = X^2$ as follows:

$$\begin{aligned} f_Y(y) &= \frac{1}{2\sqrt{y}} [f_X(\sqrt{y}) + f_X(-\sqrt{y})] \\ &= \frac{1}{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-y/2} \\ &= \frac{1}{2^{1/2}\sqrt{\pi}} y^{1/2-1} e^{-y/2}, \ y > 0. \end{aligned}$$

Non-monotonic Transformation: Normal vs Chi-Square II

- The gamma distribution has the pdf

$$f(x;\gamma,\lambda) = \frac{\lambda^\gamma x^{\gamma-1} e^{-\lambda x}}{\Gamma(\gamma)} = \frac{\lambda}{\Gamma(\gamma)}(\lambda x)^{\gamma-1} e^{-\lambda x},\ x > 0.$$

When $\lambda = 1/2$ and $\gamma = 1/2, 1, 3/2, 2, \ldots$, it becomes chi-square distribution:

$$f(x;\gamma,1/2) = \frac{1}{\Gamma(\gamma)}\left(\frac{1}{2}\right)^\gamma x^{\gamma-1} e^{-x/2},\ x > 0.$$

- Since $\sqrt{\pi} = \Gamma(\frac{1}{2})$, we can rewrite $f_Y(y)$ as

$$f_Y(y) = \frac{1}{\Gamma(\frac{1}{2})}\left(\frac{1}{2}\right)^{1/2} y^{1/2-1} e^{-y/2}$$

Y has a chi-square distribution.

General Transformation

Theorem

Suppose that X is a continuous random variable with probability distribution $f_X(x)$, and $Y = h(X)$ is a transformation that is not one-to-one. If the interval over which X is defined can be partitioned into m mutually exclusive disjoint sets such that each of the inverse functions $x_1 = u_1(y), x_2 = u_2(y), \ldots, x_m = u_m(y)$ of $y = h(x)$ is one-to-one, the probability distribution of Y is

$$f_Y(y) = \sum_{i=0}^{m} f_X[u_i(y)]|J_i| \quad (6)$$

Function of Two Random Variables

Theorem

Suppose that X_1 *and* X_2 *are two continuous random variables with joint probability distribution* $f_{X_1X_2}(x_1, x_2)$ *(or* $f(x_1, x_2)$*). Let* $Y_1 = h_1(X_1, X_2)$ *and* $Y_2 = h_2(X_1, X_2)$ *define one-to-one transformations between the points* (x_1, x_2) *and* (y_1, y_2) *so that equations* $y_1 = h_1(x_1, x_2)$ *and* $y_2 = h_2(x_1, x_2)$ *can be solved uniquely for* x_1 *and* x_2 *in terms of* y_1 *and* y_2*, i.e.,* $x_1 = u_1(y_1, y_2)$ *and* $x_2 = u_2(y_1, y_2)$*. Then the joint probability distribution of* Y_1 *and* Y_2 *is*

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1X_2}[u_1(y_1, y_2), u_2(y_1, y_2)] \cdot |\mathbf{J}| \qquad (7)$$

where $\mathbf{J}$ *is the Jacobian and is given by the following determinant:*

$$\mathbf{J} = \begin{vmatrix} \partial x_1/\partial y_1 & \partial x_1/\partial y_2 \\ \partial x_2/\partial y_1 & \partial x_2/\partial y_2 \end{vmatrix} \qquad (8)$$

and the absolute value of the determinant is used.

Function of Two Random Variables I

Example

Suppose that X_1 and X_2 are two independent exponential random variables with $f_{X_1}(x_1) = 2e^{-2x_1}$ and $f_{X_2}(x_2) = 2e^{-2x_2}$. Find the probability distribution of $Y = X_1/X_2$.

- The joint probability distribution of X_1 and X_2 is

$$f_{X_1X_2}(x_1, x_2) = 4e^{-2(x_1+x_2)}$$

- Let $Y = Y_1 = h_1(X_1, X_2) = X_1/X_2$ and $Y_2 = h_2(X_1, X_2) = X_1 + X_2$.

