# Autocorrelation and Partial Autocorrelation Functions

Ching-Kang Ing

National Tsing Hua University

*cking@stat.nthu.edu.tw*

# Overview

1. Autocorrelatoin Function

2. Partial Autocorrelation Function

# Autocorrelatoin Function

In the following, we consider $Z_t$ as a stationary process, and, without loss of generality, we assume $\mathrm{E}(Z_t) = 0$.

**Definition**

The autocovariance function of $Z_t$ at lag $k$, $k = 0, 1, \ldots$, is defined by $\gamma_k = \mathrm{E}(Z_t Z_{t+k})$.

**Definition**

The autocorrelation function (ACF) of $Z_t$ at lag $k$, $k = 0, 1, \ldots$, is defined by $\rho_k = \gamma_k / \gamma_0 = \mathrm{Corr}(Z_t, Z_{t+k})$.

# Partial Autocorrelation Function

## Definition

The partial autocorrelation function (PACF) of $Z_t$ at lag $k$, $k = 0, 1, \ldots,$ is defined by

$$P_k = \begin{cases} \rho_k, & \text{if } k = 0, 1, \\ \mathrm{Corr}(Z_t - Z_t^*, Z_{t+k} - Z_{t+k}^*), & \text{if } k \geq 2, \end{cases}$$

where $Z_t^*$ and $Z_{t+k}^*$ are the best linear predictor of $Z_t$ and $Z_{t+k}$, respectively based on $Z_{t+1}, \ldots, Z_{t+k-1}$.

**Remark 1.** We say $Z^*_{t+k}$ is the best linear predictor of $Z_{t+k}$ based on $Z_{t+1},\ldots,Z_{t+k-1}$ if

$$
\begin{aligned}
Z^*_{t+k} &= \alpha_1 Z_{t+k-1} + \cdots + \alpha_{k-1} Z_{t+1}, \\
&:= \boldsymbol{\alpha}_{k-1}^\top \mathbf{Z}_{t+k-1}(k-1),
\end{aligned}
$$

where $\mathbf{Z}_{t+k-1}(k-1) = (Z_{t+k-1},\ldots,Z_{t+1})^\top$ and

$$
\boldsymbol{\alpha}_{k-1} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{k-1} \end{pmatrix} = \arg\min_{\boldsymbol{\eta}\in\mathbb{R}^{k-1}} \mathrm{E}(Z_{t+k} - \boldsymbol{\eta}^\top \mathbf{Z}_{t+k-1}(k-1))^2.
$$

**Remark 2.** To express $\boldsymbol{\alpha}_{k-1}$ in terms of $\gamma_k$, $k = 0, 1, \ldots,$ define

$$f(\boldsymbol{\eta}) = \mathrm{E}[Z_{t+k} - \boldsymbol{\eta}^\top \mathbf{Z}_{t+k-1}(k-1)]^2.$$

It is easy to verify that

$$f(\boldsymbol{\eta}) = \gamma_0 - 2\boldsymbol{\gamma}_{k-1}^\top \boldsymbol{\eta} + \boldsymbol{\eta}^\top \mathbf{R}_{k-1} \boldsymbol{\eta},$$

where $\boldsymbol{\gamma}_{k-1} = (\gamma_1, \ldots, \gamma_{k-1})^\top$ and

$$\mathbf{R}_{k-1} = \begin{pmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_{k-2} \\ \gamma_1 & \gamma_0 & \cdots & \gamma_{k-3} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{k-2} & \gamma_{k-3} & \cdots & \gamma_0 \end{pmatrix}.$$