Function of Two Random Variables II

- The inverse solutions of $y_1 = x_1/x_2$ and $y_2 = x_1 + x_2$ are $x_1 = y_1y_2/(1+y_1)$ and $x_2 = y_2/(1+y_1)$, and it follows that

$$\mathbf{J} = \begin{vmatrix} \frac{y_2}{(1+y_1)^2} & \frac{y_1}{(1+y_1)} \\ \frac{-y_2}{(1+y_1)^2} & \frac{1}{(1+y_1)} \end{vmatrix} = \frac{y_2}{(1+y_1)^2}$$

- Then the joint probability distribution of Y_1 and Y_2 is

$$\begin{aligned} f_{Y_1,Y_2}(y_1,y_2) &= f_{X_1X_2}[u_1(y_1,y_2),u_2(y_1,y_2)] \cdot |\mathbf{J}| \\ &= 4e^{-2(y_1y_2/(1+y_1)+y_2/(1+y_1))} \cdot \frac{y_2}{(1+y_1)^2} \\ &= 4e^{-2y_2}\frac{y_2}{(1+y_1)^2} \\ & \quad y_1 > 0, y_2 > 0. \end{aligned}$$

Function of Two Random Variables III

- We need to find the distribution of $Y = Y_1 = X_1/X_2$. The marginal distribution of Y_1 is

$$
\begin{aligned}
f_{Y_1}(y_1) &= \int_0^\infty f_{Y_1,Y_2}(y_1, y_2)dy_2 \\
&= \int_0^\infty 4e^{-2y_2}\frac{y_2}{(1+y_1)^2}dy_2 \\
&= \frac{1}{(1+y_1)^2},\ y > 0 \\
&= f_Y(y)
\end{aligned}
$$

Example: Function of Two $N(0,1)$ I

Example

Let X_1 and X_2 be two independent normal random variables and both have a $N(0,1)$ distribution. Let $Y_1 = \frac{X_1+X_2}{2}$ and $Y_2 = \frac{X_2-X_1}{2}$. Show that Y_1 and Y_2 are independent.

- Since X_1 and X_2 are independent, the joint pdf is

$$f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1) \cdot f_{X_2}(x_2) = \frac{1}{2\pi} \exp^{\frac{-1}{2}\left(x_1^2+x_2^2\right)}.$$

- $X_1 = u_1(Y_1,Y_2) = Y_1 - Y_2$ and $X_2 = u_2(Y_1,Y_2) = Y_1 + Y_2$.
- The Jacobian

$$\mathbf{J} = \begin{vmatrix} \partial x_1/\partial y_1 & \partial x_1/\partial y_2 \\ \partial x_2/\partial y_1 & \partial x_2/\partial y_2 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2$$

Example: Function of Two $N(0,1)$ II

- The joint pdf of Y_1 and Y_2 is

$$
\begin{aligned}
f_{Y_1,Y_2}(y_1,y_2) &= f_{X_1X_2}[u_1(y_1,y_2),u_2(y_1,y_2)]\cdot|\mathbf{J}| \\
&= \frac{2}{2\pi}\exp\left\{\frac{-1}{2}\left[(y_1-y_2)^2+(y_1+y_2)^2\right]\right\} \\
&= \frac{1}{\left(\sqrt{2\pi}\right)^2\left(\frac{1}{\sqrt{2}}\right)^2}\exp^{\frac{-1}{2}\left[\frac{y_1^2}{\left(\frac{1}{\sqrt{2}}\right)^2}+\frac{y_2^2}{\left(\frac{1}{\sqrt{2}}\right)^2}\right]} \\
&= \frac{1}{\sqrt{2\pi}\cdot\left(\frac{1}{\sqrt{2}}\right)}\exp^{\frac{-1}{2}\left[\frac{y_1^2}{\left(\frac{1}{\sqrt{2}}\right)^2}\right]} \\
&\cdot \frac{1}{\sqrt{2\pi}\cdot\left(\frac{1}{\sqrt{2}}\right)}\exp^{\frac{-1}{2}\left[\frac{y_2^2}{\left(\frac{1}{\sqrt{2}}\right)^2}\right]}.
\end{aligned}
$$

Example: Function of Two $N(0,1)$ III

- Therefore, $f_{Y_1,Y_2}(y_1,y_2) = f_{Y_1}(y_1) \cdot f_{Y_2}(y_2)$.
- Y_1 and Y_2 both are normal distribution with $N(0, \frac{1}{2})$.

Example: Function of Multiple Random Variables I

Example

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables from a uniform distribution on $[0, 1]$. Let $Y = \max\{X_1, X_2, \ldots, X_n\}$ and $Z = \min\{X_1, X_2, \ldots, X_n\}$. What are the pdf's of Y and Z?

- The pdf of Y:

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(\max\{X_1, X_2, \ldots, X_n\} \leq y) \\ &= P(X_1 \leq y, \ldots, X_n \leq y) \\ &= P(X_1 \leq y) \times \cdots \times P(X_n \leq y) \\ &= y^n. \end{aligned}$$

Example: Function of Multiple Random Variables II

Hence $f_y(y) = \frac{dF_Y(y)}{dy} = ny^{n-1},\ 0 \leq y \leq 1.$

$$\begin{aligned} F_Z(z) &= P(Z \leq z) = P(\min\{X_1, X_2, \ldots, X_n\} \leq z) \\ &= 1 - P(X_1 > z, \ldots, X_n > z) \\ &= 1 - P(X_1 > z) \times \cdots \times P(X_n > z) \\ &= 1 - (1 - z)^n. \end{aligned}$$

Hence $f_Z(z) = \frac{dF_Z(z)}{dz} = n(1 - z)^{n-1},\ 0 \leq z \leq 1.$

Moment Generating Function (mgf)

Definition (Discrete Random Variable)

Let X be a discrete random variable with probability mass function $P(X = x_i) = f(x_i), i = 1, 2, \ldots$. The function M_X is called the moment generating function of X is defined by

$$M_X(t) = E(e^{tX}) = \sum_{i=1}^{\infty} e^{tx_i} f(x_i) \tag{9}$$

Definition (Continuous Random Variable)

If X be a continuous random variable with probability density function $f(x)$, we define the moment generating function of X by

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x) dx \tag{10}$$

MGF of Poisson Distribution

Example

If X is a Poisson random variable, $f(x) = \frac{e^{-\lambda}\lambda^x}{x!}$, with parameter $\lambda > 0$, find its moment generating function.

Solution

$$\begin{aligned} M_X(t) &= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!} \\ &= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} \\ &= e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t - 1)} \end{aligned} \quad (11)$$

MGF of Exponential Distribution

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Example

If X is an exponential random variable, $f(x) = \lambda e^{-\lambda x}$, with parameter $\lambda > 0$, find its moment generating function.

Solution

$$\begin{aligned} M_X(t) &= \int_0^\infty e^{tx}\lambda e^{-\lambda x}dx \\ &= \lambda \int_0^\infty e^{-x(\lambda - t)}dx \\ &= \frac{\lambda}{\lambda - t},\ t < \lambda \end{aligned} \tag{12}$$

MGF of Normal Distribution(I)

Example

If $X \sim N(0,1)$, find its moment generating function.