Setting $\frac{\partial f(\boldsymbol{\eta})}{\partial \boldsymbol{\eta}} = 0$, one gets $-2\gamma_{k-1} + 2\mathbf{R}_{k-1}\boldsymbol{\eta} = 0$, which yields

$$\boldsymbol{\eta}^* = \mathbf{R}_{k-1}^{-1}\gamma_{k-1}.$$

Furthermore, since $\frac{\partial^2 f(\boldsymbol{\eta})}{\partial \boldsymbol{\eta}^2} = 2\mathbf{R}_{k-1}$ is positive definite, we conclude that

$$\boldsymbol{\alpha}_{k-1} = \boldsymbol{\eta}^* = \mathbf{R}_{k-1}^{-1}\gamma_{k-1}.$$

## Fact

*$P_k$ can be expressed as*

$$P_k = \frac{\gamma_k - \tilde{\boldsymbol{\alpha}}_{k-1}^\top \boldsymbol{\gamma}_{k-1}}{\mathcal{C}},$$

*where $\tilde{\boldsymbol{\alpha}}_{k-1} = (\alpha_{k-1}, \ldots, \alpha_1)^\top$ and $\mathcal{C} = \gamma_0 - \boldsymbol{\gamma}_{k-1}^\top \mathbf{R}_{k-1}^{-1} \boldsymbol{\gamma}_{k-1}$.*

*proof.* First notice that

$$Z_{t+k}^* = \boldsymbol{\alpha}_{k-1}^\top \begin{pmatrix} Z_{t+k-1} \\ \vdots \\ Z_{t+1} \end{pmatrix} \text{ and } Z_t^* = \boldsymbol{\alpha}_{k-1}^\top \begin{pmatrix} Z_{t+1} \\ \vdots \\ Z_{t+k-1} \end{pmatrix}.$$

The above equations give

$$
\begin{aligned}
&\mathrm{Cov}(Z_t - Z_t^*, Z_{t+k} - Z_{t+k}^*) = \mathrm{E}[(Z_t - Z_t^*)(Z_{t+k} - Z_{t+k}^*)] \\
&= \gamma_k - \boldsymbol{\alpha}_{k-1}^\top \begin{pmatrix} \gamma_{k-1} \\ \vdots \\ \gamma_1 \end{pmatrix} - \boldsymbol{\alpha}_{k-1}^\top \begin{pmatrix} \gamma_{k-1} \\ \vdots \\ \gamma_1 \end{pmatrix} + \boldsymbol{\alpha}_{k-1}^\top R_{k-1} \begin{pmatrix} \alpha_{k-1} \\ \vdots \\ \alpha_1 \end{pmatrix} \\
&= \gamma_k - \boldsymbol{\alpha}_{k-1}^\top \begin{pmatrix} \gamma_{k-1} \\ \vdots \\ \gamma_1 \end{pmatrix}.
\end{aligned}
$$

Similarly, we have

$$
\mathrm{E}(Z_t - Z_t^*)^2 = \mathrm{E}(Z_{t+k} - Z_{t+k}^*)^2 = \gamma_0 - \boldsymbol{\gamma}_{k-1}^\top R_{k-1}^{-1} \boldsymbol{\gamma}_{k-1}.
$$

Then the required result follows immediately from the definition of correlation coefficient. $\square$

## Definition

Define

$$\phi_k = (\phi_{k1}, \ldots, \phi_{kk})^\top = \arg\min_{\mathbf{c} \in \mathbb{R}^k} \mathrm{E}[Z_{t+k} - \mathbf{c}^\top \mathbf{Z}_{t+k-1}(k)]^2.$$

## Fact

$P_k = \phi_{kk}$.

*proof.* Consider the following two minimization problem,

$$g_1(\mathbf{c}) = \mathrm{E}[Z_{t+k} - (c_1 Z_{t+k-1} + \cdots + c_{k-1} Z_{t-1} + c_k Z_t)]^2,$$
$$g_2(\mathbf{c}) = \mathrm{E}[Z_{t+k} - (c_1 Z_{t+k-1} + \cdots + c_{k-1} Z_{t-1} + c_k (Z_t - Z_t^*))]^2.$$