Solution

$$\begin{aligned}
M_X(t) &= E\{e^{tX}\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-x^2/2} dx \\
&= \frac{1}{\sqrt{2\pi}} e^{t^2/2} \int_{-\infty}^{\infty} e^{-t^2/2} e^{tx} e^{-x^2/2} dx \\
&= e^{t^2/2} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-t)^2/2} dx \right] \\
&= e^{t^2/2} \qquad (13)
\end{aligned}$$

MGF of Gamma Distribution I

Example

If X is a Gamma distribution with pdf

$$f(x;\gamma,\lambda) = \begin{cases} \frac{\lambda^\gamma}{\Gamma(\gamma)} x^{\gamma-1} e^{-\lambda x}, & 0 < x < \infty \\ 0, & \text{elsewhere} \end{cases},$$

find its moment generating function.

$$\begin{aligned} M_X(t) &= E\{e^{tX}\} = \int_0^\infty e^{tx} \frac{\lambda^\gamma}{\Gamma(\gamma)} x^{\gamma-1} e^{-\lambda x} dx \\ &= \int_0^\infty \frac{\lambda^\gamma}{\Gamma(\gamma)} x^{\gamma-1} e^{-(\lambda-t)x} dx \end{aligned}$$

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MGF of Gamma Distribution II

Let $y = (\lambda - t)x$, then $x = y/(\lambda - t)$ and $dx = dy/(\lambda - t)$. We have

$$
\begin{aligned}
M_X(t) &= \int_0^\infty \frac{\lambda^\gamma}{\Gamma(\gamma)} x^{\gamma-1} e^{-(\lambda-t)x} dx \\
&= \int_0^\infty \frac{\lambda^\gamma}{\Gamma(\gamma)} \left(\frac{y}{(\lambda - t)}\right)^{\gamma-1} e^{-y} \frac{1}{(\lambda - t)} dy \\
&= \left(\frac{\lambda}{\lambda - t}\right)^\gamma \int_0^\infty \frac{1}{\Gamma(\gamma)} y^{\gamma-1} e^{-y} dy \\
&= \left(\frac{\lambda}{\lambda - t}\right)^\gamma \\
&= \frac{1}{(1 - t/\lambda)^\gamma}
\end{aligned}
$$

Properties of MGF

- Recall the Taylor series of the function e^x:

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

  Thus

$$e^{tx} = 1 + tx + \frac{(tx)^2}{2!} + \cdots + \frac{(tx)^n}{n!} + \cdots$$

- The moment generating of the random variable X is

$$\begin{aligned} M_X(t) &= E(e^{tX}) \\ &= E\left(1 + tX + \frac{(tX)^2}{2!} + \cdots + \frac{(tX)^n}{n!} + \cdots\right) \\ &= 1 + tE(X) + \frac{t^2E(X^2)}{2!} + \cdots + \frac{t^nE(X^n)}{n!} + \cdots \end{aligned}$$

Properties of MGF

- Consider the first derivative of $M_X(t)$ with respect to t and obtain

$$M_X'(t) = E(X) + tE(X^2) + \cdots + nt^{n-1}\frac{E(X^n)}{n!} + \cdots$$

Taking $t = 0$, we have

$$M_X'(t)|_{t=0} = M_X'(0) = E\{X\} = \mu_X$$

- Consider the second derivative of $M_X(t)$ and obtain

$$M_X''(t) = E(X^2) + tE(X^3) \cdots + n(n-1)t^{n-2}\frac{E(X^n)}{n!} + \cdots$$

The second moment of X is $M_X''(0) = E\{X^2\}$. Then, the variance of X is given by

$$V(X) = E\{X^2\} - E^2\{X\} = M_X''(0) - (M_X'(0))^2$$

Properties of MGF

- Differentiating k times and then setting $t = 0$ yields

$$M_X^{(k)}(t)|_{t=0} = E\{X^k\} \tag{14}$$

- If X is a random variable and α is a constant, then

$$M_{X+\alpha}(t) = e^{\alpha t} M_X(t) \tag{15}$$

$$M_{\alpha X}(t) = M_X(\alpha t) \tag{16}$$

- Let X and Y be two random variables with moment generating functions $M_X(t)$ and $M_Y(t)$, respectively. If $M_X(t) = M_Y(t)$ for all values of t, then X and Y have the same probability distribution.

MGF of Normal Distribution(II)

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Example

Use Eq. (13) to find the first four moments of the normal rv X.

Solution

- $M'_X(t) = \frac{d}{dt} e^{t^2/2} = e^{t^2/2} \cdot t = t \cdot M_X(t)$.
  $E\{X^1\} = M'_X(0) = 0$.
- $M''_X(t) = M_X(t) + t \cdot M'_X(t) = M_X(t) + t^2 \cdot M_X(t)$.
  $E\{X^2\} = M''_X(0) = 1$.
- $M_X^{(3)}(t) = (2t) \cdot M_X(t) + (1 + t^2) \cdot M'_X(t) = (3t + t^3) \cdot M_X(t)$.
  $E\{X^3\} = M_X^{(3)}(0) = 0$.
- $M_X^{(4)}(t) = (3 + 3t^2) \cdot M_X(t) + (3t + t^3) \cdot M'_X(t)$.
  $E\{X^4\} = M_X^{(4)}(0) = 3$.

MGF of Normal Distribution(III)

Example

Use Eqs. (14) and (15) to find the mgf of a normal distribution with mean μ and variance σ^2.

- Assume X be a normal distribution with mean μ and variance σ^2.
- If $Z = \frac{X-\mu}{\sigma}$, Z is a standard normal random variable.
- The mgf of X is

$$\begin{aligned} M_X(t) &= M_{\sigma Z+\mu}(t) = e^{\mu t} M_{\sigma Z}(t) \\ &= e^{\mu t} M_Z(\sigma t) \\ &= e^{\mu t} e^{(\sigma t)^2/2} \\ &= e^{\mu t + \sigma^2 t^2/2} \end{aligned} \tag{17}$$

Sum of Independent RVs: MGF I

Theorem

If $X_1, X_2, \ldots, X_n$ are independent random variables with moment generating functions $M_{X_1}(t), M_{X_2}(t), \ldots, M_{X_n}(t)$, respectively, and if $Y = X_1 + X_2 + \cdots + X_n$ then the moment generating function of Y is

$$M_Y(t) = M_{X_1}(t) \cdot M_{X_2}(t) \cdot \cdots \cdot M_{X_n}(t) \tag{18}$$

- For the continuous case,

$$\begin{aligned} M_Y(t) &= E(e^{tY}) = E(e^{t(X_1+X_2+\cdots+X_n)}) \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{t(x_1+x_2+\cdots+x_n)} f(x_1, x_2, \ldots, x_n) \\ &\quad dx_1 dx_2 \cdots dx_n \end{aligned}$$

Sum of Independent RVs: MGF II

- Since the variables are independent, we have

$$f(x_1, x_2, \ldots, x_n) = f(x_1)f(x_2)\cdots f(x_n)$$

- Therefore,

$$\begin{aligned} M_Y(t) &= \int_{-\infty}^{\infty} e^{tx_1} f(x_1)dx_1 \cdot \int_{-\infty}^{\infty} e^{tx_2} f(x_2)dx_2 \\ \cdots &\cdot \int_{-\infty}^{\infty} e^{tx_n} f(x_n)dx_n \\ &= M_{X_1}(t) \cdot M_{X_2}(t) \cdot \cdots \cdot M_{X_n}(t) \end{aligned}$$

MGF of Sum Two Independent Poisson RVs

Example

Suppose that X_1 and X_2 are two independent Poisson random variables with parameters λ_1 and λ_2. Find the probability distribution of $Y = X_1 + X_2$.

- The mgf's of X_1 and X_2 are $M_{X_1}(t) = e^{\lambda_1(e^t-1)}$ and $M_{X_2}(t) = e^{\lambda_2(e^t-1)}$, respectively.
- The mgf of Y is

$$\begin{aligned} M_Y(t) &= M_{X_1}(t) \cdot M_{X_2}(t) = e^{\lambda_1(e^t-1)} \cdot e^{\lambda_2(e^t-1)} \\ &= e^{(\lambda_1+\lambda_2)(e^t-1)} \end{aligned} \tag{19}$$

- Therefore, Y is also a Poisson random variable with parameters $\lambda_1 + \lambda_2$.