Let $\mathbf{d} = \arg\min_{\mathbf{c}\in\mathbb{R}^k} g_2(\mathbf{c})$. Using the similar argument as Remark 2, we obtain

$$\mathbf{d} = \left[ \mathrm{E}\left( \begin{pmatrix} Z_{t+k-1} \\ \vdots \\ Z_{t+1} \\ Z_t - Z_t^* \end{pmatrix} \begin{pmatrix} Z_{t+k-1} & \dots & Z_{t+1} & Z_t - Z_t^* \end{pmatrix} \right) \right]^{-1} \mathrm{E} \begin{pmatrix} Z_{t+k-1}Z_{t+k} \\ \vdots \\ Z_{t+1}Z_{t+k} \\ (Z_t - Z_t^*)Z_{t+k} \end{pmatrix}$$

In addition, by the fact that $\mathrm{E}(Z_{t+i}(Z_t - Z_t^*)) = 0$, $i = 1, \dots, k-1$, and $\mathrm{E}((Z_t - Z_t^*)Z_{t+k}) = P_k \mathrm{E}(Z_t - Z_t^*)^2$, we have

$$\mathbf{d} = \begin{pmatrix} \mathbf{R}_{k-1}^{-1} & \mathbf{0} \\ \mathbf{0}^\top & 1/\mathrm{E}[(Z_t - Z_t^*)^2] \end{pmatrix} \begin{pmatrix} \gamma_{k-1} \\ P_k \mathrm{E}[(Z_t - Z_t^*)^2] \end{pmatrix} = \begin{pmatrix} \alpha_{k-1} \\ P_k \end{pmatrix}. \quad (1)$$

We next show that $g_1(\phi_k) = g_2(\mathbf{d})$. Define

$$\mathbf{A} = \begin{pmatrix} \mathbf{I}_{k-1} & \mathbf{0} \\ -\tilde{\boldsymbol{\alpha}}_{k-1}^\top & 1 \end{pmatrix}.$$

Then, by definition of $Z_t^*$ and $\phi_k$, one gets

$$\begin{aligned} g_2(\mathbf{d}) &= \mathrm{E}[Z_{t+k} - (d_1 Z_{t+k-1} + \cdots + d_k(Z_t - Z_t^*))]^2 \\ &= \mathrm{E}[Z_{t+k} - \mathbf{d}^\top \mathbf{A} \mathbf{Z}_{t+k-1}(k)]^2 \\ &\geq \mathrm{E}[Z_{t+k} - \phi_k^\top \mathbf{Z}_{t+k-1}(k)]^2 = g_1(\phi_k). \end{aligned}$$

Similarly, it can be shown that

$$g_1(\phi_k) \geq g_2(\mathbf{d}).$$

As a result, $g_1(\phi_k) = g_2(\mathbf{d})$.

By the fact that $\mathrm{E}[Z_{t+k-i}(Z_{t+k} - \phi_k^\top \mathbf{Z}_{t+k-1}(k))] = 0$, $i = 1, \ldots, k$, we have

$$
\begin{aligned}
g_2(\mathbf{d}) &= \mathrm{E}[Z_{t+k} - (d_1 Z_{t+k-1} + \cdots + d_k(Z_t - Z_t^*))]^2 \\
&= \mathrm{E}[Z_{t+k} - \mathbf{d}^\top \mathbf{A}\mathbf{Z}_{t+k-1}(k)]^2 \\
&= \mathrm{E}[Z_{t+k} - \phi_k^\top \mathbf{Z}_{t+k-1}(k) - (\mathbf{A}^\top \mathbf{d} - \phi_k)^\top \mathbf{Z}_{t+k-1}(k)]^2 \\
&= g_1(\phi_k) + (\mathbf{A}^\top \mathbf{d} - \phi_k)^\top \mathbf{R}_k (\mathbf{A}^\top \mathbf{d} - \phi_k).
\end{aligned}
$$

Since $\mathbf{R}_k$ is positive definite, $\mathbf{A}^\top \mathbf{d} - \phi_k = \mathbf{0}$ and the desired result follows. $\square$